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ESTIMATION OF THE COVARIANCE FUNCTION OF A HOMOGENEOUS PROCESS ON THE SPHERE

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A homogeneous random process on the sphere $\{X(P): P \in S_2\}$ is a process whose mean is zero and whose covariance function depends only on the angular distance θ between the two points, i.e. $E[X(P)] \equiv 0$ and $E[X(P)X(Q)] = R(\theta)$. Given T independent realizations of a Gaussian homogeneous process $X(P)$, we first derive the exact distribution of the spectral estimates introduced by Jones (1963 b). Further, an estimate $R^{(T)}(\theta)$ of the covariance function $R(\theta)$ is proposed. Exact expressions for its first- and second-order moments are derived and it is shown that the sequence of processes $\{T^{\frac{1}{2}}[R^{(T)}(\theta) - R(\theta)]\}_{T=1}^{\infty}$ converges weakly in $C[0, \pi]$ to a given Gaussian process.

1. Introduction. Let $\{X(P): P \in S_2\}$ be a real-valued process on the unit sphere S_2 of the three-dimensional space $\mathbb{R}^3$, which has finite second-order moment and which is continuous in quadratic mean (q.m.). Under these conditions, the process $X(P)$ can be expanded in a spherical harmonics series which is convergent in q.m. If for each integer $n \geqq 0$, $\{Y_{nk}(P): -n \leqq k \leqq n\}$ denotes an orthonormal basis for the real spherical harmonics of order n, we can write

$$X(P) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} Z_{nk} Y_{nk}(P), \qquad \text{all } P \in S_2 \tag{1.1}$$

where

$$Z_{nk} = \int_{S_2} X(P) Y_{nk}(P) \sigma(dP), \tag{1.2}$$

σ being the surface measure on S_2. The integral (1.2) is the integral in the q.m. sense and the series (1.1) converges in q.m.

The process $X(P)$ is said to be homogeneous if its first and second-order moments are invariant under the group of rotations of the sphere. This is equivalent to saying that the mean $E[X(P)]$ is constant (and without loss of generality, we can assume that $E[X(P)] \equiv 0$) and that the covariance function $E[X(P)X(Q)]$ depends only on the angular distance θ_{PQ} between the points P and Q. $E[X(P)] \equiv 0$ implies that $E[Z_{nk}] = 0$ and from Obukhov (1947) (see Yaglom (1961)), the process $X(P)$ is homogeneous if and only if

$$E[Z_{nk} Z_{mh}] = \delta_{nm} \delta_{kh} a_n \geqq 0 \tag{1.3}$$

for $-n \leqq k \leqq n$, $-m \leqq h \leqq m$ and $m, n \geqq 0$, δ_{ij} being the Kronecker delta.

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From (1.1) and (1.3), it is easily deduced that

$$E[X(P)X(Q)] = R(\theta_{PQ}) = (4\pi)^{-1} \sum_{n=0}^{\infty} (2n+1) a_n P_n(\cos\theta_{PQ}) \tag{1.4}$$

where $P_n(\cdot)$ is the Legendre polynomial of degree n and the spectral parameters a_n are such that

$$\sum_{n=0}^{\infty} n a_n < \infty . \tag{1.5}$$

Jones (1963b) has proposed estimates of the spectral parameters a_n, given one realization of the process on the sphere. The estimates considered are unbiased and for a Gaussian process, the variance is calculated. In Section 2 of this paper, the exact distribution of these estimates is obtained for a Gaussian process. In Section 3, an estimate of the covariance function $R(\theta)$ is presented and some of its statistical properties are studied. For numerical analysis of random processes on the sphere, the reader is referred to Jones (1963a), Cohen and Jones (1969).

2. Spectral estimates. In the following, we will say that the process $X(P)$ is Gaussian if for any finite collection of points $P_1, P_2, \cdots, P_k \in S_2$, $X(P_1), \cdots, X(P_k)$ has a joint normal distribution. For a Gaussian process, the coefficients Z_{nk} have a joint normal distribution since they are defined as q.m. integrals and since the q.m. limit of a sequence of finite linear combinations of normal variables is always normal. Then we have,

LEMMA 2.1. *If $X(P)$ is a Gaussian homogeneous process whose mean is zero, then the coefficients Z_{nk} are mutually independent and Z_{nk}, $-n \leqq k \leqq n$ are $N(0, a_n)$ if $a_n > 0$, degenerate at* 0 *if $a_n = 0$.*

Given one realization of the process $X(P)$, one can compute the coefficients Z_{nk} by (1.2) and since $E[Z_{nk}^2] = a_n$ for $-n \leqq k \leqq n$, a natural unbiased estimate of a_n is given by

$$\hat{a}_n = (2n+1)^{-1} \sum_{k=-n}^{n} Z_{nk}^2 . \tag{2.1}$$

If $a_n = 0$, $\hat{a}_n$ is degenerate at 0 and for $a_n > 0$, $(2n+1)\hat{a}_n/a_n$ is $\chi^2_{(2n+1)}$ (chi-square with $(2n+1)$ degrees of freedom). Also, the random variables $\hat{a}_n$, $n = 0, 1, 2, \cdots$ are mutually independent.

By replacing Z_{nk} by its definition (1.2) in (2.1) and using the addition formula for spherical harmonics (see Sansome (1959), page 268), one can write

$$\hat{a}_n = (4\pi)^{-1} \int_{S_2} \int_{S_2} P_n(\cos\theta_{PQ}) X(P) X(Q) \sigma(dP)\sigma(dQ) ,$$

which is precisely the estimate studied by Jones (1963b). Besides computing its variance in the case of a Gaussian process, Jones has shown that $\hat{a}_n$ is the unbiased estimate of minimum variance in the class of the quadratic estimates of the form $a_n^* = \int_{S_2} \int_{S_2} W(P, Q) X(P) X(Q) \sigma(dP)\sigma(dQ)$ where $W(P, Q)$ is a symmetric square integrable function.

Suppose now that we have T independent realizations of the process:

$\{X(P, t): P \in S_2\}$, $t = 1, 2, \cdots, T$. Let $\hat{a}_{n,t}$ be the estimate (2.1) corresponding to the tth realization, then

$$a_n^{(T)} = T^{-1} \textstyle\sum_{t=1}^{T} \hat{a}_{n,t} \tag{2.2}$$

is unbiased for a_n, and by the previous remarks we have the following results:

THEOREM 2.2. *Let $X(P)$ be a Gaussian homogeneous process whose mean is zero and let $a_n^{(T)}$ be given by* (2.2). *Then, if $a_n = 0$, $a_n^{(T)}$ is degenerate at* 0 *and if $a_n > 0$, $T(2n + 1)a_n^{(T)}/a_n$ is $\chi^2_{T(2n+1)}$. Also, $a_n^{(T)}$, $n = 0, 1, 2, \cdots$ are mutually independent.*

From the previous theorem, we see that $\operatorname{Var}(a_n^{(T)}) = 2a_n^2/(2n + 1)T$, which means that $a_n^{(T)}$ is consistent for a_n. Also, an advantage of $a_n^{(T)}$, $T > 1$, is its increased stability for small n. From Jones (1963b), we conclude that $a_n^{(T)}$ is the estimate of minimum variance in the class of the unbiased estimates of the form: $\int_{S_2} \int_{S_2} W(P, Q)(\sum_{t=1}^{T} \alpha_t X(P, t)X(Q, t))\sigma(dP)\sigma(dQ)$ where the α_t's are real and $W(P, Q)$ is symmetric square integrable and such that

$$E[\textstyle\int_{S_2} \int_{S_2} W(P, Q)X(P)X(Q)\sigma(dP)\sigma(dQ)] = a_n .$$

3. Estimation of the covariance function. As an estimate of the covariance function $R(\theta)$, we will consider

$$R^{(T)}(\theta) = (4\pi)^{-1} \textstyle\sum_{n=0}^{N_T} (2n + 1)P_n(\cos\theta)a_n^{(T)} , \qquad 0 \leqq \theta \leqq \pi \tag{3.1}$$

where for each T, N_T is a positive integer. By the paper of Schoenberg (1942), $R^{(T)}(\theta)$ represents a positive definite function on the sphere since $a_n^{(T)} \geqq 0$ for all $n \geqq 0$. For a homogeneous process on the sphere,

$$E[R^{(T)}(\theta)] = (4\pi)^{-1} \textstyle\sum_{n=0}^{N_T} (2n + 1)P_n(\cos\theta)a_n \tag{3.2}$$

which means that $R^{(T)}(\theta)$ is asymptotically unbiased for $R(\theta)$ if $N_T \to \infty$ as $T \to \infty$. Furthermore, for a Gaussian process by Theorem 2.2,

$$\operatorname{Cov}(R^{(T)}(\theta_1), R^{(T)}(\theta_2)) = (8\pi^2 T)^{-1} \textstyle\sum_{n=0}^{N_T} (2n + 1)P_n(\cos\theta_1)P_n(\cos\theta_2)a_n^2 . \tag{3.3}$$

So,

$$\lim_{T\to\infty} T \operatorname{Cov}(R^{(T)}(\theta_1), R^{(T)}(\theta_2)) = (8\pi^2)^{-1} \textstyle\sum_{n=0}^{\infty} (2n + 1)P_n(\cos\theta_1)P_n(\cos\theta_2)a_n^2$$

which is well defined since $|P_n(x)| \leqq 1$ for $|x| \leqq 1$, $n \geqq 0$ and $\sum_{n=0}^{\infty} na_n^2 < \infty$, by (1.5).

The bias of our estimate is given by

$$R(\theta) - E[R^{(T)}(\theta)] = (4\pi)^{-1} \textstyle\sum_{n>N_T} (2n + 1)P_n(\cos\theta)a_n ,$$

which implies that $|R(\theta) - E[R^{(T)}(\theta)]| \leqq (4\pi)^{-1} \sum_{n>N_T} (2n + 1)a_n$, and by a suitable choice of the N_T's the bias can be uniformly reduced to the order of magnitude we want.

For the following, let us define the processes

$$\xi_T(\theta) = T^{\frac{1}{2}}(R^{(T)}(\theta) - E[R^{(T)}(\theta)]) , \qquad 0 \leqq \theta \leqq \pi .$$

If the N_T's are such that

$$T^{\frac{1}{2}} \sum_{n>N_T} na_n \to 0 \qquad \text{as } T \to \infty , \tag{3.4}$$

the results obtained for the processes $\xi_T(\theta)$ will be valid also for the processes $T^{\frac{1}{2}}(R^{(T)}(\theta) - R(\theta))$, since $\sup_{0\le\theta\le\pi} \{T^{\frac{1}{2}}(R(\theta) - E[R^{(T)}(\theta)])\} \to 0$ as $T \to \infty$.

THEOREM 3.1. *Under the assumptions of Theorem* 2.2 *and if $N_T \to \infty$ as $T \to \infty$, then for any $\theta_1, \cdots, \theta_k \in [0, \pi]$, $(\xi_T(\theta_1), \cdots, \xi_T(\theta_k))$ is asymptotically normal with mean zero and covariance matrix*

$$[(8\pi^2)^{-1} \sum_{n=0}^{\infty} (2n+1)P_n(\cos\theta_i)P_n(\cos\theta_j)a_n^2]_{i,j=1}^k .$$

PROOF. If $\mathrm{cum}\,(X_1, \cdots, X_h)$ denotes the joint cumulant of order h of $X_1, \cdots, X_h$, we have only to show that $\mathrm{cum}\,(\xi_T(\theta_{i_1}), \cdots, \xi_T(\theta_{i_h})) \to 0$ for $i_1, \cdots, i_h \in \{1, 2, \cdots, k\}$ and $h > 2$. By elementary properties of cumulants (see for example Brillinger (1970), Section 1), for $h > 2$,

$$\begin{aligned}\mathrm{cum}\,&\{\xi_T(\theta_{i_1}), \cdots, \xi_T(\theta_{i_h})\} \\ &= T^{h/2}(4\pi)^{-h} \sum_{n=0}^{N_T} (2n+1)^h P_n(\cos\theta_{i_1}) \cdots P_n(\cos\theta_{i_h}) \cdot \mathrm{cum}_h(a_n^{(T)})\end{aligned}$$

where $\mathrm{cum}_h(X)$ is the cumulant of order h of the random variable X. Furthermore, $\mathrm{cum}_h(\chi_n^2) = (h-1)!\, 2^{h-1}n$ for $h \geqq 1$ and since $a_n^{(T)} = a_n \chi^2_{(2n+1)T}/(2n+1)T$ by Theorem 2.2, we see that $\mathrm{cum}_h(a_n^{(T)}) = a_n^h(h-1)!\, 2^{h-1}(T(2n+1))^{-h+1}$ which implies

$$|\mathrm{cum}\,\{\xi_T(\theta_{i_1}), \cdots, \xi_T(\theta_{i_h})\}| \leqq MT^{(1-h/2)} \sum_{n=0}^{\infty} (2n+1)a_n^h$$

where the constant M is independent of T. Thus the proof is complete, since $\sum_{n=0}^{\infty} (2n+1)a_n^h < \infty$ by (1.5).

From the previous theorem, we have the convergence of the finite dimensional distributions of $\{\xi_T(\theta) : 0 \leqq \theta \leqq \pi\}$. Under a stronger assumption on the process on the sphere $X(P)$, i.e.

$$\sum_{n=0}^{\infty} n^{\frac{5}{2}} a_n < \infty , \tag{3.5}$$

we can establish the weak convergence of the processes $\xi_T(\theta)$. Condition (3.5) means that the spectral parameters a_n decrease faster to zero than under the existence of the second moment of $X(P)$ which is equivalent to (1.5).

THEOREM 3.2. *Under the assumptions of Theorem* 2.2 *and if* (3.5) *is satisfied, the processes $\{\xi_T(\theta) : 0 \leqq \theta \leqq \pi\}$ converge weakly in $C[0, \pi]$ to a Gaussian process $\{\xi(\theta) : 0 \leqq \theta \leqq \pi\}$ with mean* 0 *and*

$$\mathrm{Cov}\,(\xi(\theta_1), \xi(\theta_2)) = (8\pi^2)^{-1} \sum_{n=0}^{\infty} (2n+1)P_n(\cos\theta_1)P_n(\cos\theta_2)a_n^2 . \tag{3.6}$$

PROOF. $\{\xi_T(\theta)\}$ is a sequence of random elements of $C[0, \pi]$. By Theorems 8.1, 12.3 and 12.4 of Billingsley (1968), since we have already the convergence of the finite dimensional distributions, it is sufficient to show that

$$E[|\xi_T(\theta_2) - \xi_T(\theta_1)|^\gamma] \leqq |F(\theta_2) - F(\theta_1)|^\alpha$$

for $\theta_1 \leqq \theta_2$ and $T \geqq 1$, where $\gamma \geqq 0$, $\alpha > 1$ and F is a non-decreasing, continuous function on $[0, \pi]$.
But

$$E[|\xi_T(\theta_2) - \xi_T(\theta_1)|^2] \\ \leqq T(4\pi)^{-2} \sum_{n_1=0}^{N_T} \sum_{n_2=0}^{N_T} (2n_1 + 1)(2n_2 + 1)|P_{n_1}(\cos\theta_2) - P_{n_1}(\cos\theta_1)| \\ \times |P_{n_2}(\cos\theta_2) - P_{n_2}(\cos\theta_1)| \cdot E[|a_{n_1}^{(T)} - a_{n_1}|\,|a_{n_2}^{(T)} - a_{n_2}|]\,.$$

By Schwarz inequality and Theorem 2.2,

$$E[|a_{n_1}^{(T)} - a_{n_1}|\,|a_{n_2}^{(T)} - a_{n_2}|] \leqq 2a_{n_1}a_{n_2}/(T(2n_1 + 1)^{\frac{1}{2}}(2n_2 + 1)^{\frac{1}{2}})\,.$$

Also, by the mean value theorem,

$$P_n(\cos\theta_2) - P_n(\cos\theta_1) = -P_n'(\cos\theta^*)\sin\theta^*(\theta_2 - \theta_1)$$

for some $\theta^* \in (\theta_1, \theta_2)$, and from Sansome (1959), page 251, $|P_n'(x)| \leqq n^2$ for $|x| \leqq 1$ and $n \geqq 0$, which implies that $|P_n(\cos\theta_2) - P_n(\cos\theta_1)| \leqq n^2|\theta_2 - \theta_1|$ for all $\theta_1, \theta_2 \in [0, \pi]$ and for every $n \geqq 0$. Then,

$$E[|\xi_T(\theta_2) - \xi_T(\theta_1)|^2] \\ \leqq (8\pi^2)^{-1}|\theta_2 - \theta_1|^2\left(\sum_{n=0}^{N_T} (2n + 1)^{\frac{1}{2}} n^2 a_n\right)^2 \\ \leqq |\theta_2 - \theta_1|^2(8\pi^2)^{-1}\left(\sum_{n=0}^{\infty} (2n + 1)^{\frac{1}{2}} n^2 a_n\right)^2 = M|\theta_2 - \theta_1|^2$$

where $M < \infty$ by (3.5). This completes the proof.

In the following corollary, the results of this section are summarized for the sequence of processes $T^{\frac{1}{2}}[R^{(T)}(\theta) - R(\theta)]$.

COROLLARY 3.3. *Let $X(P)$ be a zero mean Gaussian homogeneous process satisfying* (3.5) *and let $R^{(T)}(\theta)$ be defined by* (3.1) *where the N_T's satisfy* (3.4). *Then, the processes $\{T^{\frac{1}{2}}[R^{(T)}(\theta) - R(\theta)] : 0 \leqq \theta \leqq \pi\}$ converge weakly in $C[0, \pi]$ to a Gaussian process $\{\xi(\theta) : 0 \leqq \theta \leqq \pi\}$ with mean* 0 *and covariance function given by* (3.6).

REMARK. If $X(P)$ has a nonzero mean, the results of this paper continue to hold if we replace $X(P)$ by the process $\bar{X}(P) = X(P) - \bar{X}$ where $\bar{X} = (4\pi)^{-1}\int_{S_2} X(P)\sigma(dP) = (2\pi)^{\frac{1}{2}}Z_{00}$. Then $\bar{X}(P)$ has mean 0 and for the computations, we have the relations

$$\begin{aligned} \bar{Z}_{nk} &= 0 \qquad \text{if} \quad n = 0\,, \qquad\qquad k = 0\,, \\ &= Z_{nk} \qquad \text{if} \quad n > 0\,, \qquad\qquad \text{all} \quad k\,, \end{aligned}$$

since $\bar{Z}_{nk} = Z_{nk} - \bar{X}\int_{S_2} Y_{nk}(P)\sigma(dP)$ and

$$\int_{S_2} Y_{nk}(P)\sigma(dP) = (4\pi)^{\frac{1}{2}} \int_{S_2} Y_{00}(P)Y_{nk}(P)\sigma(dP) = 0$$

by the orthogonality of the spherical harmonics. Also

$$E[\bar{X}(P)\bar{X}(Q)] = (4\pi)^{-1} \sum_{n=1}^{\infty} (2n + 1)a_n P_n(\cos\theta)\,,$$

i.e. the constant term is removed from the covariance function.

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