The Annals of Statistics
2003, Vol. 31, No. 4, 1186–1219

REGRESSION M-ESTIMATORS WITH NON-I.I.D. DOUBLY CENSORED DATA[1]

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Considering the linear regression model with fixed design, the usual M-estimator with a complete sample of the response variables is expressed as a functional of a *generalized weighted bivariate empirical process*, and its asymptotic normality is directly derived through the Hadamard differentiability property of this functional and the weak convergence of this generalized weighted empirical process. The result reveals the direct relationship between the M-estimator and the distribution function of the error variables in the linear model, which leads to the construction of the M-estimator when the response variables are subject to double censoring. For this proposed regression M-estimator with non-i.i.d. doubly censored data, strong consistency and asymptotic normality are established.

1. Introduction. In statistical analysis, one of the most widely used tools is the linear regression model

$$X_i = t_i\beta + e_i, \qquad i = 1, 2, \dots, n, \tag{1.1}$$

where X_i are the response variables, t_i are the fixed design points, β is the unknown regression parameter, and e_i are the independently and identically distributed (i.i.d.) error random variables (r.v.'s) with an unknown continuous distribution function (d.f.) F. To properly use this model with incomplete response observations, which are frequently encountered in medical research and reliability studies, the right-censored linear regression model has been studied over the past two decades by Buckley and James (1979), Koul, Susarla and Van Ryzin (1981), Leurgans (1987), Ritov (1990), Lai and Ying (1991) and Zhou (1992), among others. In Lai and Ying (1994), the linear regression model with left-truncated and right-censored response variables was considered. More recently, Zhang and Li (1996) extended Buckley–James–Ritov-type regression estimators from the right-censored case to the linear regression model with random design and doubly censored response observations, and Ren and Gu (1997) constructed and studied M-estimators for the same model using a functional of a Campbell-type estimator for a bivariate d.f. based on data which are doubly censored in one coordinate. In this article, we consider the *doubly censored linear regression model with fixed design* (1.1), and construct and study an M-estimator for this model.

Received July 2000; revised August 2002.
[1]Supported in part by NSF Grants DMS-96-26532, DMS-97-96229 and DMS-02-04182.
AMS 2000 subject classifications. 62J05, 62N02, 62E20.
Key words and phrases. Asymptotic normality, generalized weighted empirical process, Hadamard differentiability, linear regression model, strong consistency, weak convergence.

To be precise, in this study we do not observe the X_i's in model (1.1), but a doubly censored sample,

$$V_i = \begin{cases} X_i, & \text{if } B_i < X_i \le C_i \text{ with } \delta_i = 1, \\ C_i, & \text{if } X_i > C_i \text{ with } \delta_i = 2, \\ B_i, & \text{if } X_i \le B_i \text{ with } \delta_i = 3, \end{cases} \tag{1.2}$$

where B_i are C_i are left- and right-censoring random variables, respectively, that satisfy $P\{B_i < C_i\} = 1$, and (B_i, C_i) are i.i.d. and independent of X_i. This means that (B_i, C_i) is independent of e_i and the problem considered here is to estimate the regression parameter β in (1.1) consistently using data (V_i, δ_i, t_i), $i = 1, 2, \dots, n$.

Note that in (1.1), X_i are independent random variables, but not i.i.d. (unless $t_i \equiv 1$) because t_i are constants. Thus, (V_i, δ_i, t_i), $i = 1, 2, \dots, n$, in (1.2) is a *non-i.i.d. doubly censored regression sample*. For the i.i.d. doubly censored sample (V_i, δ_i), $1 \le i \le n$, in (1.2) with $t_i \equiv 1$, examples encountered in practice have been given by Gehan (1965), Turnbull (1974) and others. In particular, Ren and Gu (1997) discussed an example of doubly censored regression data (V_i, δ_i, t_i) with random design, that is, t_i are i.i.d. r.v.'s, that occurred in recent research on primary breast cancer [Peer, Van Dijck, Hendriks, Holland and Verbeek (1993) and Ren and Peer (2000)]. Since the sample (V_i, δ_i, t_i) considered in their paper, as well as in Zhang and Li (1996), is an *i.i.d. doubly censored regression sample*, the methods developed in Zhang and Li (1996) and Ren and Gu (1997) do not have direct extensions to the problem we are considering here. It is precisely the non-i.i.d. property of our doubly censored regression sample (V_i, δ_i, t_i) in (1.2) that causes considerable difficulties in the construction and study of a consistent estimator of β in (1.1).

In Section 2, to allow construction of an M-estimator with data (1.2), we first express the usual M-estimator $\widehat{\beta}_n$ as a functional of a *generalized weighted bivariate empirical process*, where a complete non-i.i.d. sample (X_i, t_i), $1 \le i \le n$, in (1.1) is used. Then in Theorem 1 (with proofs given in the Appendix) we derive its asymptotic normality via the Hadamard differentiability property of this functional and the weak convergence of this empirical process. The implication of Theorem 1 is twofold: (1) It reveals the direct relationship between the M-estimator and the d.f. F of the error variables e_i in the linear model (1.1), which in Section 2 leads to the construction of the M-estimator β_n for β using the non-i.i.d. doubly censored regression sample (V_i, δ_i, t_i), $1 \le i \le n$, given by (1.2). (2) Whereas in the literature, the Hadamard differentiability approach has been successfully used to study the asymptotic properties of various important statistics based on i.i.d. random samples by such researchers as Bickel and Freedman (1981), Fernholz (1983), Sen (1988), Gill (1989), Groeneboom and Wellner (1992), Ren and Sen (1995, 2001), van der Vaart and Wellner (1996) and Ren and Gu (1997), among others, Theorem 1 shows that this attractive formulation can also be used to deal with problems based on non-i.i.d. samples.

In Section 3, the strong consistency and the asymptotic normality of the proposed regression M-estimator β_n with non-i.i.d. doubly censored regression data are established; the proofs are deferred to Sections 4 and 5.

Due to the complexity of the problem studied in this article, the extension of the proposed M-estimators to multiple regression models is considered in a separate article.

2. *M*-estimator with non-i.i.d. doubly censored regression sample. When there is no censoring on the response variables in (1.1), the robust M-estimator $\widehat{\beta}_n$ for β is given by the solution of the equation

$$\sum_{i=1}^{n} t_i \psi(X_i - t_i\theta) = 0, \tag{2.1}$$

where ψ is the score function [Huber (1981)]. Considering the case $t_i \in [0, 1]$, $1 \le i \le n$, throughout, we let

$$\eta = \theta - \beta, \qquad \widehat{\eta}_n = \widehat{\beta}_n - \beta, \qquad Y_i = X_i - t_i\beta. \tag{2.2}$$

Then Y_i are i.i.d. with d.f. F, and for

$$W_n(y,t) = n^{-1}\sum_{i=1}^{n} I\{Y_i \le y, t_i \le t\}, \qquad EW_n(y,t) = F(y)\mu_n(t), \tag{2.3}$$

$$\Psi(G,\eta) = \iint_{0\le t\le 1, y\in\mathbb{R}} t\psi(y - t\eta)\, dG(y,t), \tag{2.4}$$

where

$$\mu_n(t) = n^{-1}\sum_{i=1}^{n} I\{t_i \le t\} \qquad \text{for constants } t_1, \dots, t_n \text{ in } [0, 1] \tag{2.5}$$

and $G \in D_2 \equiv \{G | G$ is a function: $\mathbb{R} \times [0, 1] \to \mathbb{R}$ with defined integral in (2.4)$\}$, straightforward algebra shows that (2.1) is equivalent to

$$\Psi(W_n,\eta) = \iint_{0\le t\le 1,\, y\in\mathbb{R}} t\psi(y - t\eta)\, dW_n(y,t) = 0. \tag{2.6}$$

Define the statistical functional $T : D_2 \to \mathbb{R}$ as the solution of $\Psi(G,\eta) = 0$, that is,

$$T(G) \text{ satisfies } \Psi(G, T(G)) = 0 \qquad \text{for } G \in D_2. \tag{2.7}$$

Then

$$T(W_n) = \widehat{\eta}_n = \widehat{\beta}_n - \beta \tag{2.8}$$

and Theorem 1 gives the asymptotic normality of $\widehat{\beta}_n$ under the following assumptions:

(A1) The function ψ is nondecreasing, bounded, continuous and piecewise differentiable with bounded derivative ψ' satisfying $\psi'(x) = 0$ for x outside of some finite interval $[A, B]$, while for x in some neighborhood of 0, $\psi(x)$ has positive and negative values with $\psi'(x) \geq c > 0$ for a constant $0 < c < \infty$.
(A2) The function ψ' is of bounded variation.
(A3) The integral $\int \psi(x)\, dF(x) = 0$.
(A4) For $i = 1, 2, \dots, n,\ 0 \leq t_i \leq 1$.
(A5) As $n \to \infty$, $\sup_{0 \leq t \leq 1} |\mu_n(t) - \mu(t)| \to 0$, where $\mu(t)$ is nondegenerate.

THEOREM 1. *Under* (A1)–(A5), *we have that as $n \to \infty$*:

(i) $\sqrt{n}(W_n - EW_n)$ *weakly converges to a centered Gaussian process $\mathbb{G}$ on a Banach space* $(\bar{D}_2, \mathcal{D}_2, \|\cdot\|)$, *where $\bar{D}_2$ is the closure of D_2, $\|\cdot\|$ stands for the uniform norm and $\mathcal{D}_2$ is the σ-field generated by open balls*;

(ii) $\sqrt{n}(\widehat{\beta}_n - \beta) = \sqrt{n}[T(W_n) - T(EW_n)] = T'_{EW_n}(\sqrt{n}[W_n - EW_n]) + o_p(1) \to_D T'_W(\mathbb{G}) =_D N(0, \sigma_0^2)$, *where $W(y, t) = F(y)\mu(t)$, $0 < \sigma_0^2 < \infty$, and T'_G is a linear functional.*

While the proof of Theorem 1(i) is given in the Appendix, it is easy to see the proof of (ii) from the following. From (A1), (A3) and (2.3), we know that for any fixed n, $\Psi(EW_n, \eta)$ is strictly decreasing in η and $T(EW_n) = 0$ is the unique solution of $\Psi(EW_n, \eta) = 0$. Noting that (A5) implies $\|EW_n - W\| \to 0$, as $n \to \infty$, a slightly modified proof of Theorem 3.1 in Ren and Gu (1997) gives that $T(\cdot)$ is Hadamard differentiable at EW_n for any fixed n and it satisfies

$$
\begin{aligned}
&\sqrt{n}[T(W_n) - T(EW_n)] \\
&\quad = T'_{EW_n}\big(\sqrt{n}[W_n - EW_n]\big) + o_p(1) \qquad \text{as } n \to \infty,
\end{aligned}
\tag{2.9}
$$

where $o_p(1)$ converges to 0 in probability as $n \to \infty$. The proof of Theorem 1(ii) follows from Theorem 1(i) and the continuity of T'_G (in G) in the neighborhood of W based on Theorem 3.1 of Ren and Gu (1997).

Theorem 1(ii) shows that the M-estimator $\widehat{\beta}_n$ and its asymptotic properties are totally determined by the *generalized weighted empirical process* W_n via Ψ given by (2.6). From Theorem 1(i), we know that

$$
\Psi(W_n, \eta) \approx \Psi(EW_n, \eta) = \int_0^1 \int_{-\infty}^{\infty} t\psi(x - t\theta)\, dF(x - t\beta)\, d\mu_n(t)
\tag{2.10}
$$

implies that if the error d.f. F can be estimated, say by $\widehat{F}_n$, based on available data, then from (2.6), an M-estimator should be given by the solution (in θ) of

$$
\int_0^1 \int_{-\infty}^{\infty} t\psi(x - t\theta)\, d\widehat{F}_n(x - t\beta)\, d\mu_n(t) = 0.
\tag{2.11}
$$

In this context, we study the estimation of F using the non-i.i.d. doubly censored regression sample (V_i, δ_i, t_i), $1 \le i \le n$, given (1.2) as follows.

First, we observe that for each $1 \le i \le n$ and any $x \in \mathbb{R}$,

$$\begin{aligned}
P_i^{(1)}(x) &= P\{V_i \le x, \delta_i = 1\} = P\{X_i \le x, B_i < X_i \le C_i\} \\
&= \int_{-\infty}^{x} \big(F_B(u) - F_C(u)\big)\, dF(u - t_i\beta), \\
P_i^{(2)}(x) &= P\{V_i \le x, \delta_i = 2\} = P\{C_i \le x, X_i > C_i\} \\
&= \int_{-\infty}^{x} [1 - F(u - t_i\beta)]\, dF_C(u), \\
P_i^{(3)}(x) &= P\{V_i \le x, \delta_i = 3\} = P\{B_i \le x, X_i \le B_i\} \\
&= \int_{-\infty}^{x} F(u - t_i\beta)\, dF_B(u), \\
P_i^{(0)}(x) &= P_i^{(1)}(x) + P_i^{(2)}(x) + P_i^{(3)}(x) \\
&= F_C(x) + [F_B(x) - F_C(x)]F(x - t_i\beta),
\end{aligned}$$

where F_B and F_C are the d.f.'s of B_i and C_i in (1.2), respectively, and from (2.7) of Gu and Zhang (1993), we know that F_{X_i} satisfies the integral equation

$$\begin{aligned}
F_{X_i}(x) = P_i^{(0)}(x) &- \int_{u \le x} \frac{1 - F_{X_i}(x)}{1 - F_{X_i}(u)}\, dP_i^{(2)}(u) \\
&+ \int_{x < u} \frac{F_{X_i}(x)}{F_{X_i}(u)}\, dP_i^{(3)}(u).
\end{aligned} \tag{2.12}$$

Noting that $F_{X_i}(x) = F(x - t_i\beta)$ for any $x \in \mathbb{R}$, (2.12) can be written as

$$\begin{aligned}
F_{X_i}(x + t_i\beta) &= P_i^{(0)}(x + t_i\beta) - \int_{u \le x + t_i\beta} \frac{1 - F_{X_i}(x + t_i\beta)}{1 - F_{X_i}(u)}\, dP_i^{(2)}(u) \\
&\quad + \int_{x + t_i\beta < u} \frac{F_{X_i}(x + t_i\beta)}{F_{X_i}(u)}\, dP_i^{(3)}(u) \\
&= P_i^{(0)}(x + t_i\beta) - \int_{y \le x} \frac{1 - F_{X_i}(x + t_i\beta)}{1 - F_{X_i}(y + t_i\beta)}\, dP_i^{(2)}(y + t_i\beta) \\
&\quad + \int_{x < y} \frac{F_{X_i}(x + t_i\beta)}{F_{X_i}(y + t_i\beta)}\, dP_i^{(3)}(y + t_i\beta),
\end{aligned} \tag{2.13}$$

which gives that for each i, $F(x)$ is the solution of the integral equation

$$\begin{aligned}
F(x) = P_i^{(0)}(x + t_i\beta) &- \int_{u \le x} \frac{1 - F(x)}{1 - F(u)}\, dP_i^{(2)}(u + t_i\beta) \\
&+ \int_{x < u} \frac{F(x)}{F(u)}\, dP_i^{(3)}(u + t_i\beta).
\end{aligned}$$

Hence, the sum of these integral equations divided by n is given by

$$
\begin{aligned}
F(x) = n^{-1} \sum_{i=1}^{n} P_i^{(0)}(x + t_i \beta) \\
- \int_{u \le x} \frac{1 - F(x)}{1 - F(u)} \, d\left(n^{-1} \sum_{i=1}^{n} P_i^{(2)}(u + t_i \beta) \right) \\
+ \int_{x < u} \frac{F(x)}{F(u)} \, d\left(n^{-1} \sum_{i=1}^{n} P_i^{(3)}(u + t_i \beta) \right).
\end{aligned} \tag{2.14}
$$

Let, for any $\theta \in \mathbb{R}$,

$$
\begin{aligned}
Q_{n,\theta}^{(j)}(x) &= n^{-1} \sum_{i=1}^{n} I\{V_i \le x + t_i \theta, \delta_i = j\}, \qquad j = 1, 2, 3, \\
Q_{n,\theta}^{(0)}(x) &= Q_{n,\theta}^{(1)}(x) + Q_{n,\theta}^{(2)}(x) + Q_{n,\theta}^{(3)}(x) = n^{-1} \sum_{i=1}^{n} I\{V_i \le x + t_i \theta\}.
\end{aligned} \tag{2.15}
$$

Then (2.14) becomes

$$
\begin{aligned}
F(x) = E Q_{n,\beta}^{(0)}(x) - \int_{u \le x} \frac{1 - F(x)}{1 - F(u)} \, d\bigl(E Q_{n,\beta}^{(2)}(u)\bigr) \\
+ \int_{x < u} \frac{F(x)}{F(u)} \, d\bigl(E Q_{n,\beta}^{(3)}(u)\bigr)
\end{aligned} \tag{2.16}
$$

and the self-consistent estimator $\widehat{F}_{n,\beta}$ for F [Mykland and Ren (1996)] should be given by the solution of the integral equation

$$
\begin{aligned}
\widehat{F}_{n,\beta}(x) = Q_{n,\beta}^{(0)}(x) - \int_{u \le x} \frac{1 - \widehat{F}_{n,\beta}(x)}{1 - \widehat{F}_{n,\beta}(u)} \, d Q_{n,\beta}^{(2)}(u) \\
+ \int_{x < u} \frac{\widehat{F}_{n,\beta}(x)}{\widehat{F}_{n,\beta}(u)} \, d Q_{n,\beta}^{(3)}(u).
\end{aligned} \tag{2.17}
$$

Equations (2.11) and (2.17) imply that the M-estimator with non-i.i.d. doubly censored regression sample in (1.2) should be given by the solution (in θ) of

$$
\int_0^1 \int_{-\infty}^{\infty} t \psi(x - t\theta) \, d\widehat{F}_{n,\beta}(x - t\beta) \, d\mu_n(t) = 0. \tag{2.18}
$$

However, in practice the parameter β in (2.18) is unknown. Thus, the equation is naturally replaced by

$$
\int_0^1 \int_{-\infty}^{\infty} t \psi(x - t\theta) \, d\widehat{F}_{n,\theta}(x - t\theta) \, d\mu_n(t) = 0,
$$

where for any θ, $\widehat{F}_{n,\theta}$ is a solution of the integral equation

$$\begin{aligned}\widehat{F}_{n,\theta}(x) = Q_{n,\theta}^{(0)}(x) - \int_{u\le x} \frac{1-\widehat{F}_{n,\theta}(x)}{1-\widehat{F}_{n,\theta}(u)}\, dQ_{n,\theta}^{(2)}(u) \\ + \int_{x<u} \frac{\widehat{F}_{n,\theta}(x)}{\widehat{F}_{n,\theta}(u)}\, dQ_{n,\theta}^{(3)}(u).\end{aligned} \tag{2.19}$$

The existence of $\widehat{F}_{n,\theta}$ is shown in the Appendix. Note that if $\widehat{F}_{n,\theta}(-\infty) = 0$ and $\widehat{F}_{n,\theta}(\infty) = 1$, then from (A1), change of variables and integration by parts, we have

$$\int_0^1 \int_{-\infty}^{\infty} t\psi(x - t\theta)\, d\widehat{F}_{n,\theta}(x - t\theta)\, d\mu_n(t) = \bar{t}_n \left[\psi(B) - \int_A^B \widehat{F}_{n,\theta}(y)\, d\psi(y)\right],$$

where $\bar{t}_n = n^{-1}\sum_{i=1}^n t_i$. Hence, we define the *$M$-estimator β_n for the doubly censored regression sample* (V_i, δ_i, t_i), $i = 1, \dots, n$, given in (1.2) by the solution of

$$M_n(\theta) \equiv \psi(B) - \int_A^B \widehat{F}_{n,\theta}(y)\, d\psi(y) \doteq 0, \tag{2.20}$$

where $\doteq$ means "as near 0 as possible."

Note that the use of $\doteq$ in (2.20) is because $M_n(\theta) = 0$ may not have any solutions due to the fact, discussed in the Appendix, that $M_n(\theta)$ is piecewise continuous and piecewise nonincreasing in θ. In practice, for each θ one may treat the sample $(V_i - t_i\theta, \delta_i)$, $1 \le i \le n$, as a usual i.i.d. doubly censored sample and compute $\widehat{F}_{n,\theta}$ as in Mykland and Ren (1996), while the M-estimator β_n defined by (2.20) can be found using the piecewise nonincreasing property of $M_n(\theta)$. See the Appendix.

REMARK 1. Assumptions (A1) and (A2) are required in Ren and Gu (1997) to show that the statistical functional $T(\cdot)$ in Theorem 1 is Hadamard differentiable, but (A2) is not needed for our asymptotic results on proposed the M-estimator β_n in Section 3.

REMARK 2. In the linear model (1.1), if all design points t_i are restricted to a compact set, the problem can be reduced to the case of (A4). Assumption (A5) is used in the proof of the weak convergence of process W_n. In practice, there are many examples which satisfy (A5). For instance, (A5) holds if the design points are evenly distributed on [0, 1], that is, $t_i = i/n$, $i = 1, \dots, n$, or if the design points are proportionally distributed on finite points $b_1, \dots, b_m$ according to weights $p_1, \dots, p_m$, that is, t_i are selected such that $(np_j - 1) \le n_j \le np_j$ and $\sum_{j=1}^m n_j = n$, where $n_j = \sum_{i=1}^n I\{t_i = b_j\}$. On the other hand, noting that $\mu(t)$ is a d.f., it is easy to see that the design points t_i may be easily selected to satisfy (A5) for a known $\mu(\cdot)$.

REMARK 3. Note that our proposed M-estimator β_n is based on the direct relationship between the usual M-estimator and the error distribution F shown by (2.10), which is revealed by Theorem 1. This idea is different from Lai and Ying's (1994) *missing information principle* for an M-estimator with incomplete regression data. In fact, (2.10) may be applied in other linear regression problems with a fixed design as long as the error d.f. F can be estimated consistently using available data.

3. Consistency and asymptotic normality. To study the asymptotic properties of the proposed M-estimator β_n, defined by (2.20), for the doubly censored regression sample (V_i, δ_i, t_i) given in (1.2), we introduce the notion of $Q_\theta^{(j)}(x)$ as follows.

Note that for any θ and x, the expectation of $Q_{n,\theta}^{(j)}(x)$ in (2.15) is given by

$$
\begin{aligned}
E\{Q_{n,\theta}^{(1)}(x)\} &= \int_0^1 \int_{-\infty}^{x+t\theta} [F_B(u) - F_C(u)]\, dF(u - t\beta)\, d\mu_n(t),\\
E\{Q_{n,\theta}^{(2)}(x)\} &= \int_0^1 \int_{-\infty}^{x+t\theta} [1 - F(u - t\beta)]\, dF_C(u)\, d\mu_n(t),\\
E\{Q_{n,\theta}^{(3)}(x)\} &= \int_0^1 \int_{-\infty}^{x+t\theta} F(u - t\beta)\, dF_B(u)\, d\mu_n(t), \qquad (3.1)\\
E\{Q_{n,\theta}^{(0)}(x)\} &= E\{Q_{n,\theta}^{(1)}(x)\} + E\{Q_{n,\theta}^{(2)}(x)\} + E\{Q_{n,\theta}^{(3)}(x)\}\\
&= \int_0^1 \{F_C(x + t\theta) + [F_B(x + t\theta) - F_C(x + t\theta)]\\
&\qquad \times F\big(x + t(\theta - \beta)\big)\}\, d\mu_n(t).
\end{aligned}
$$

Based on assumption (A5), we define for the d.f. $\mu(t)$ on [0, 1],

$$
\begin{aligned}
Q_\theta^{(1)}(x) &= \int_0^1 \int_{-\infty}^{x+t\theta} [F_B(u) - F_C(u)]\, dF(u - t\beta)\, d\mu(t),\\
Q_\theta^{(2)}(x) &= \int_0^1 \int_{-\infty}^{x+t\theta} [1 - F(u - t\beta)]\, dF_C(u)\, d\mu(t),\\
Q_\theta^{(3)}(x) &= \int_0^1 \int_{-\infty}^{x+t\theta} F(u - t\beta)\, dF_B(u)\, d\mu(t), \qquad (3.2)\\
Q_\theta^{(0)}(x) &= Q_\theta^{(1)}(x) + Q_\theta^{(2)}(x) + Q_\theta^{(3)}(x)\\
&= \int_0^1 \{F_C(x + t\theta) + [F_B(x + t\theta) - F_C(x + t\theta)]\\
&\qquad \times F\big(x + t(\theta - \beta)\big)\}\, d\mu(t).
\end{aligned}
$$

Thus, letting f, f_B and f_C be the density functions of F, F_B and F_C, respectively, we have

$$
\begin{aligned}
q_{n,\theta}^{(1)}(x) &= \frac{d\{E Q_{n,\theta}^{(1)}(x)\}}{dx} \\
&= \int_0^1 [F_B(x+t\theta) - F_C(x+t\theta)] f\big(x + t(\theta-\beta)\big)\, d\mu_n(t), \\
q_{n,\theta}^{(2)}(x) &= \frac{d\{E Q_{n,\theta}^{(2)}(x)\}}{dx} = \int_0^1 \big[1 - F\big(x+t(\theta-\beta)\big)\big] f_C(x+t\theta)\, d\mu_n(t), \\
q_{n,\theta}^{(3)}(x) &= \frac{d\{E Q_{n,\theta}^{(3)}(x)\}}{dx} = \int_0^1 F\big(x+t(\theta-\beta)\big) f_B(x+t\theta)\, d\mu_n(t), \\
q_{\theta}^{(1)}(x) &= \frac{d Q_{\theta}^{(1)}(x)}{dx} \\
&= \int_0^1 [F_B(x+t\theta) - F_C(x+t\theta)] f\big(x + t(\theta-\beta)\big)\, d\mu(t), \\
q_{\theta}^{(2)}(x) &= \frac{d Q_{\theta}^{(2)}(x)}{dx} = \int_0^1 \big[1 - F\big(x+t(\theta-\beta)\big)\big] f_C(x+t\theta)\, d\mu(t), \\
q_{\theta}^{(3)}(x) &= \frac{d Q_{\theta}^{(3)}(x)}{dx} = \int_0^1 F\big(x+t(\theta-\beta)\big) f_B(x+t\theta)\, d\mu(t),
\end{aligned}
\tag{3.3}
$$

and for any θ, we define [0, 1]-valued nondecreasing $F_{n,\theta}$ and F_θ as solutions of

$$
\begin{aligned}
F_{n,\theta}(x) &= E Q_{n,\theta}^{(0)}(x) - \int_{u \le x} \frac{1 - F_{n,\theta}(x)}{1 - F_{n,\theta}(u)}\, dE\, Q_{n,\theta}^{(2)}(u) \\
&\quad + \int_{x<u} \frac{F_{n,\theta}(x)}{F_{n,\theta}(u)}\, dE\, Q_{n,\theta}^{(3)}(u), \\
F_\theta(x) &= Q_\theta^{(0)}(x) - \int_{u \le x} \frac{1 - F_\theta(x)}{1 - F_\theta(u)}\, d Q_\theta^{(2)}(u) \\
&\quad + \int_{x<u} \frac{F_\theta(x)}{F_\theta(u)}\, d Q_\theta^{(3)}(u),
\end{aligned}
\tag{3.4}
$$

respectively. While the existence of $F_{n,\theta}$ and F_θ is shown in the Appendix, the next proposition, with proofs deferred to Section 4, gives some basic results on $Q_{n,\theta}^{(j)}$, $Q_\theta^{(j)}$, $F_{n,\theta}$ and F_θ under some of the following conditions:

(B1) The function F has support $(-\infty, \infty)$ and has a continuous and bounded density function f.

(B2) For any $x \in \mathbb{R}$, $F_B(x) - F_C(x) > 0$.

(B3) The functions F_B and F_C have bounded density functions f_B and f_C, respectively.

(B4) The density functions f_B and f_C both satisfy the Lipschitz condition of order 1 on $\mathbb{R}$.

(B5) There exist constants M_B and M_C such that $f_C(x) \equiv 0$ for $x \le M_C$ and $f_B(x) \equiv 0$ for $x \ge M_B$.

(B6) If $\theta \ne \beta$, $F_\theta \ne F_\beta$.

PROPOSITION 1. *Under* (A5) *and* (B1)–(B5), *for* $j = 0, 1, 2, 3$ *and* $0 < \rho, M < \infty$ *given by* (3.5) *and* (3.6), *we have*:

(i) $\sup_{\theta,x} |E\{Q_{n,\theta}^{(j)}(x)\} - Q_\theta^{(j)}(x)| \to 0$ *as* $n \to \infty$;

(ii) $F_{n,\beta} = F = F_\beta$;

(iii) $\|F_\theta - F_\beta\| \to 0$ *as* $\theta \to \beta$;

(iv) $\|F_{n,\theta} - F_{n,\beta}\|_M \le M_0|\theta - \beta|$, *where* $n \ge 1$, $|\theta| \le \rho$, M_0 *is a constant and* $\|\cdot\|_M$ *denotes the uniform norm on* $[-M, M]$;

(v) $\sup_{|\theta|\le\rho} \|Q_{n,\theta}^{(j)} - EQ_{n,\theta}^{(j)}\| \xrightarrow{a.s.} 0$ *as* $n \to \infty$;

(vi) $\sup_{|\theta|\le\rho} n^\lambda \|Q_{n,\theta}^{(j)} - EQ_{n,\theta}^{(j)}\|_M \xrightarrow{a.s.} 0$ *as* $n \to \infty$, *where* $0 < \lambda < \frac{1}{2}$.

To state Theorem 2, which is proved in Section 5, we let ρ be a large constant such that $|\beta| < \rho$, and we note that under (B5) and (A1), we have for any $|\theta| \le \rho$,

$$\begin{aligned} &q_{n,\theta}^{(2)}(x) = 0 \qquad \text{for } x \le -M \quad \text{and} \quad q_{n,\theta}^{(3)}(x) = 0 \qquad \text{for } x \ge M, \\ &q_\theta^{(2)}(x) = 0 \qquad \text{for } x \le -M \quad \text{and} \quad q_\theta^{(3)}(x) = 0 \qquad \text{for } x \ge M, \end{aligned} \tag{3.5}$$

where

$$M = \rho + \max\{|M_B|, |M_C|, |A|, |B|\}. \tag{3.6}$$

Since (3.5) implies that for $x \in [-M, M]$ and $|\theta| \le \rho$, (3.4) is equivalent to

$$\begin{aligned} F_{n,\theta}(x) &= EQ_{n,\theta}^{(0)}(x) - \int_{-M}^{x} \frac{1 - F_{n,\theta}(x)}{1 - F_{n,\theta}(u)}\, dE\, Q_{n,\theta}^{(2)}(u) \\ &\quad + \int_x^M \frac{F_{n,\theta}(x)}{F_{n,\theta}(u)}\, dE\, Q_{n,\theta}^{(3)}(u), \\ F_\theta(x) &= Q_\theta^{(0)}(x) - \int_{-M}^{x} \frac{1 - F_\theta(x)}{1 - F_\theta(u)}\, dQ_\theta^{(2)}(u) \\ &\quad + \int_x^M \frac{F_\theta(x)}{F_\theta(u)}\, dQ_\theta^{(3)}(u). \end{aligned} \tag{3.7}$$

Thus, Proposition 1(v) leads us to treat $\widehat{F}_{n,\theta}$ in Theorem 2 as a solution of

$$\begin{aligned} \widehat{F}_{n,\theta}(x) &= Q_{n,\theta}^{(0)}(x) - \int_{-M}^{x} \frac{1 - \widehat{F}_{n,\theta}(x)}{1 - \widehat{F}_{n,\theta}(u)}\, dQ_{n,\theta}^{(2)}(u) \\ &\quad + \int_x^M \frac{\widehat{F}_{n,\theta}(x)}{\widehat{F}_{n,\theta}(u)}\, dQ_{n,\theta}^{(3)}(u). \end{aligned} \tag{3.8}$$

Moreover, we define

$$A_\beta = \int_A^B \left((I - \widetilde{K}_\beta^M)^{-1} \frac{\widetilde{A}_\beta}{1 - C_\beta} \right)(x)\, d\psi(x), \tag{3.9}$$

where for

$$C_\beta(x) = \int_{u \le x} \frac{1}{1 - F(u)}\, dQ_\beta^{(2)}(u) + \int_{x<u} \frac{1}{F(u)}\, dQ_\beta^{(3)}(u),$$

$$K_\beta(x,u) = -\left\{ \frac{1 - F(x)}{[1 - F(u)]^2} I\{u \le x\} q_\beta^{(2)}(u) + \frac{F(x)}{[F(u)]^2} I\{x < u\} q_\beta^{(3)}(u) \right\},$$

$$\begin{aligned} \widetilde{A}_\beta(x) = {} & f(x) \int_0^1 [F_B(x + t\beta) - F_C(x + t\beta)] t\, d\mu(t) \\ & + \int_{u \le x} \int_0^1 [1 - F(u)] f_C(u + t\beta) t\, d\mu(t)\, d\left\{ \frac{1 - F(x)}{1 - F(u)} \right\} \\ & - \int_{x<u} \int_0^1 F(u) f_B(u + t\beta) t\, d\mu(t)\, d\left\{ \frac{F(x)}{F(u)} \right\} \end{aligned} \tag{3.10}$$

and

$$\big(\widetilde{K}_\beta^M h\big)(x) = \int_{-M}^{M} \frac{K_\beta(x,u)}{1 - C_\beta(x)} h(u)\, du, \qquad x \in [-M, M], \tag{3.11}$$

$(I - \widetilde{K}_\beta^M)^{-1}$ denotes the inverse of the operator $I - \widetilde{K}_\beta^M$, the existence of which is established in Lemma 1 of Section 4.

THEOREM 2. *Assume* (A1), (A3)–(A5), (B1)–(B6) *and* $A_\beta \neq 0$. *Then for the doubly censored regression M-estimator β_n given by the solution of* (2.20) *in the interval* $[-\rho, \rho]$, *where $\widehat{F}_{n,\theta}$ is a solution of* (3.8), *we have*:

(i) $n^\lambda |\beta_n - \beta| \stackrel{a.s.}{\to} 0$ *as* $n \to \infty$, *where* $0 < \lambda < \frac{1}{2}$;

(ii) $\sqrt{n}(\beta_n - \beta + \eta_n) \to_D N(0, \sigma^2)$ *as* $n \to \infty$, *where* $0 < \sigma^2 < \infty$ *and* η_n *is some quantity satisfying* $n^\lambda |\eta_n| \stackrel{a.s.}{\to} 0$ *as* $n \to \infty$, *for any* $0 < \lambda < \frac{1}{2}$.

REMARK 4. In Theorem 2, condition (B2) is usually required in the studies of asymptotic properties with i.i.d. doubly censored data; see Gu and Zhang (1993), Ren (1995) and Ren and Gu (1997), among others. Assumption (B5) is needed to avoid some technical difficulties in the proofs. In practice, it means that there is no right (left) censoring when X_i is sufficiently small (large), which is not an unreasonable assumption in many situations. Also, it is worth noting that the strong consistency of β_n in Theorem 2(i) was not studied in Zhang and Li (1996) and Ren and Gu (1997).

REMARK 5. From the proofs of Theorem 2, we know that we have $\sqrt{n}(\beta_n - \beta) \to_D N(0, \sigma^2)$ as $n \to \infty$ if we can show $\sqrt{n}|M_n(\beta_n)| \to_P 0$ as $n \to \infty$. Since the current article is already considerably technical, this detail is not studied.

4. Proof of Proposition 1.

PROOF OF PROPOSITION 1(i). Noting that $\mu_n(0) = \mu(0) = 0$ and $\mu_n(1) = \mu(1) = 1$, from (B1), (B3) and integration by parts, we have for $j = 1$,

$$\begin{aligned} E\{Q_{n,\theta}^{(1)}(x)\} = &\int_{-\infty}^{x+\theta} [F_B(u) - F_C(u)]\, dF(u-\beta) \\ &- \int_0^1 \mu_n(t)[F_B(x+t\theta) - F_C(x+t\theta)]\, dF\big(x + t(\theta-\beta)\big) \\ &- \beta \int_0^1 \mu_n(t)\bigg(\int_{-\infty}^{x+t\theta} f(u-t\beta)\, d[F_B(u) - F_C(u)]\bigg)\, dt \end{aligned}$$

and

$$\begin{aligned} Q_\theta^{(1)}(x) = &\int_{-\infty}^{x+\theta} [F_B(u) - F_C(u)]\, dF(u-\beta) \\ &- \int_0^1 \mu(t)[F_B(x+t\theta) - F_C(x+t\theta)]\, dF\big(x + t(\theta-\beta)\big) \\ &- \beta \int_0^1 \mu(t)\bigg(\int_{-\infty}^{x+t\theta} f(u-t\beta)\, d[F_B(u) - F_C(u)]\bigg)\, dt. \end{aligned}$$

Thus, the proof for $j = 1$ follows from (A5) and

$$\sup_{\theta,x} \big|E\{Q_{n,\theta}^{(1)}(x)\} - Q_\theta^{(1)}(x)\big| \le \|\mu_n - \mu\|(1 + 2|\beta|\|f\|).$$

Similarly, we can complete the proof for $j = 2$ or 3. $\square$

PROOF OF PROPOSITION 1(ii). From Theorem 1 of Gu and Zhang (1993), we know that under (B2), F_{X_i} is the unique solution of (2.12). Thus, as derived in Section 2, F satisfies (2.16) for any n. In turn, (2.16) and (3.4) imply $F_{n,\beta} = F$.

To show $F_\beta = F$, we let $n \to \infty$ in (2.16), which from Proposition 1(i) gives

$$F(x) = Q_\beta^{(0)}(x) - \int_{u \le x} \frac{1 - F(x)}{1 - F(u)}\, dQ_\beta^{(2)}(u) + \int_{x<u} \frac{F(x)}{F(u)}\, dQ_\beta^{(3)}(u). \tag{4.1}$$

Hence, it suffices to show that (4.1) has a unique solution. Denote

$$\begin{aligned} h &= F_\beta - F, \\ H_B(x) &= \int_0^1 f_B(x + t\beta)\, d\mu(t), \\ H_C(x) &= \int_0^1 f_C(x + t\beta)\, d\mu(t). \end{aligned} \tag{4.2}$$

Then from (3.3) we have

$$q_\beta^{(2)}(x) = [1 - F(x)]H_C(x) \quad \text{and} \quad q_\beta^{(3)}(x) = F(x)H_B(x). \tag{4.3}$$

To show $h = 0$, we subtract the integral equation (4.1) from (3.4) with $\theta = \beta$ to obtain

$$\begin{aligned} h(x)K(x) = &-\int_{u\le x} \frac{1-F_\beta(x)}{1-F_\beta(u)} h(u)H_C(u)\,du \\ &-\int_{x<u} \frac{F_\beta(x)}{F_\beta(u)} h(u)H_B(u)\,du, \end{aligned} \tag{4.4}$$

where

$$K(x) = 1 - \int_{u\le x} H_C(u)\,du - \int_{x<u} H_B(u)\,du. \tag{4.5}$$

Noting that

$$\int_{-\infty}^{\infty} H_B(u)\,du = \int_0^1 \int_{-\infty}^{\infty} dF_B(u+t\beta)\,d\mu(t) = 1,$$

by (B2) and (A5), we have that for any x,

$$\begin{aligned} K(x) &= \int_{-\infty}^{x} H_B(u)\,du - \int_{-\infty}^{x} H_C(u)\,du \\ &= \int_0^1 [F_B(x+t\beta) - F_C(x+t\beta)]\,d\mu(t) > 0. \end{aligned} \tag{4.6}$$

From (4.6), we know that (4.4) above is the same as (3.1) of Gu and Zhang (1993), where their density functions F_Y and F_Z correspond to H_B and H_C, respectively. Moreover, since F is continuous and F_β is nondecreasing, then $h(t+) \ne h(t) \Rightarrow F_\beta(t+) \ne F_\beta(t) \Rightarrow F_\beta(t+) > F_\beta(t)$, which gives (3.2) of Gu and Zhang (1993). Hence, from (4.6) and Lemma 1(i) of Gu and Zhang (1993), we have $h = 0$ if we can establish Gu and Zhang's (3.3), that is, $F_\beta(x) = 1 \Rightarrow F(x) = 1$ and $F_\beta(x) = 0 \Rightarrow F(x) = 0$.

If $F_\beta(x) = 1$, then (3.4) with $\theta = \beta$ becomes $1 = Q_\beta^{(0)}(x) + Q_\beta^{(3)}(\infty) - Q_\beta^{(3)}(x) = Q_\beta^{(1)}(x) + Q_\beta^{(2)}(x) + Q_\beta^{(3)}(\infty)$, which along with $Q_\beta^{(0)}(\infty) = 1$ implies $Q_\beta^{(1)}(\infty) + Q_\beta^{(2)}(\infty) = Q_\beta^{(1)}(x) + Q_\beta^{(2)}(x)$. Since $Q_\beta^{(j)}(x)$, $j = 1, 2, 3$, are nonnegative and nondecreasing, we have $Q_\beta^{(2)}(\infty) = Q_\beta^{(2)}(x)$ and

$$0 = Q_\beta^{(1)}(\infty) - Q_\beta^{(1)}(x) = \int_0^1 \int_x^{\infty} [F_B(u+t\beta) - F_C(u+t\beta)]\,dF(u)\,d\mu(t),$$

which by (B1) and (B2) gives $dF(u) = 0$ for $u \ge x$ and, in turn,

$$0 = Q_\beta^{(2)}(\infty) - Q_\beta^{(2)}(x) = [1 - F(x)] \int_0^1 [1 - F_C(x+t\beta)]\,d\mu(t).$$

Note that (B2) implies $[1-F_C(u)] \ge [F_B(u) - F_C(u)] > 0$ for any $u \in \mathbb{R}$. Hence $F(x) = 1$.

If $F_\beta(x) = 0$, then (3.4) with $\theta = \beta$ becomes $0 = Q_\beta^{(0)}(x) - Q_\beta^{(2)}(x) = Q_\beta^{(1)}(x) + Q_\beta^{(3)}(x)$, which implies $Q_\beta^{(1)}(x) = 0$ and $Q_\beta^{(3)}(x) = 0$. Since

$$0 = Q_\beta^{(1)}(x) = \int_0^1 \int_{-\infty}^x [F_B(u + t\beta) - F_C(u + t\beta)]\, dF(u)\, d\mu(t),$$

we know that $dF(u) = 0$ for $u \le x$, which gives

$$0 = Q_\beta^{(3)}(x) = \int_0^1 \int_{-\infty}^x F(u)\, dF_B(u + t\beta)\, d\mu(t) = F(x) \int_0^1 F_B(x + t\beta)\, d\mu(t).$$

Note that (B2) implies $F_B(u) \ge [F_B(u) - F_C(u)] > 0$ for any u. Hence, $F(x) = 0$. $\square$

PROOF OF PROPOSITION 1(iii). First, it is easy to see that from (B1), (B3), (B4) and (3.1)–(3.3), there exists a constant M_{BC} such that

$$\|EQ_{n,\theta}^{(j)} - EQ_{n,\beta}^{(j)}\| \le M_{BC}|\theta - \beta|, \qquad \|Q_\theta^{(j)} - Q_\beta^{(j)}\| \le M_{BC}|\theta - \beta|, \tag{4.7}$$

where $j = 0, 1, 2, 3$ and

$$\|q_{n,\theta}^{(j)} - q_{n,\beta}^{(j)}\| \le M_{BC}|\theta - \beta|, \qquad \|q_\theta^{(j)} - q_\beta^{(j)}\| \le M_{BC}|\theta - \beta|, \tag{4.8}$$

where $j = 2, 3$. Let $\theta_n \to \beta$ as $n \to \infty$. Then from Helly's theorem F_{θ_n} has a convergent subsequence $F_{\theta_{n_k}}$ such that for any x, $\lim_{k\to\infty} F_{\theta_{n_k}}(x) = H_0(x)$. Since $F_{\theta_{n_k}}$ satisfies

$$F_{\theta_{n_k}}(x) = Q_{\theta_{n_k}}^{(0)}(x) - \int_{u \le x} \frac{1 - F_{\theta_{n_k}}(x)}{1 - F_{\theta_{n_k}}(u)}\, dQ_{\theta_{n_k}}^{(2)}(u) + \int_{x<u} \frac{F_{\theta_{n_k}}(x)}{F_{\theta_{n_k}}(u)}\, dQ_{\theta_{n_k}}^{(3)}(u),$$

from (4.7) the limit of this equation for each fixed x as $k \to \infty$ is given by

$$H_0(x) = Q_\beta^{(0)}(x) - \int_{u \le x} \frac{1 - H_0(x)}{1 - H_0(u)}\, dQ_\beta^{(2)}(u) + \int_{x<u} \frac{H_0(x)}{H_0(u)}\, dQ_\beta^{(3)}(u). \tag{4.9}$$

From the uniqueness of the solution of (4.9) or (4.1), shown in the proof of Proposition 1(ii), we know $H_0 = F_\beta = F$. Thus, $\lim_{k\to\infty} F_{\theta_{n_k}}(x) = F_\beta(x)$ for any fixed x. Since $F_\beta = F$ is continuous, we have $\|F_{\theta_{n_k}} - F_\beta\| \to 0$, as $k \to \infty$. Hence, $\|F_{\theta_n} - F_\beta\| \to 0$, as $n \to \infty$, which gives the proof. $\square$

Before proving Proposition 1(iv), we first establish the following lemma. Let

$$\begin{aligned}
K_{n,\beta}(x, u) &= -\left\{\frac{1 - F(x)}{[1 - F(u)]^2} I\{u \le x\} q_{n,\beta}^{(2)}(u)\right. \\
&\qquad\qquad \left. + \frac{F(x)}{[F(u)]^2} I\{x < u\} q_{n,\beta}^{(3)}(u)\right\}, \\
C_{n,\beta}(x) &= \int_{u \le x} \frac{1}{1 - F(u)}\, d\{EQ_{n,\beta}^{(2)}(u)\} \\
&\qquad + \int_{x<u} \frac{1}{F(u)}\, d\{EQ_{n,\beta}^{(3)}(u)\}
\end{aligned} \tag{4.10}$$

and

$$(\widetilde{K}_{n,\beta}^{M}h)(x) = \int_{-M}^{M} \frac{K_{n,\beta}(x,u)}{1 - C_{n,\beta}(x)} h(u)\,du, \qquad x \in [-M, M]. \tag{4.11}$$

LEMMA 1. *Assume* (A5), (B1)–(B5), *and let* $D[-M, M]$ *denote the space of all functions on* $[-M, M]$ *which are right continuous and have left-hand limits, where* M *is given by* (3.6). *Then,*

(i) $\widetilde{K}_{\beta}^{M}$ *given by* (3.11), *there exists a bounded measurable function* $\widetilde{\Gamma}_{\beta}^{M}$ *on* $[-M, M]$, *such that for any* $g \in D[-M, M]$, *the integral equation*

$$(I - \widetilde{K}_{\beta}^{M})h = g \tag{4.12}$$

has the unique solution

$$h(x) = (I - \widetilde{K}_{\beta}^{M})^{-1} g(x) = g(x) + \int_{-M}^{M} \widetilde{\Gamma}_{\beta}^{M}(x,u)\, g(u)\,du; \tag{4.13}$$

(ii) *for any* n, *part* (i) *holds for* $\widetilde{K}_{n,\beta}^{M}$ *given by* (4.11);

(iii) *for any sequence* $g_n \in D[-M, M]$ *satisfying* $\sup_{n\ge 1} \|g_n\|_M < \infty$, *we have*

$$\|(I - \widetilde{K}_{n,\beta}^{M})^{-1} g_n - (I - \widetilde{K}_{\beta}^{M})^{-1} g_n\|_M \to 0 \qquad \text{as } n \to \infty. \tag{4.14}$$

PROOF OF (i). From the proof of Theorem 2.1 in Chang (1990), it suffices to show that for any $h \in D[-M, M]$, $(I - \widetilde{K}_{\beta}^{M})h = 0$ if and only if $h \equiv 0$ on $[-M, M]$.

First, we notice that from (3.10), (4.2), (4.3), (4.5) and (4.6) we have

$$\begin{aligned} 1 - C_\beta(x) &= \int_0^1 [F_B(x + t\beta) - F_C(x + t\beta)]\,d\mu(t) \\ &= K(x) > 0, \qquad x \in \mathbb{R}, \end{aligned} \tag{4.15}$$

and that $(I - \widetilde{K}_{\beta}^{M})h = 0$ implies $\widetilde{K}_{\beta}^{M} h = h$. Hence, from (3.2), (3.3), (3.10), (3.11) and (4.15) we know that h is continuous. Note that $(I - \widetilde{K}_{\beta}^{M})h = 0$, (4.15), (3.5), (3.10), (4.2) and (4.3) imply

$$\begin{aligned} h(x)K(x) &= \int_{-M}^{M} K_\beta(x,u)h(u)\,du = \int_{-\infty}^{\infty} K_\beta(x,u)h(u)\,du \\ &= -\int_{u \le x} \frac{1 - F(x)}{1 - F(u)} h(u) H_C(u)\,du - \int_{x<u} \frac{F(x)}{F(u)} h(u) H_B(u)\,du, \end{aligned}$$

which is the same as (3.1) of Gu and Zhang (1993). Since h is continuous and the support of F is $(-\infty, \infty)$, Gu and Zhang's (3.2) and (3.3) also hold. Hence, from (4.15) and Lemma 1(i) of Gu and Zhang (1993), we have $h = 0$. $\square$

PROOF OF (ii). From (4.10), (3.3), (B2) and (A5), we have

$$
\begin{aligned}
1 - C_{n,\beta}(x) &= \int_0^1 [F_B(x+t\beta) - F_C(x+t\beta)]\, d\mu_n(t) \\
&= K_n(x) > 0, \qquad x \in \mathbb{R},
\end{aligned}
\tag{4.16}
$$

where, for the derivation, we use

$$
\begin{aligned}
q_{n,\beta}^{(2)}(x) &= \int_0^1 [1 - F(x)] f_C(x+t\beta)\, d\mu_n(t) = [1 - F(x)] H_{n,C}(x), \\
q_{n,\beta}^{(3)}(x) &= \int_0^1 F(x) f_B(x+t\beta)\, d\mu_n(t) = F(x) H_{n,B}(x), \\
H_{n,B}(x) &= \int_0^1 f_B(x+t\beta)\, d\mu_n(t), \\
H_{n,C}(x) &= \int_0^1 f_C(x+t\beta)\, d\mu_n(t).
\end{aligned}
\tag{4.17}
$$

Since $(I - \widetilde{K}_{n,\beta}^M)h = 0$, (4.10), (4.11), (4.16), (4.17) and (3.5) imply

$$
\begin{aligned}
h(x)K_n(x) &= \int_{-M}^{M} K_{n,\beta}(x,u)h(u)\, du = \int_{-\infty}^{\infty} K_{n,\beta}(x,u)h(u)\, du \\
&= -\int_{u \le x} \frac{1 - F(x)}{1 - F(u)} h(u) H_{n,C}(u)\, du - \int_{x<u} \frac{F(x)}{F(u)} h(u) H_{n,B}(u)\, du;
\end{aligned}
$$

thus the rest is the same as the proof of part (i). $\square$

PROOF OF (iii). Let $(I - \widetilde{K}_{n,\beta}^M)^{-1} g_n = h_n$ and $(I - \widetilde{K}_{\beta}^M)^{-1} g_n = \bar{h}_n$. Then we need to show

$$
\|h_n - \bar{h}_n\|_M \to 0 \qquad \text{as } n \to \infty. \tag{4.18}
$$

Note that in (3.10) and (4.10), from (4.15), (4.16), integration by parts, (A5) and (B2), and from (4.2), (4.3), (4.17), integration by parts, (A5) and (B1), we have

$$
\begin{aligned}
&\|C_{n,\beta} - C_\beta\| \to 0 \qquad \text{as } n \to \infty, \\
\sup_{x,u \in [-M,M]} &\left| \frac{K_{n,\beta}(x,u)}{1 - C_{n,\beta}(x)} - \frac{K_\beta(x,u)}{1 - C_\beta(x)} \right| \to 0 \qquad \text{as } n \to \infty,
\end{aligned}
\tag{4.19}
$$

where $\inf_{|x| \le M} |1 - C_\beta(x)| > 0$. Also note that (4.13) implies $\sup_{n \ge 1} \|\bar{h}_n\|_M < \infty$. Thus, from

$$
\begin{aligned}
0 &= (I - \widetilde{K}_{n,\beta}^M)h_n - (I - \widetilde{K}_\beta^M)\bar{h}_n \\
&= (I - \widetilde{K}_{n,\beta}^M)(h_n - \bar{h}_n) + [(I - \widetilde{K}_{n,\beta}^M) - (I - \widetilde{K}_\beta^M)]\bar{h}_n \\
&= (I - \widetilde{K}_{n,\beta}^M)(h_n - \bar{h}_n) - (\widetilde{K}_{n,\beta}^M - \widetilde{K}_\beta^M)\bar{h}_n,
\end{aligned}
$$

(3.11), (4.11) and (4.19) we know

$$
\|(I - \widetilde{K}_{n,\beta}^M)(h_n - \bar{h}_n)\|_M \to 0 \qquad \text{as } n \to \infty. \tag{4.20}
$$

If $\sup_{n\ge 1}\{\|h_n - \bar{h}_n\|_M\} < \infty$, then let $(h_{n_k} - \bar{h}_{n_k})$ be any convergent subsequence with limit v_0. From (4.19), we know that $(I - \widetilde{K}^M_{n_k,\beta})(h_{n_k} - \bar{h}_{n_k})$ converges to $(I - \widetilde{K}^M_\beta)v_0$. Thus, (4.20) gives $(I - \widetilde{K}^M_\beta)v_0 = 0$, which implies $v_0 = 0$. Hence, we have (4.18).

If $\sup_{n\ge 1}\{\|h_n - \bar{h}_n\|_M\} = \infty$, then there exists a subsequence such that $\|h_{n_k} - \bar{h}_{n_k}\|_M \to \infty$ as $k \to \infty$. Let $v_{n_k} = (h_{n_k} - \bar{h}_{n_k})/\|h_{n_k} - \bar{h}_{n_k}\|_M$. Then $\|v_{n_k}\|_M \equiv 1$ and from (4.20) we have $\|(I - \widetilde{K}^M_{n_k,\beta})v_{n_k}\|_M \to 0$ as $k \to \infty$. From above, we know that v_{n_k} has limit 0, which contradicts $\|v_{n_k}\|_M \equiv 1$. □

PROOF OF PROPOSITION 1(iv). From Proposition 1(ii), we subtract (2.16) from (3.4) to obtain

$$\begin{aligned}
&F_{n,\theta}(x) - F_{n,\beta}(x)\\
&\quad = EQ^{(0)}_{n,\theta}(x) - EQ^{(0)}_{n,\beta}(x)\\
&\qquad - \int_{u\le x} \frac{1 - F_{n,\beta}(x)}{1 - F_{n,\beta}(u)}\, d\big[EQ^{(2)}_{n,\theta}(u) - EQ^{(2)}_{n,\beta}(u)\big]\\
&\qquad - \int_{u\le x} \left\{\frac{1 - F_{n,\theta}(x)}{1 - F_{n,\theta}(u)} - \frac{1 - F_{n,\beta}(x)}{1 - F_{n,\beta}(u)}\right\} dEQ^{(2)}_{n,\theta}(u)\\
&\qquad + \int_{x<u} \left\{\frac{F_{n,\theta}(x)}{F_{n,\theta}(u)} - \frac{F_{n,\beta}(x)}{F_{n,\beta}(u)}\right\} dEQ^{(3)}_{n,\theta}(u)\\
&\qquad + \int_{x<u} \frac{F_{n,\beta}(x)}{F_{n,\beta}(u)}\, d\big[EQ^{(3)}_{n,\theta}(u) - EQ^{(3)}_{n,\beta}(u)\big].
\end{aligned} \tag{4.21}$$

Letting

$$\begin{aligned}
B^{(2)}_{n,\theta,\beta}(x) &= [F_{n,\theta}(x) - F_{n,\beta}(x)]\\
&\qquad \times \int_{u\le x} \left\{\frac{1}{1 - F_{n,\theta}(u)} - \frac{1}{1 - F_{n,\beta}(u)}\right\} dEQ^{(2)}_{n,\theta}(u)\\
&\quad + [F_{n,\theta}(x) - F_{n,\beta}(x)]\\
&\qquad \times \int_{u\le x} \frac{1}{1 - F_{n,\beta}(u)}\, d\big[EQ^{(2)}_{n,\theta}(u) - EQ^{(2)}_{n,\beta}(u)\big]\\
&\quad - [1 - F_{n,\beta}(x)]\\
&\qquad \times \int_{u\le x} \left\{\frac{1}{1 - F_{n,\theta}(u)} - \frac{1}{1 - F_{n,\beta}(u)}\right\} d\big[EQ^{(2)}_{n,\theta}(u) - EQ^{(2)}_{n,\beta}(u)\big]\\
&\quad - [1 - F_{n,\beta}(x)]\\
&\qquad \times \int_{u\le x} \frac{[F_{n,\theta}(u) - F_{n,\beta}(u)]^2}{[1 - F_{n,\theta}(u)][1 - F_{n,\beta}(u)]^2}\, dEQ^{(2)}_{n,\beta}(u)
\end{aligned} \tag{4.22}$$

and

$$
\begin{aligned}
B^{(3)}_{n,\theta,\beta}(x) = {} & [F_{n,\theta}(x) - F_{n,\beta}(x)] \\
& \qquad \times \int_{x<u} \left\{ \frac{1}{F_{n,\theta}(u)} - \frac{1}{F_{n,\beta}(u)} \right\} d E Q^{(3)}_{n,\theta}(u) \\
& + [F_{n,\theta}(x) - F_{n,\beta}(x)] \\
& \qquad \times \int_{x<u} \frac{1}{F_{n,\beta}(u)} d\big[E Q^{(3)}_{n,\theta}(u) - E Q^{(3)}_{n,\beta}(u)\big] \\
& + F_{n,\beta}(x) \\
& \qquad \times \int_{x<u} \left\{ \frac{1}{F_{n,\theta}(u)} - \frac{1}{F_{n,\beta}(u)} \right\} d\big[E Q^{(3)}_{n,\theta}(u) - E Q^{(3)}_{n,\beta}(u)\big] \\
& + F_{n,\beta}(x) \\
& \qquad \times \int_{x<u} \frac{[F_{n,\theta}(u) - F_{n,\beta}(u)]^2}{F_{n,\theta}(u) F^2_{n,\beta}(u)} d E Q^{(3)}_{n,\beta}(u),
\end{aligned}
\tag{4.23}
$$

we can easily derive

$$
\begin{aligned}
& -\int_{u \le x} \left\{ \frac{1 - F_{n,\theta}(x)}{1 - F_{n,\theta}(u)} - \frac{1 - F_{n,\beta}(x)}{1 - F_{n,\beta}(u)} \right\} d E Q^{(2)}_{n,\theta}(u) \\
& \qquad = [F_{n,\theta}(x) - F_{n,\beta}(x)] \int_{u \le x} \frac{1}{1 - F_{n,\beta}(u)} d E Q^{(2)}_{n,\beta}(u) \\
& \qquad\quad - [1 - F_{n,\beta}(x)] \int_{u \le x} \frac{[F_{n,\theta}(u) - F_{n,\beta}(u)]}{[1 - F_{n,\beta}(u)]^2} d E Q^{(2)}_{n,\beta}(u) + B^{(2)}_{n,\theta,\beta}(x)
\end{aligned}
\tag{4.24}
$$

and

$$
\begin{aligned}
& \int_{x<u} \left\{ \frac{F_{n,\theta}(x)}{F_{n,\theta}(u)} - \frac{F_{n,\beta}(x)}{F_{n,\beta}(u)} \right\} d E Q^{(3)}_{n,\theta}(u) \\
& \qquad = [F_{n,\theta}(x) - F_{n,\beta}(x)] \int_{x<u} \frac{1}{F_{n,\beta}(u)} d E Q^{(3)}_{n,\beta}(u) \\
& \qquad\quad - F_{n,\beta}(x) \int_{x<u} \frac{F_{n,\theta}(u) - F_{n,\beta}(u)}{F^2_{n,\beta}(u)} d E Q^{(3)}_{n,\beta}(u) + B^{(3)}_{n,\theta,\beta}(x).
\end{aligned}
\tag{4.25}
$$

Hence, from (3.5) and Proposition 1(ii) we can write (4.21) as

$$
\begin{aligned}
& F_{n,\theta}(x) - F_{n,\beta}(x) \\
& \qquad = A_{n,\theta,\beta}(x) + B^{(2)}_{n,\theta,\beta}(x) + B^{(3)}_{n,\theta,\beta}(x) + [F_{n,\theta}(x) - F_{n,\beta}(x)] C_{n,\beta}(x) \\
& \qquad\quad + \int_{-M}^{M} K_{n,\beta}(x,u)[F_{n,\theta}(u) - F_{n,\beta}(u)]\, du,
\end{aligned}
\tag{4.26}
$$

where $x \in [-M, M]$, $C_{n,\beta}$ and $K_{n,\beta}$ are given by (4.10) and

$$A_{n,\theta,\beta}(x) = EQ^{(0)}_{n,\theta}(x) - EQ^{(0)}_{n,\beta}(x) - \int_{u \le x} \frac{1 - F(x)}{1 - F(u)} d\big[EQ^{(2)}_{n,\theta}(u) - EQ^{(2)}_{n,\beta}(u)\big] + \int_{x<u} \frac{F(x)}{F(u)} d\big[EQ^{(3)}_{n,\theta}(u) - EQ^{(3)}_{n,\beta}(u)\big]. \tag{4.27}$$

Moreover, from (4.7) and integration by parts in (4.27), we have

$$\|A_{n,\theta,\beta}\|_M \le 3M_{BC}|\theta - \beta|, \tag{4.28}$$

and from (3.5), (4.22), (4.23) and (4.8) we have

$$\|B^{(2)}_{n,\theta,\beta}\|_M \le \frac{2\|F_{n,\theta} - F_{n,\beta}\|^2_M + 4MM_{BC}\|F_{n,\theta} - F_{n,\beta}\|_M|\theta - \beta|}{[1 - F_{n,\theta}(M)][1 - F_{n,\beta}(M)]}, \tag{4.29}$$

$$\|B^{(3)}_{n,\theta,\beta}\|_M \le \frac{2\|F_{n,\theta} - F_{n,\beta}\|^2_M + 4MM_{BC}\|F_{n,\theta} - F_{n,\beta}\|_M|\theta - \beta|}{F_{n,\theta}(-M)F_{n,\beta}(-M)}. \tag{4.30}$$

Suppose that Proposition 1(iv) is false. Then there exists a sequence $|\theta_k| \le \rho$ such that

$$\frac{\|F_{n_k,\theta_k} - F_{n_k,\beta}\|_M}{|\theta_k - \beta|} \to \infty \qquad \text{as } k \to \infty, \tag{4.31}$$

which implies

$$|\theta_k - \beta| \to 0 \qquad \text{as } k \to \infty. \tag{4.32}$$

From Proposition 1(i), (4.7) and (4.32), we have

$$\|EQ^{(j)}_{n_k,\theta_k} - Q^{(j)}_\beta\| \le \|EQ^{(j)}_{n_k,\theta_k} - Q^{(j)}_{\theta_k}\| + \|Q^{(j)}_{\theta_k} - Q^{(j)}_\beta\| \to 0$$

as $k \to \infty$. Thus, Proposition 1(ii) and the proof of Proposition 1(iii) give

$$\|F_{n_k,\theta_k} - F_\beta\|_M = \|F_{n_k,\theta_k} - F_{n_k,\beta}\|_M \to 0 \qquad \text{as } k \to \infty. \tag{4.33}$$

Let

$$v_k(x) = \frac{F_{n_k,\theta_k}(x) - F_{n_k,\beta}(x)}{\|F_{n_k,\theta_k} - F_{n_k,\beta}\|_M} \quad \Longrightarrow \quad \|v_k\|_M \equiv 1. \tag{4.34}$$

Then noting that (4.26) can be written as

$$(I - \widetilde{K}^M_{n,\beta})(F_{n,\theta} - F_{n,\beta}) = (1 - C_{n,\beta})^{-1}\big(A_{n,\theta,\beta} + B^{(2)}_{n,\theta,\beta} + B^{(3)}_{n,\theta,\beta}\big), \tag{4.35}$$

where $\widetilde{K}^M_{n,\beta}$ is given by (4.11), we have

$$(I - \widetilde{K}^M_{n_k,\beta})v_k = (1 - C_{n_k,\beta})^{-1}\left(\frac{A_{n_k,\theta_k,\beta} + B^{(2)}_{n_k,\theta_k,\beta} + B^{(3)}_{n_k,\theta_k,\beta}}{\|F_{n_k,\theta_k} - F_{n_k,\beta}\|_M}\right). \tag{4.36}$$

Since from (4.28)–(4.33) we have

$$\frac{\|A_{n_k,\theta_k,\beta}\|_M + \|B^{(2)}_{n_k,\theta_k,\beta}\|_M + \|B^{(3)}_{n_k,\theta_k,\beta}\|_M}{\|F_{n_k,\theta_k} - F_{n_k,\beta}\|_M} \to 0 \qquad \text{as } k \to \infty,$$

then letting v_0 be the limit of any convergent subsequence of v_k, from (4.19), (3.11), (4.11) and (4.36) we have $(I - \widetilde{K}^M_\beta)v_0 = 0$. However, Lemma 1 implies $v_0 = 0$, which contradicts (4.34). $\square$

PROOF OF PROPOSITION 1(v). Following the notation in Pollard (1990), we have $\xi_i = (V_i, \delta_i, t_i)$, $i = 1, 2, \dots, n$, as the independent random vectors. Since $f_i^{(j)}(\omega, t, \theta) = I\{F(V_i(\omega) - t_i\theta) \le t, \delta_i = j\}$ for $t \in [0, 1]$ and $|\theta| \le \rho$ is an indicator function with envelope $|f_i^{(j)}(\omega, \cdot)| \le F_i^{(j)}(\omega) = 1$, the process $\{f_i^{(j)}(\omega, t, \theta)\}$ is manageable [see examples on VC index in Chapter 2.6 of van der Vaart and Wellner (1996)]. The proof follows from $n^{-1}\sum_{i=1}^n f_i^{(j)}(\omega, t, \theta) = \sum_{i=1}^n Q^{(j)}_{n,\theta}(F^{-1}(t))$ and Theorem 8.3 of Pollard (1990). $\square$

PROOF OF PROPOSITION 1(vi). Without loss of generality, we only need to consider the case $0 \le \theta \le 1$ and $0 \le x \le 1$. Let k be a positive integer such that $\lambda < \frac{k-1}{2k+2} < \frac{1}{2}$ and let γ be a constant satisfying $k - 2(k+1)\lambda - 1 > 2\gamma > 0$. Then, for any $\theta, x \in [0, 1]$ and positive integer $m = n^{\lambda+\gamma}$, there exist $0 \le p, q \le (m-1)$ such that $\frac{p}{m} < x \le \frac{p+1}{m}$ and $\frac{q}{m} < \theta \le \frac{q+1}{m}$. Thus, from (B1), (B3), (3.1) and (4.7), we can show for $j = 1, 2, 3$,

$$\begin{aligned} Z_n^{(j)}(x, \theta) &\equiv [Q^{(j)}_{n,\theta}(x) - EQ^{(j)}_{n,\theta}(x)] \\ &\le Z_n^{(j)}\left(\frac{p+1}{m}, \frac{q+1}{m}\right) \\ &\quad + E\left\{Q^{(j)}_{n,(q+1)/m}\left(\frac{p+1}{m}\right)\right\} - E\{Q^{(j)}_{n,\theta}(x)\} \\ &\le Z_n^{(j)}\left(\frac{p+1}{m}, \frac{q+1}{m}\right) + 2M_{BC}m^{-1}, \end{aligned}$$

and from a similar lower bound for $Z_n^{(j)}(x, \theta)$, we have

$$\begin{aligned} &\sup_{0\le\theta\le1} \|Q^{(j)}_{n,\theta} - EQ^{(j)}_{n,\theta}\|_{[0,1]} \\ &\qquad \le \max\left\{\left|Z_n^{(j)}\left(\frac{p}{m}, \frac{q}{m}\right)\right|; 0 \le p, q \le m\right\} + 2M_{BC}m^{-1}, \end{aligned} \tag{4.37}$$

where $\|\cdot\|_{[0,1]}$ stands for the uniform norm on [0, 1]. Note that for any p and q, it can be shown that $E|Z_n^{(j)}(\frac{p}{m},\frac{q}{m})|^{2k} \le n^{-k}C_k$ for $C_k = k(8k)^k$. For instance,

$$\begin{aligned}
E|Z_n^{(j)}(x,\theta)|^4 \le n^{-4}\Biggl\{&\sum_{i=1}^{n} E\bigl[I\{V_i \le x + t_i\theta, \delta_i = j\}\\
&\qquad - P\{V_i \le x + t_i\theta, \delta_i = j\}\bigr]^4\\
&+ \sum\sum_{p\ne q} E\bigl[I\{V_p \le x + t_p\theta, \delta_p = j\}\\
&\qquad\qquad - P\{V_p \le x + t_p\theta, \delta_p = j\}\bigr]^2\\
&\qquad \times E\bigl[I\{V_q \le x + t_q\theta, \delta_q = j\}\\
&\qquad\qquad - P\{V_q \le x + t_q\theta, \delta_q = j\}\bigr]^2\Biggr\}\\
\le n^{-4}(n2^4 + n^2 2^2 2^2) &= 16n^{-3} + 16n^{-2} \le 32n^{-2}.
\end{aligned}$$

Hence, for $k - 2(k+1)\lambda - 2\gamma > 1$ and any $\varepsilon > 0$, the Markov inequality gives

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left\{\max\left\{n^{\lambda}\left|Z_n^{(j)}\left(\frac{p}{m},\frac{q}{m}\right)\right|; 0 \le p, q \le m\right\} > \varepsilon\right\}\\
&\quad \le \sum_{n=1}^{\infty}\sum_{p=0}^{m}\sum_{q=0}^{m} P\left\{n^{\lambda}\left|Z_n^{(j)}\left(\frac{p}{m},\frac{q}{m}\right)\right| > \varepsilon\right\}\\
&\quad \le \sum_{n=1}^{\infty}\sum_{p=0}^{m}\sum_{q=0}^{m} \varepsilon^{-2k} n^{2k\lambda} E\left|Z_n^{(j)}\left(\frac{p}{m},\frac{q}{m}\right)\right|^{2k}\\
&\quad \le C_k\varepsilon^{-2k}\sum_{n=1}^{\infty}(m+1)^2 n^{2k\lambda} n^{-k} \le 4C_k\varepsilon^{-2k}\sum_{n=1}^{\infty} n^{2(\lambda+\gamma)} n^{2k\lambda - k} < \infty
\end{aligned}$$

and, in turn, from the theorem in Section 1.3.4 of Serfling [(1980), page 10], we know that

$$\max\left\{n^{\lambda}\left|Z_n^{(j)}\left(\frac{p}{m},\frac{q}{m}\right)\right|; 0 \le p, q \le m\right\} \xrightarrow{\text{a.s.}} 0 \qquad \text{as } n \to \infty. \tag{4.38}$$

Therefore, the proof follows from (4.37), (4.38) and $n^{\lambda}m^{-1} = n^{-\gamma} \to 0$ as $n \to \infty$. $\square$

5. Proof of Theorem 2. In this section, $EQ_{n,\beta_n}^{(j)}$ always denotes the expectation that treats β_n as a constant in (3.1) and (3.3). We begin by establishing the following lemma, which is needed to prove Theorem 2(i).

LEMMA 2. *Assume* (A5) *and* (B1)–(B5). *For M given by* (3.6) *and* $0 < \lambda < \frac{1}{2}$, *if*

$$\theta_n \to \beta, \qquad n^\lambda \| Q_{n,\theta_n}^{(j)} - E Q_{n,\theta_n}^{(j)} \|_M \to 0, \qquad \| Q_{n,\theta_n}^{(j)} - E Q_{n,\theta_n}^{(j)} \| \to 0 \qquad \textit{as } n \to \infty, \tag{5.1}$$

where $j = 0, 1, 2, 3$, *then*

$$n^\lambda \| \widehat{F}_{n,\theta_n} - F_{n,\theta_n} \|_M \to 0 \qquad \textit{as } n \to \infty. \tag{5.2}$$

PROOF. First, we note that (5.1), Proposition 1(i) and (4.7) give

$$\| Q_{n,\theta_n}^{(j)} - Q_\beta^{(j)} \| \to 0 \qquad \text{as } n \to \infty, \tag{5.3}$$

where $j = 0, 1, 2, 3$. Thus, from the proof of Proposition 1(iii) and from Proposition 1(ii) and (iv), we have that as $n \to \infty$,

$$\begin{aligned} &\| \widehat{F}_{n,\theta_n} - F_\beta \| \to 0, \\ &\| F_{n,\theta_n} - F_{n,\beta} \|_M \to 0, \\ &\| \widehat{F}_{n,\theta_n} - F_{n,\theta_n} \|_M \to 0. \end{aligned} \tag{5.4}$$

From (3.5), (3.7), (3.8) and the derivation of (4.26), we have for any $x \in [-M, M]$,

$$\begin{aligned} &\widehat{F}_{n,\theta}(x) - F_{n,\theta}(x) \\ &\quad = A_{n,\theta}(x) + B_{n,\theta}^{(2)}(x) + B_{n,\theta}^{(3)}(x) + [\widehat{F}_{n,\theta}(x) - F_{n,\theta}(x)] C_{n,\theta}(x) \\ &\qquad + \int_{-M}^{M} K_{n,\theta}(x, u) [\widehat{F}_{n,\theta}(u) - F_{n,\theta}(u)] \, du, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} A_{n,\theta}(x) &= Q_{n,\theta}^{(0)}(x) - E Q_{n,\theta}^{(0)}(x) - \int_{-M}^{x} \frac{1 - F_{n,\theta}(x)}{1 - F_{n,\theta}(u)} \, d[Q_{n,\theta}^{(2)}(u) - E Q_{n,\theta}^{(2)}(u)] \\ &\quad + \int_{x}^{M} \frac{F_{n,\theta}(x)}{F_{n,\theta}(u)} \, d[Q_{n,\theta}^{(3)}(u) - E Q_{n,\theta}^{(3)}(u)], \\ B_{n,\theta}^{(2)}(x) &= [\widehat{F}_{n,\theta}(x) - F_{n,\theta}(x)] \int_{-M}^{x} \left\{ \frac{1}{1 - \widehat{F}_{n,\theta}(u)} - \frac{1}{1 - F_{n,\theta}(u)} \right\} dQ_{n,\theta}^{(2)}(u) \\ &\quad + [\widehat{F}_{n,\theta}(x) - F_{n,\theta}(x)] \int_{-M}^{x} \frac{1}{1 - F_{n,\theta}(u)} \, d[Q_{n,\theta}^{(2)}(u) - E Q_{n,\theta}^{(2)}(u)] \\ &\quad - [1 - F_{n,\theta}(x)] \\ &\qquad \times \int_{-M}^{x} \left\{ \frac{1}{1 - \widehat{F}_{n,\theta}(u)} - \frac{1}{1 - F_{n,\theta}(u)} \right\} d[Q_{n,\theta}^{(2)}(u) - E Q_{n,\theta}^{(2)}(u)] \end{aligned}$$

$$- [1 - F_{n,\theta}(x)] \int_{-M}^{x} \frac{[\widehat{F}_{n,\theta}(u) - F_{n,\theta}(u)]^2}{[1 - \widehat{F}_{n,\theta}(u)][1 - F_{n,\theta}(u)]^2} \, dE Q_{n,\theta}^{(2)}(u),$$

$$B_{n,\theta}^{(3)}(x) = [\widehat{F}_{n,\theta}(x) - F_{n,\theta}(x)] \int_{x}^{M} \left\{ \frac{1}{\widehat{F}_{n,\theta}(u)} - \frac{1}{F_{n,\theta}(u)} \right\} dQ_{n,\theta}^{(3)}(u)$$

$$+ \big[\widehat{F}_{n,\theta}(x) - F_{n,\theta}(x)\big] \int_{x}^{M} \frac{1}{F_{n,\theta}(u)} \, d\big[Q_{n,\theta}^{(3)}(u) - E Q_{n,\theta}^{(3)}(u)\big]$$

$$+ F_{n,\theta}(x) \int_{x}^{M} \left\{ \frac{1}{\widehat{F}_{n,\theta}(u)} - \frac{1}{F_{n,\theta}(u)} \right\} d\big[Q_{n,\theta}^{(3)}(u) - E Q_{n,\theta}^{(3)}(u)\big]$$

$$+ F_{n,\theta}(x) \int_{x}^{M} \frac{[\widehat{F}_{n,\theta}(u) - F_{n,\theta}(u)]^2}{\widehat{F}_{n,\theta}(u) \, F_{n,\theta}^2(u)} \, dE Q_{n,\theta}^{(3)}(u),$$

$$C_{n,\theta}(x) = \int_{-M}^{x} \frac{1}{1 - F_{n,\theta}(u)} \, dE Q_{n,\theta}^{(2)}(u) + \int_{x}^{M} \frac{1}{F_{n,\theta}(u)} \, dE Q_{n,\theta}^{(3)}(u),$$

$$K_{n,\theta}(x, u) = -\left\{ \frac{1 - F_{n,\theta}(x)}{[1 - F_{n,\theta}(u)]^2} I\{u \le x\} q_{n,\theta}^{(2)}(u) + \frac{F_{n,\theta}(x)}{[F_{n,\theta}(u)]^2} I\{x < u\} q_{n,\theta}^{(3)}(u) \right\}.$$

Equivalently, (5.5) can be written as

$$\begin{aligned} \widehat{F}_{n,\theta}(x) - F_{n,\theta}(x) &= A_{n,\theta}(x) + B_{n,\theta}^{(2)}(x) + B_{n,\theta}^{(3)}(x) + K_{n,\theta,\beta}(x) \\ &\quad + \big[\widehat{F}_{n,\theta}(x) - F_{n,\theta}(x)\big] C_{n,\beta}(x) \\ &\quad + \int_{-M}^{M} K_{n,\beta}(x, u) \big[\widehat{F}_{n,\theta}(u) - F_{n,\theta}(u)\big] \, du, \end{aligned}$$

which gives

$$(I - \widetilde{K}_{n,\beta}^{M})\big(\widehat{F}_{n,\theta} - F_{n,\theta}\big) = (1 - C_{n,\beta})^{-1}\big(A_{n,\theta} + B_{n,\theta}^{(2)} + B_{n,\theta}^{(3)} + K_{n,\theta,\beta}\big), \tag{5.6}$$

where

$$\begin{aligned} K_{n,\theta,\beta}(x) &= \int_{-M}^{M} [K_{n,\theta}(x, u) - K_{n,\beta}(x, u)] \big[\widehat{F}_{n,\theta}(u) - F_{n,\theta}(u)\big] \, du \\ &\quad + \big[\widehat{F}_{n,\theta}(x) - F_{n,\theta}(x)\big][C_{n,\theta}(x) - C_{n,\beta}(x)]. \end{aligned}$$

Since $\theta_n \to \beta$ as $n \to \infty$, we have from Proposition 1(ii), (5.4) and (4.8),

$$\|C_{n,\theta_n} - C_{n,\beta}\|_M \to 0 \quad \text{and} \quad \|K_{n,\theta_n} - K_{n,\beta}\|_{M^2} \to 0 \qquad \text{as } n \to \infty. \tag{5.7}$$

Next, we establish (5.2) by discussing the cases with bounded and unbounded $\{n^{\lambda} \|\widehat{F}_{n,\theta_n} - F_{n,\theta_n}\|_M\}$ separately.

CASE (a). $\sup_{n \ge 1}\{n^{\lambda} \|\widehat{F}_{n,\theta_n} - F_{n,\theta_n}\|_M\} < \infty$. Note that (5.7) gives

$$n^{\lambda} \|K_{n,\theta_n,\beta}\|_M \to 0 \qquad \text{as } n \to \infty. \tag{5.8}$$

Also, from (5.1), (5.4), Proposition 1(ii) and (B1) we have that

$$n^{\lambda}\|A_{n,\theta_n}\|_M \to 0 \qquad \text{as } n\to\infty, \tag{5.9}$$

because from integration by parts,

$$\begin{aligned}
A_{n,\theta_n}(x) = {} & [Q^{(1)}_{n,\theta_n}(x) - EQ^{(1)}_{n,\theta_n}(x)] \\
& + [Q^{(2)}_{n,\theta_n}(-M) - EQ^{(2)}_{n,\theta_n}(-M)]\frac{1-F_{n,\theta_n}(x)}{1-F_{n,\theta_n}(-M)} \\
& + \int_{-M}^{x}[Q^{(2)}_{n,\theta_n}(u) - EQ^{(2)}_{n,\theta_n}(u)]\,d\left\{\frac{1-F_{n,\theta_n}(x)}{1-F_{n,\theta_n}(u)}\right\} \\
& + [Q^{(3)}_{n,\theta_n}(M) - EQ^{(3)}_{n,\theta_n}(M)]\frac{F_{n,\theta_n}(x)}{F_{n,\theta_n}(M)} \\
& - \int_{x}^{M}[Q^{(3)}_{n,\theta_n}(u) - EQ^{(3)}_{n,\theta_n}(u)]\,d\left\{\frac{F_{n,\theta_n}(x)}{F_{n,\theta_n}(u)}\right\},
\end{aligned} \tag{5.10}$$

and we have that as $n\to\infty$,

$$\begin{aligned}
n^{\lambda}\|B^{(2)}_{n,\theta_n}\|_M \le {} & \frac{n^{\lambda}\|\widehat{F}_{n,\theta_n} - F_{n,\theta_n}\|_M^2[Q^{(2)}_{n,\theta_n}(M) - Q^{(2)}_{n,\theta_n}(-M)]}{[1-\widehat{F}_{n,\theta_n}(M)][1-F_{n,\theta_n}(M)]} \\
& + \frac{2n^{\lambda}\|\widehat{F}_{n,\theta_n} - F_{n,\theta_n}\|_M\|Q^{(2)}_{n,\theta_n} - EQ^{(2)}_{n,\theta_n}\|_M}{[1-F_{n,\theta_n}(M)]} \\
& + \frac{2n^{\lambda}\|\widehat{F}_{n,\theta_n} - F_{n,\theta_n}\|_M\|Q^{(2)}_{n,\theta_n} - EQ^{(2)}_{n,\theta_n}\|_M}{[1-\widehat{F}_{n,\theta_n}(M)]} \\
& + \frac{n^{\lambda}\|\widehat{F}_{n,\theta_n} - F_{n,\theta_n}\|_M^2[EQ^{(2)}_{n,\theta_n}(M) - EQ^{(2)}_{n,\theta_n}(-M)]}{[1-\widehat{F}_{n,\theta_n}(M)][1-F_{n,\theta_n}(M)]} \to 0
\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}
n^{\lambda}\|B^{(3)}_{n,\theta_n}\|_M \le {} & \frac{n^{\lambda}\|\widehat{F}_{n,\theta_n} - F_{n,\theta_n}\|_M^2[Q^{(3)}_{n,\theta_n}(M) - Q^{(3)}_{n,\theta_n}(-M)]}{\widehat{F}_{n,\theta_n}(-M)F_{n,\theta_n}(-M)} \\
& + \frac{2n^{\lambda}\|\widehat{F}_{n,\theta_n} - F_{n,\theta_n}\|_M\|Q^{(3)}_{n,\theta_n} - EQ^{(3)}_{n,\theta_n}\|_M}{F_{\theta_n}(-M)} \\
& + \frac{2n^{\lambda}\|\widehat{F}_{n,\theta_n} - F_{n,\theta_n}\|_M\|Q^{(3)}_{n,\theta_n} - EQ^{(3)}_{n,\theta_n}\|_M}{\widehat{F}_{n,\theta_n}(-M)} \\
& + \frac{n^{\lambda}\|\widehat{F}_{n,\theta_n} - F_{n,\theta_n}\|_M^2[EQ^{(3)}_{n,\theta_n}(M) - EQ^{(3)}_{n,\theta_n}(-M)]}{\widehat{F}_{n,\theta_n}(-M)F_{n,\theta_n}(-M)} \to 0.
\end{aligned} \tag{5.12}$$

Hence, (5.6), (5.8)–(5.12) and (4.19) give

$$\|(I - \widetilde{K}^M_{n,\beta})[n^\lambda(\widehat{F}_{n,\theta_n} - F_{n,\theta_n})]\|_M \to 0 \qquad \text{as } n \to \infty. \tag{5.13}$$

Since $\sup_{n\ge 1}\{n^\lambda \|\widehat{F}_{n,\theta_n} - F_{n,\theta_n}\|_M\} < \infty$, then (5.13), (3.11), (4.11) and (4.19) imply that if a convergent subsequence of $n^\lambda(\widehat{F}_{n,\theta_n} - F_{n,\theta_n})$ has limit v_0, then we have $(I - \widetilde{K}^M_\beta)v_0 = 0$, and Lemma 1 implies $v_0 = 0$. Hence, we have (5.2).

CASE (b). $\sup_{n\ge 1}\{n^\lambda \|\widehat{F}_{n,\theta_n} - F_{n,\theta_n}\|_M\} = \infty$. Let $u_n = n^\lambda(\widehat{F}_{n,\theta_n} - F_{n,\theta_n})$. Then there exists a subsequence u_{n_k} such that $\|u_{n_k}\|_M \to \infty$ as $k \to \infty$ and for

$$v_{n_k} = \frac{u_{n_k}}{\|u_{n_k}\|_M} \quad \Longrightarrow \quad \|v_{n_k}\|_M \equiv 1, \tag{5.14}$$

equation (5.6) becomes

$$(I - \widetilde{K}^M_{n_k,\beta})v_{n_k} = \frac{n_k^\lambda(A_{n_k,\theta_{n_k}} + B^{(2)}_{n_k,\theta_{n_k}} + B^{(3)}_{n_k,\theta_{n_k}} + K_{n_k,\theta_{n_k},\beta})}{\|u_{n_k}\|_M(1 - C_{n_k,\beta})}. \tag{5.15}$$

Clearly, from (5.10), (5.1), (5.4) and (5.7) we have that as $k \to \infty$,

$$\frac{n_k^\lambda \|A_{n_k,\theta_{n_k}}\|_M}{\|u_{n_k}\|_M} \to 0 \quad \text{and} \quad \frac{n_k^\lambda \|K_{n_k,\theta_{n_k},\beta}\|_M}{\|u_{n_k}\|_M} \to 0. \tag{5.16}$$

Moreover, (5.1), (5.4), (5.11) and (5.12) give

$$\frac{n_k^\lambda \|B^{(2)}_{n_k,\theta_{n_k}}\|_M}{\|u_{n_k}\|_M} \to 0 \quad \text{and} \quad \frac{n_k^\lambda \|B^{(3)}_{n_k,\theta_{n_k}}\|_M}{\|u_{n_k}\|_M} \to 0 \qquad \text{as } k \to \infty. \tag{5.17}$$

Hence, (5.15)–(5.17) and (4.19) imply $(I - \widetilde{K}^M_{n_k,\beta})v_{n_k} \to 0$ as $k \to \infty$. From the proof of Case (a), we know that $v_{n_k} \to 0$ as $k \to \infty$ which contradicts (5.14). □

PROOF OF THEOREM 2(i). From (A1), (A3) and (2.20), we know for any $\theta \in [-\rho, \rho]$,

$$\begin{aligned} M_n(\theta) &= -\int_A^B [\widehat{F}_{n,\theta}(x) - F(x)]\,d\psi(x) \\ &= -\int_A^B [\widehat{F}_{n,\theta}(x) - \widehat{F}_{n,\beta}(x)]\,d\psi(x) \\ &\quad - \int_A^B [\widehat{F}_{n,\beta}(x) - F_{n,\beta}(x)]\,d\psi(x), \end{aligned} \tag{5.18}$$

and from Lemma 2 and Proposition 1(v) and (vi), we have

$$n^\lambda \|\widehat{F}_{n,\beta} - F_{n,\beta}\|_M \overset{\text{a.s.}}{\to} 0 \qquad \text{as } n \to \infty. \tag{5.19}$$

Since $|M_n(\beta_n)| \le |M_n(\beta)|$, we know that (5.18), (5.19) and Proposition 1(ii) imply

$$n^\lambda |M_n(\beta_n)| \overset{\text{a.s.}}{\to} 0 \qquad \text{as } n \to \infty \tag{5.20}$$

and, in turn,

$$n^\lambda \int_A^B \big[\widehat{F}_{n,\beta_n}(x) - \widehat{F}_{n,\beta}(x)\big]\, d\psi(x) \overset{\text{a.s.}}{\to} 0 \qquad \text{as } n \to \infty. \tag{5.21}$$

Because of Proposition 1(v) and (vi), it suffices to show that for $j = 0, 1, 2, 3$,

$$n^\lambda \| Q^{(j)}_{n,\beta_n} - E Q^{(j)}_{n,\beta_n} \|_M \to 0 \quad \text{and} \quad \| Q^{(j)}_{n,\beta_n} - E Q^{(j)}_{n,\beta_n} \| \to 0 \quad \text{as } n \to \infty \tag{5.22}$$

implies $n^\lambda |\beta_n - \beta| \to 0$ as $n \to \infty$.

Let $\{\beta_{n_k}\}$ be a subsequence of $\{\beta_n\}$ such that

$$\lim_{k\to\infty} n_k^\lambda |\beta_{n_k} - \beta| = \limsup_{n\to\infty} n^\lambda |\beta_n - \beta| \tag{5.23}$$

and denote

$$\xi = \inf_{n_k \ge 1} \frac{\|\widehat{F}_{n_k,\beta_{n_k}} - \widehat{F}_{n_k,\beta}\|_M}{|\beta_{n_k} - \beta|}. \tag{5.24}$$

Then

$$\left| \int_A^B \frac{\widehat{F}_{n_k,\beta_{n_k}}(x) - \widehat{F}_{n_k,\beta}(x)}{\beta_{n_k} - \beta}\, d\psi(x) \right| \ge \xi[\psi(B) - \psi(A)],$$

and from (5.21) we have

$$\begin{aligned} &\lim_{k\to\infty} n_k^\lambda |\beta_{n_k} - \beta| \xi[\psi(B) - \psi(A)] \\ &\quad \le \lim_{k\to\infty} n_k^\lambda |\beta_{n_k} - \beta| \left| \int_A^B \frac{\widehat{F}_{n_k,\beta_{n_k}}(x) - \widehat{F}_{n_k,\beta}(x)}{\beta_{n_k} - \beta}\, d\psi(x) \right| = 0. \end{aligned} \tag{5.25}$$

Moreover, we know from (5.23) the proof follows by showing

$$\lim_{k\to\infty} n_k^\lambda |\beta_{n_k} - \beta| = 0.$$

If $\lim_{k\to\infty} n_k^\lambda |\beta_{n_k} - \beta| \ne 0$, then we have $\xi = 0$ in (5.25), which by (5.24) means that there exists a subsequence, still denoted as $\{n_k^\lambda |\beta_{n_k} - \beta|\}$, such that

$$\lim_{k\to\infty} \frac{\|\widehat{F}_{n_k,\beta_{n_k}} - \widehat{F}_{n_k,\beta}\|_M}{|\beta_{n_k} - \beta|} = 0 \quad \text{and} \quad \lim_{k\to\infty} \beta_{n_k} = \beta_0. \tag{5.26}$$

If $\beta_0 \ne \beta$, then $\lim_{k\to\infty} \|\widehat{F}_{n_k,\beta_{n_k}} - \widehat{F}_{n_k,\beta}\|_M = 0$ and, in turn, (5.19) implies

$$\lim_{k\to\infty} \|\widehat{F}_{n_k,\beta_{n_k}} - F_\beta\|_M = 0. \tag{5.27}$$

On the other hand, from (5.22), Proposition 1(i), (4.7) and $\lim_{k\to\infty}\beta_{n_k}=\beta_0$, we know for $j=0,1,2,3$, $\lim_{k\to\infty}\|Q^{(j)}_{n_k,\beta_{n_k}}-Q^{(j)}_{\beta_0}\|=0$, which from the proof of Proposition 1(iii) gives that for any convergent subsequence of $\widehat{F}_{n_k,\beta_{n_k}}$, still denoted as $\{\widehat{F}_{n_k,\beta_{n_k}}\}$, and for any fixed $x\in[-M,M]$, we have $\lim_{k\to\infty}\widehat{F}_{n_k,\beta_{n_k}}(x)=F_{\beta_0}(x)$. However, (5.27) implies $\lim_{k\to\infty}\widehat{F}_{n_k,\beta_{n_k}}(x)=F_\beta(x)$, which contradicts (B6) because $\beta_0\neq\beta$.

If $\beta_0=\beta$, from (5.22) and Lemma 2 we have

$$\lim_{k\to\infty} n_k^\lambda\|\widehat{F}_{n_k,\beta_{n_k}}-F_{n_k,\beta_{n_k}}\|_M=0, \tag{5.28}$$

which from the assumption $\lim_{k\to\infty}n_k^\lambda|\beta_{n_k}-\beta|\neq 0$, (5.26) and (5.19) implies

$$\lim_{k\to\infty} n_k^\lambda\|F_{n_k,\beta_{n_k}}-F_\beta\|_M=0. \tag{5.29}$$

Note that (5.18), (5.20), (5.28), (4.35), Lemma 1, (4.29), (4.30), (5.29) and (4.19) imply

$$\begin{aligned}
o(n_k^{-\lambda})&=-\int_A^B\big[F_{n_k,\beta_{n_k}}(x)-F_{n_k,\beta}(x)\big]\,d\psi(x)\\
&=-\int_A^B(I-\widetilde{K}^M_{n_k,\beta})^{-1}\\
&\qquad\times\left(\frac{A_{n_k,\beta_{n_k},\beta}+B^{(2)}_{n_k,\beta_{n_k},\beta}+B^{(3)}_{n_k,\beta_{n_k},\beta}}{1-C_{n_k,\beta}}\right)(x)\,d\psi(x)\\
&=-\int_A^B(I-\widetilde{K}^M_{n_k,\beta})^{-1}\left(\frac{A_{n_k,\beta_{n_k},\beta}}{1-C_{n_k,\beta}}\right)(x)\,d\psi(x)+o(n_k^{-\lambda}).
\end{aligned}\tag{5.30}$$

From (3.1), (3.10), integration by parts in (4.27), $\lim_{k\to\infty}\beta_{n_k}=\beta$, (B1), (B3), (B5) and (A5) we have

$$\|(\beta_{n_k}-\beta)^{-1}A_{n_k,\beta_{n_k},\beta}-\widetilde{A}_\beta\|_M\to 0\qquad\text{as } k\to\infty. \tag{5.31}$$

Hence, the dominated convergence theorem, (5.31), (4.19), Lemma 1 and (3.9) give

$$(\beta_{n_k}-\beta)^{-1}\int_A^B(I-\widetilde{K}^M_{n_k,\beta})^{-1}\left(\frac{A_{n_k,\beta_{n_k},\beta}}{1-C_{n_k,\beta}}\right)(x)\,d\psi(x)\to A_\beta\neq 0\qquad\text{as } k\to\infty. \tag{5.32}$$

Hence, by (5.30) and (5.32) we have $\lim_{k\to\infty}n_k^\lambda|\beta_{n_k}-\beta|=0$, which contradicts the assumption $\lim_{k\to\infty}n_k^\lambda|\beta_{n_k}-\beta|\neq 0$. $\square$

The following lemma is needed in the proof of Theorem 2(ii).

LEMMA 3. *Under the assumptions of Theorem* 2, *we have for* M *given by* (3.6),

$$\sqrt{n}(Q^{(j)}_{n,\beta_n} - EQ^{(j)}_{n,\beta_n}) \text{ weakly converges to } \mathbb{G}^{(j)}_{\beta} \text{ as } n \to \infty \tag{5.33}$$

on $[-M, M]$, *where* $\mathbb{G}^{(j)}_{\beta}$ *is a centered Gaussian process for* $j = 1, 2, 3$.

PROOF. From the notation in the proofs of Proposition 1(v) and (vi), we know that $f_{ni}(\omega, x, \theta) = n^{-1/2} I\{V_i(\omega) \le x + t_i\theta, \delta_i = j\}$ for $|x| \le M$ and $|\theta| \le \rho$ is an indicator function with envelope $|f_{ni}(\omega, \cdot)| \le F_{ni}(\omega) = n^{-1/2}$. Thus the triangular array of processes $\{f_{ni}(\omega, x, \theta)\}$ is manageable. Noting that

$$\sqrt{n}\, Z^{(j)}_n(x, \theta) = \sum_{i=1}^{n} [f_{ni}(\omega, x, \theta) - Ef_{ni}(\cdot, x, \theta)],$$

from (A5) and $P^{(j)}_i$ in (2.12), straightforward verification of the sufficient conditions of Theorem 10.6 in Pollard (1990) shows that $\sqrt{n}\, Z^{(j)}_n(x, \theta)$ weakly converges to a centered Gaussian process on $[-M, M] \times [-\rho, \rho]$. This means that $\sqrt{n} Z^{(j)}_n(\cdot, \beta) = \sqrt{n}(Q^{(j)}_{n,\beta} - EQ^{(j)}_{n,\beta})$ weakly converges to a centered Gaussian process on $[-M, M]$ as $n \to \infty$ and that from Neuhaus [(1971), page 1291] we have that for any $\varepsilon > 0$, when $\delta \to 0$,

$$\limsup_{n\to\infty} P\Big\{\sup\{\sqrt{n}|Z^{(j)}_n(x, \theta) - Z^{(j)}_n(y, \eta)|; |x - y| < \delta, |\theta - \eta| < \delta\} \ge \varepsilon\Big\} \to 0.$$

Since from $\sqrt{n} Z^{(j)}_n(x, \theta) = \sqrt{n}[Q^{(j)}_{n,\theta}(x) - EQ^{(j)}_{n,\theta}(x)]$, we have that for $n^{-1/3} \le \delta$,

$$\begin{aligned} &\sup\{\sqrt{n}|[Q^{(j)}_{n,\theta}(x) - EQ^{(j)}_{n,\theta}(x)] \\ &\qquad - [Q^{(j)}_{n,\beta}(x) - EQ^{(j)}_{n,\beta}(x)]|; |x| \le M, n^{1/3}|\theta - \beta| \le 1\} \\ &\qquad \le \sup\{\sqrt{n}|Z^{(j)}_n(x, \theta) - Z^{(j)}_n(y, \beta)|; |x - y| < \delta, |\theta - \beta| < \delta\}, \end{aligned}$$

thus as $n \to \infty$,

$$\sup_{|x|\le M, n^{1/3}|\theta-\beta|\le 1} \{\sqrt{n}|[Q^{(j)}_{n,\theta}(x) - EQ^{(j)}_{n,\theta}(x)] - [Q^{(j)}_{n,\beta}(x) - EQ^{(j)}_{n,\beta}(x)]|\} \xrightarrow{P} 0.$$

From Theorem 2(i), we know that $n^{1/3}|\beta_n - \beta| \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. Hence, we have that $\sqrt{n}\,(Q^{(j)}_{n,\beta_n} - EQ^{(j)}_{n,\beta_n})$ weakly converges to the same centered Gaussian process as $\sqrt{n}(Q^{(j)}_{n,\beta} - EQ^{(j)}_{n,\beta})$. $\square$

PROOF OF THEOREM 2(ii). From (5.6) we have that on $[-M, M]$,

$$(I - \widetilde{K}^M_{n,\beta})[\sqrt{n}(\widehat{F}_{n,\beta_n} - F_{n,\beta_n})] = \sqrt{n}\left(\frac{A_{n,\beta_n} + B^{(2)}_{n,\beta_n} + B^{(3)}_{n,\beta_n} + K_{n,\beta_n,\beta}}{1 - C_{n,\beta}}\right), \tag{5.34}$$

and from Theorem 2(i), Proposition 1(iv)–(vi) and Lemma 2, we have

$$n^{\lambda}\|\widehat{F}_{n,\beta_n} - F_{n,\beta_n}\|_M \stackrel{\text{a.s.}}{\to} 0 \quad \text{and} \quad n^{\lambda}\|F_{n,\beta_n} - F_{n,\beta}\|_M \stackrel{\text{a.s.}}{\to} 0 \qquad \text{as } n \to \infty, \tag{5.35}$$

where $0 < \lambda < \frac{1}{2}$. By (5.11), (5.12) and Proposition 1(vi), we know (5.35) implies

$$\sqrt{n}\|B^{(2)}_{n,\beta_n}\|_M \stackrel{\text{a.s.}}{\to} 0 \quad \text{and} \quad \sqrt{n}\|B^{(3)}_{n,\beta_n}\|_M \stackrel{\text{a.s.}}{\to} 0 \qquad \text{as } n \to \infty. \tag{5.36}$$

From (3.3), (4.8) and Proposition 1(iv) there exists $0 < \widetilde{M}_0 < \infty$ such that

$$\|C_{n,\beta_n} - C_{n,\beta}\|_M \le \frac{\widetilde{M}_0|\beta_n - \beta|}{[1 - F_{n,\beta_n}(M)][1 - F_{n,\beta}(M)]} + \frac{\widetilde{M}_0|\beta_n - \beta|}{F_{n,\beta_n}(-M)F_{n,\beta}(-M)}$$

and similarly

$$\|K_{n,\beta_n} - K_{n,\beta}\|_{M^2} \le \frac{\widetilde{M}_0|\beta_n - \beta|}{[1 - F_{n,\beta_n}(M)]^2[1 - F_{n,\beta}(M)]^2} + \frac{\widetilde{M}_0|\beta_n - \beta|}{F^2_{n,\beta_n}(-M)F^2_{n,\beta}(-M)}.$$

In turn, from Theorem 2(i) and (5.35) we have, in (5.6),

$$\sqrt{n}\|K_{n,\beta_n,\beta}\|_M \stackrel{\text{a.s.}}{\to} 0 \qquad \text{as } n \to \infty. \tag{5.37}$$

Hence, from (5.36), (5.37), Lemma 1 and (4.19), equation (5.34) gives

$$\sqrt{n}(\widehat{F}_{n,\beta_n} - F_{n,\beta_n}) = (I - \widetilde{K}^M_{n,\beta})^{-1}\left(\frac{\sqrt{n}A_{n,\beta_n}}{1 - C_{n,\beta}}\right) + o_{\text{a.s.}}(1),$$

where $o_{\text{a.s.}}(1)$ converges to 0 almost surely as $n \to \infty$. Thus, (5.10), Lemma 3, (5.35), (4.19) and Lemma 1 imply that on $[-M, M]$,

$$\sqrt{n}(\widehat{F}_{n,\beta_n} - F_{n,\beta_n}) \text{ weakly converges to } \mathbb{G}_\beta \text{ as } n \to \infty, \tag{5.38}$$

where $\mathbb{G}_\beta$ is a centered Gaussian process.

Now, (5.18), (4.35), Lemma 1, (4.29), (4.30), (5.35), Theorem 2(i) and (4.19) give

$$
\begin{aligned}
\sqrt{n} M_n(\beta_n) &= -\sqrt{n} \int_A^B \big[\widehat{F}_{n,\beta_n}(x) - F_{n,\beta_n}(x)\big]\, d\psi(x) \\
&\quad - \sqrt{n} \int_A^B \big[F_{n,\beta_n}(x) - F_{n,\beta}(x)\big]\, d\psi(x) \\
&= -\sqrt{n} \int_A^B \big[\widehat{F}_{n,\beta_n}(x) - F_{n,\beta_n}(x)\big]\, d\psi(x) \\
&\quad - \sqrt{n} \int_A^B (I - \widetilde{K}^M_{n,\beta})^{-1} \\
&\qquad \times \left(\frac{A_{n,\beta_n,\beta} + B^{(2)}_{n,\beta_n,\beta} + B^{(3)}_{n,\beta_n,\beta}}{1 - C_{n,\beta}}\right)(x)\, d\psi(x) \\
&= -\sqrt{n} \int_A^B \big[\widehat{F}_{n,\beta_n}(x) - F_{n,\beta_n}(x)\big]\, d\psi(x) \\
&\quad - \sqrt{n} \int_A^B (I - \widetilde{K}^M_{n,\beta})^{-1} \left(\frac{A_{n,\beta_n,\beta}}{1 - C_{n,\beta}}\right)(x)\, d\psi(x) + o_{\text{a.s.}}(1) \\
&= -\sqrt{n} \int_A^B \big[\widehat{F}_{n,\beta_n}(x) - F_{n,\beta_n}(x)\big]\, d\psi(x) \\
&\quad - \sqrt{n}(\beta_n - \beta) A_n + o_{\text{a.s.}}(1),
\end{aligned}
\tag{5.39}
$$

where

$$
A_n = (\beta_n - \beta)^{-1} \int_A^B (I - \widetilde{K}^M_{n,\beta})^{-1} \left(\frac{A_{n,\beta_n,\beta}}{1 - C_{n,\beta}}\right)(x)\, d\psi(x). \tag{5.40}
$$

If we let $\eta_n = M_n(\beta_n)/A_n$, then from Theorem 2(i) and (5.32) we have

$$
A_n \stackrel{\text{a.s.}}{\to} A_\beta \neq 0 \qquad \text{as } n \to \infty. \tag{5.41}
$$

In turn, (5.20) implies $n^\lambda |\eta_n| \stackrel{\text{a.s.}}{\to} 0$ as $n \to \infty$, where $0 < \lambda < \frac{1}{2}$. Hence, from (5.38)–(5.41) we have

$$
\begin{aligned}
\sqrt{n}(\beta_n - \beta + \eta_n) &= -\sqrt{n} A_n^{-1} \int_A^B \big[\widehat{F}_{n,\beta_n}(x) - F_{n,\beta_n}(x)\big]\, d\psi(x) + o_{\text{a.s.}}(1) \\
&\stackrel{D}{\to} -A_\beta^{-1} \int_A^B \mathbb{G}_\beta\, d\psi \stackrel{D}{=} N(0, \sigma^2) \qquad \text{as } n \to \infty. \qquad \square
\end{aligned}
$$

APPENDIX.

PROOF OF THEOREM 1(i). If we denote $Z_n(s,t) = \sqrt{n}[W_n(F^{-1}(s), t) - EW_n(F^{-1}(s), t)]$, where $s, t \in [0, 1]$, then it suffices to show that $Z_n(s,t)$ weakly converges to a centered Gaussian process. Following the notation in Pollard (1990), we have $\xi_i = (F(Y_i), t_i)$, $i = 1, \dots, n$, as the independent random vectors and $f_{ni}(\omega, s, t) = n^{-1/2} I\{F(Y_i(\omega)) \le s, t_i \le t\}$ is an indicator function with envelope $|f_{ni}(\omega, \cdot)| \le F_{ni}(\omega) = n^{-1/2}$. Thus the triangular array of processes $\{f_{ni}(\omega, s, t)\}$ is manageable [see examples on VC index in Chapter 2.6 of van der Vaart and Wellner (1996)]. Noting that $Z_n(\omega, s, t) = \sum_{i=1}^n [f_{ni}(\omega, s, t) - Ef_{ni}(\cdot, s, t)]$, the proof follows from straightforward verification of the sufficient conditions of Theorem 10.6 in Pollard (1990). $\square$

Computation of $\widehat{F}_{n,\theta}$ and properties of $M_n(\theta)$. For any θ, let

$$V_i(\theta) = V_i - t_i\theta, \qquad i = 1, \dots, n. \tag{A.1}$$

Then one can compute $\widehat{F}_{n,\theta}$ as in Mykland and Ren (1996), treating $(V_i(\theta), \delta_i)$, $1 \le i \le n$, like their doubly censored sample, because the integral equation (2.2) in Mykland and Ren (1996) is exactly the same as (2.19) in Section 2.

Furthermore, from their paper we know that for any θ, $\widehat{F}_{n,\theta}$ is given by

$$\widehat{F}_{n,\theta}(x) = \sum_{i=1}^n p_{ni}(\theta) I\{V_i(\theta) \le x\}, \tag{A.2}$$

where $0 \le p_{ni}(\theta) \le 1$, $1 \le i \le n$ and $0 < \sum_{i=1}^n p_{ni}(\theta) \le 1$, and it is easy to see that $p_{n1}(\theta), \dots, p_{nn}(\theta)$ are determined by the ranks of $V_i(\theta)$ among $V_1(\theta), \dots, V_n(\theta)$. This means that for any $\underset{\sim}{\theta} < \widetilde{\theta}$, if the ranks of $V_i(\underset{\sim}{\theta})$ and $V_i(\widetilde{\theta})$ are exactly the same for every i, then we should have $p_{ni}(\underset{\sim}{\theta}) = p_{ni}(\widetilde{\theta})$ for every i.

Now, without loss of generality, assume $V_1 \le V_2 \le \cdots \le V_n$. Let

$$\Gamma_n = \left\{ \frac{V_i - V_j}{t_i - t_j} \,\middle|\, t_i \ne t_j,\ i \ne j,\ 1 \le i, j \le n \right\} \tag{A.3}$$

and let $\theta_1, \dots, \theta_N$ be all distinct points of Γ_n with $-\infty = \theta_0 < \theta_1 < \cdots < \theta_N < \theta_{N+1} = \infty$. It is straightforward to show that for any $0 \le k \le N$ and any $\theta_k < \underset{\sim}{\theta} < \widetilde{\theta} < \theta_{k+1}$, we have {rank of $V_i(\underset{\sim}{\theta})$} = {rank of $V_i(\widetilde{\theta})$}, $i = 1, \dots, n$. Thus for each i, $p_{ni}(\theta)$ is a constant on every interval (θ_k, θ_{k+1}), $0 \le k \le N$. Since by (2.20), (A1) and (A.2),

$$M_n(\theta) = \left[1 - \sum_{i=1}^n p_{ni}(\theta)\right] \psi(B) + \sum_{i=1}^n p_{ni}(\theta) \psi(V_i - t_i\theta), \tag{A.4}$$

then for each $0 \le k \le N$ and any $\theta \in (\theta_k, \theta_{k+1})$, we have

$$M_n'(\theta) = -\sum_{i=1}^n p_{ni}(\theta) t_i \psi'(V_i - t_i\theta) \le 0.$$

Hence, $M_n(\theta)$ is continuous and nonincreasing in θ on every interval (θ_k, θ_{k+1}).

Existence of $F_{n,\theta}$ and F_θ. Because of (3.1), (3.2) and Proposition 1(i), the arguments for showing the existence of $F_{n,\theta}$ and F_θ are similar. Thus, we consider only the case of F_θ.

For any θ, clearly all $Q_\theta^{(j)}(x)$ are nonnegative, continuous and nondecreasing in x with $Q_\theta^{(j)}(\infty) = \alpha_j < 1$, $j = 1, 2, 3$, and $Q_\theta^{(0)}(\infty) = \alpha_1 + \alpha_2 + \alpha_3 = 1$. Let N be any large positive integer and let $n_j = \alpha_j N$, $j = 1, 2, 3$. Then $n_1 + n_2 + n_3 = N$. For $\alpha_1^{-1} Q_\theta^{(1)}(x)$ we can choose points $v_1, \dots, v_{n_1}$ such that the uniform distance between $G_{n_1}(x) = n_1^{-1} \sum_{i=1}^{n_1} I\{v_i \le x\}$ and $\alpha_1^{-1} Q_\theta^{(1)}(x)$ is no larger than n_1^{-1}. Similarly we can choose points $v_{n_1+1}, \dots, v_{n_1+n_2}$ and $v_{n_1+n_2+1}, \dots, v_{n_1+n_2+n_3}$ for $G_{n_2}(x)$ and $G_{n_3}(x)$, respectively. Let $\delta_i = 1$, $1 \le i \le n_1$; 2, $n_1 + 1 \le i \le n_1 + n_2$; 3, $n_1 + n_2 + 1 \le i \le N$. Then

$$Q_N^{(1)}(x) = N^{-1} \sum_{i=1}^{N} I\{v_i \le x, \delta_i = 1\} = \alpha_1 n_1^{-1} \sum_{i=1}^{n_1} I\{v_i \le x\} = \alpha_1 G_{n_1}(x),$$

$$Q_N^{(2)}(x) = N^{-1} \sum_{i=1}^{N} I\{v_i \le x, \delta_i = 2\} = \alpha_2 G_{n_2}(x),$$

$$Q_N^{(3)}(x) = N^{-1} \sum_{i=1}^{N} I\{v_i \le x, \delta_i = 3\} = \alpha_3 G_{n_3}(x),$$

$$Q_N^{(0)}(x) = Q_N^{(1)}(x) + Q_N^{(2)}(x) + Q_N^{(3)}(x) = N^{-1} \sum_{i=1}^{N} I\{v_i \le x\}.$$

Let H_N be the solution of

$$\begin{aligned} H_N(x) = Q_N^{(0)}(x) &- \int_{u \le x} \frac{1 - H_N(x)}{1 - H_N(u)}\, dQ_N^{(2)}(u) \\ &+ \int_{x < u} \frac{H_N(x)}{H_N(u)}\, dQ_N^{(3)}(u). \end{aligned} \tag{A.5}$$

Then from Mykland and Ren (1996), we know that $H_N(x)$ is a $[0, 1]$-valued nondecreasing function. From Helly's theorem, H_N has a convergent subsequence H_{N_k} such that for any x, $\lim_{k\to\infty} H_{N_k}(x) = H_0(x)$. Since for $j = 1, 2, 3$, we have $Q_N^{(j)} = \alpha_j G_{n_j}$ converges to $Q_\theta^{(j)}(x)$ uniformly, we have that for each fixed x, the limit of equation (A.5) is

$$H_0(x) = Q_\theta^{(0)}(x) - \int_{u \le x} \frac{1 - H_0(x)}{1 - H_0(u)}\, dQ_\theta^{(2)}(u) + \int_{x < u} \frac{H_0(x)}{H_0(u)}\, dQ_\theta^{(3)}(u),$$

which shows the existence of $F_\theta = H_0$, a $[0, 1]$-valued nondecreasing function.

Acknowledgments. I thank the Associate Editor and the referee for their valuable comments and suggestions on the original version of this manuscript which helped strengthen the results and shorten some of the proofs.

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