The Annals of Statistics
2003, Vol. 31, No. 2, 613–641

A NECESSARY AND SUFFICIENT CONDITION FOR ASYMPTOTIC INDEPENDENCE OF DISCRETE FOURIER TRANSFORMS UNDER SHORT- AND LONG-RANGE DEPENDENCE

BY S. N. LAHIRI

Iowa State University

Let $\{X_t\}$ be a stationary time series and let $d_T(\lambda)$ denote the discrete Fourier transform (DFT) of $\{X_0, \ldots, X_{T-1}\}$ with a data taper. The main results of this paper provide a characterization of asymptotic independence of the DFTs in terms of the distance between their arguments under both short- and long-range dependence of the process $\{X_t\}$. Further, asymptotic joint distributions of the DFTs $d_T(\lambda_{1T})$ and $d_T(\lambda_{2T})$ are also established for the cases $T(\lambda_{1T} - \lambda_{2T}) = O(1)$ as $T \to \infty$ (asymptotically close ordinates) and $|T(\lambda_{1_T} - \lambda_{2_T})| \to \infty$ as $T \to \infty$ (asymptotically distant ordinates). Some implications of the main results on the estimation of the index of dependence are also discussed.

1. Introduction. Suppose that $\{X_t\}$ is a sequence of stationary random variables (r.v.s) with mean μ and spectral density function

$$f(\lambda) = |\lambda|^{-2d} L(\lambda), \qquad \lambda \in \Pi, \tag{1.1}$$

$d \in (-1/2, 1/2)$, where $\Pi = (-\pi, \pi)$ and where $L(\cdot)$ is an even function that is bounded on every compact subinterval of $(0, \pi]$ and is slowly varying at zero, that is,

$$\lim_{\lambda \to 0} L(\lambda a)/L(\lambda) = 1 \qquad \text{for all } a \in (0, \infty). \tag{1.2}$$

We classify the process $\{X_t\}$ as short-range dependent or long-range dependent depending on the value of the parameter d and the behavior of the slowly varying function $L(\cdot)$ near the origin. When $d = 0$ and the function $L(\cdot)$ is bounded with $L(0) \neq 0$, then the process $\{X_t\}$ will be called short-range dependent. The process $\{X_t\}$ will be called long-range dependent if it is *not* short-range dependent. Thus, under this definition, the sum of autocovariances of the process $\{X_t\}$ converges to a *positive* real number under short-range dependence (SRD), while under long-range dependence (LRD), the sum of autocovariances either diverges or converges to *zero*, provided some standard regularity conditions on $L(\cdot)$ hold. Note that two popular models for LRD data, namely the fractional Gaussian process of Mandelbrot and Van Ness (1968) and the fractional autoregressive integrated

Received May 1999; revised January 2002.
Supported in part by NSF Grants DMS-95-05124 and DMS-00-72571.
AMS 2000 subject classifications. Primary 62M10; secondary 62M15, 62E20.
Key words and phrases. Asymptotic independence, discrete Fourier transform, long-range dependence, stationarity.

moving average (FARIMA) models of Adenstedt (1974), Granger and Joyeux (1980) and Hosking (1981), are special cases of the present framework.

Next, let $h:[0,1] \to \mathbb{R}$ be a function of bounded variation. Then the discrete Fourier transform (DFT) of $X_0, \dots, X_{T-1}$ under the "data-taper" function $h(\cdot)$ is defined as

$$d_T(\lambda) = \sum_{t=0}^{T-1} h(t/T) X_t \exp(-it\lambda), \qquad \lambda \in \Pi, \tag{1.3}$$

where $i = \sqrt{-1}$. A data taper is typically used for handling missing data, for reducing leakage [see Brillinger (1981); Zurbenko (1986)] and in band-spectrum regression [see Robinson (1986)]. In this paper, we establish the asymptotic distribution of the DFTs at the discrete ordinates $\lambda_j \equiv 2\pi j/T$ and obtain a *characterization* of asymptotic independence of the DFTs $d_T(\lambda_{j_T})$ and $d_T(\lambda_{k_T})$, given by (1.3), in terms of the distance between the sequences $\{\lambda_{j_T}\}$ and $\{\lambda_{k_T}\}$, under both SRD and LRD.

To put the results in historical perspective, define the nontapered version of the DFT,

$$d_{1T}(\lambda) = \sum_{t=0}^{T-1} X_t e^{-it\lambda}, \qquad \lambda \in \Pi, \tag{1.4}$$

and for simplicity of discussion in this section, suppose that $\mu = 0$. Under SRD, asymptotic behavior of the nontapered DFTs $d_{1T}(\lambda_j)$ and their tapered version $d_T(\lambda_j)$ has been investigated by many authors. See, for example, Kawata (1966, 1969), Fuller (1976) and Brockwell and Davis (1991) for the nontapered case and Hannan (1970), Hannan and Thomson (1971), Brillinger (1981), Thomson (1982) and Zurbenko (1986) for the tapered case. When the time series $\{X_t\}$ is SRD, a classical result on the DFTs $d_{1T}(\cdot)$ states [Brockwell and Davis (1991); Fuller (1976)] that under some regularity conditions, if $\lambda_{j_T} \to \lambda$, $\lambda_{k_T} \to w$, as $T \to \infty$ with $\lambda, w \in [0, \pi]$ and $j_T \neq k_T$, then $T^{-1/2} d_{1T}(\lambda_{j_T})$ and $T^{-1/2} d_{1T}(\lambda_{k_T})$ are asymptotically independent (complex Gaussian) random variables. [Here and in the following, two sequences of random vectors $\{U_n\}$ in $\mathbb{R}^p$ and $\{V_n\}$ in $\mathbb{R}^q$, defined on a common probability space, are called *asymptotically independent* if there exist constants $a_n > 0$, $b_n > 0$ and vectors $c_n \in \mathbb{R}^p$ and $d_n \in \mathbb{R}^q$ such that the random vector $(a_n[U_n - c_n]', b_n[V_n - d_n]')'$ converges in distribution to some random vector $(U', V')'$, and U and V are independent.] Thus, the DFTs $d_{1T}(\lambda_{j_T})$ and $d_{1T}(\lambda_{k_T})$ are asymptotically independent under SRD even when the discrete ordinates λ_{j_T} and λ_{k_T} tend to the same limit $w = \lambda$, as long as λ_{j_T} and λ_{k_T} are *distinct*. For DFTs with a *general* taper function, the asymptotic joint distribution of the DFTs $d_T(\lambda_{jT})$ and $d_T(\lambda_{kT})$ appears to be known only for a smaller class of sequences $\{\lambda_{j_T}\}$ and $\{\lambda_{k_T}\}$. For example, the asymptotic distribution of the DFTs at ordinates λ_{j_T} and λ_{k_T} tending to *a given frequency* is known [Brillinger (1981);

Hannan (1970)]. However, it is not clear how far apart the ordinates λ_{j_T} and λ_{k_T} have to be for asymptotic independence of the corresponding DFTs. The main results of this paper present a systematic study of asymptotic independence of the tapered DFTs and provide a complete answer to this problem.

For LRD processes, the asymptotic behavior of the DFTs has received a good amount of attention in recent years. Among other reasons, this may be attributed to the important role played by the DFTs in the semiparametric estimation of the long memory parameter d. See, for example, Geweke and Porter-Hudak (1983), Robinson (1995), Hurvich, Deo and Brodsky (1998) and the references therein. However, under LRD, asymptotic independence of the DFTs is still *not* very clearly understood. For processes $\{X_t\}$ having the spectral density (1.1) with $d \in (0, 1/2)$ and with a bounded $L(\cdot)$, Yajima (1989) and Pham and Guégan (1994) established asymptotic normality and asymptotic independence of the DFTs at a finite set of ordinates that are asymptotically distant. In an important work, Robinson (1995) proved that for a stationary process $\{X_t\}$ having spectral density f of the form $f(\lambda) \sim C|\lambda|^{-2d}$ as $\lambda \to 0$, with $C > 0$, $|d| < 1/2$,

$$\operatorname{Cov}\left(d_{1T}(\lambda_{j_T})/\sqrt{Tf(\lambda_{j_T})}, d_{1T}(\lambda_{k_T})/\sqrt{Tf(\lambda_{k_T})}\right) = O\left(\frac{\log j_T}{k_T}\right) \quad \text{as } T \to \infty, \tag{1.5}$$

for any sequence of positive integers $\{j_T\}$, $\{k_T\}$ satisfying $j_T > k_T$, $j_T/T \to 0$ as $T \to \infty$. Consequently, if $\{X_t\}$ is further assumed to be *Gaussian*, then the *nontapered* DFTs $d_{1T}(\lambda_{j_T})$ and $d_{1T}(\lambda_{k_T})$ are asymptotically independent for any such frequencies λ_{j_T} and λ_{k_T} satisfying $(\log j_T)/k_T \to 0$ as $T \to \infty$. Thus, the results of Robinson (1995) imply that asymptotic independence of the nontapered DFTs $d_{1T}(\lambda_{j_T})$ and $d_{1T}(\lambda_{k_T})$ may hold under LRD, even when λ_{j_T} and λ_{k_T} tend to the *same* number (viz., to zero). On the other hand, results of Hurvich and Beltrao (1993) and Robinson (1995) [also see Künsch (1986)] show that for a stationary process $\{X_t\}$ with $f(\lambda) \sim C|\lambda|^{-2d}$ as $\lambda \to 0$, with $C > 0$, $0 < |d| < 1/2$,

$$\lim_{T\to\infty} \operatorname{Cov}\left(d_{1T}(\lambda_{j_T})/\sqrt{Tf(\lambda_{j_T})}, d_{1T}(\lambda_{k_T})/\sqrt{Tf(\lambda_{k_T})}\right) \neq 0,$$

provided $j_T \equiv j$ and $k_T \equiv k$ for all T, for some *fixed* integers j, k. Thus, for the *very low* frequencies $\lambda_j = 2\pi j/T$ and $\lambda_k = 2\pi k/T$, again both tending to the *same* limit (viz., zero), the DFTs $d_{1T}(\lambda_{j_T})$ and $d_{1T}(\lambda_{k_T})$ have, asymptotically, a nonzero correlation and thus cannot be asymptotically independent. This naturally leads to the following question:

(1.6) For which sequences $\{\lambda_{j_T}\}$ and $\{\lambda_{k_T}\}$ are the DFTs $\{d_{1T}(\lambda_{j_T})\}$ and $\{d_{1T}(\lambda_{k_T})\}$ asymptotically independent?

Apart from being an important problem in its own right, it has significant statistical implications. A number of inference procedures for time series data have been developed in the literature to exploit the "approximate independence" property of the DFTs. For example, the validity of Fisher's test for hidden periodicities and Tukey's approximate simultaneous confidence intervals for the spectral density [cf. Brockwell and Davis (1991)] relies on the approximate independence of the nontapered DFTs under SRD, as does the validity of the frequency domain bootstrap [see Hurvich and Zeger (1987); Franke and Härdle (1992); Dahlhaus and Janas (1996)]. Extensions of these methods for LRD time series data depend very much upon the asymptotic behavior of the DFTs for LRD processes. In this paper, we address the problem posed in (1.6) for the more general class of SRD and LRD processes that satisfy (1.1) and we obtain a *necessary and sufficient* condition (on the sequences $\{\lambda_{j_T}\}$ and $\{\lambda_{k_T}\}$) for asymptotic independence of the *tapered* DFTs $d_T(\lambda_{j_T})$ and $d_T(\lambda_{k_T})$.

The main results of this paper show that for a stationary process $\{X_t\}$ that has a spectral density of the form (1.1), the tapered DFTs $d_T(\lambda_{j_T})$ and $d_T(\lambda_{k_T})$ are asymptotically independent whenever

$$|T(\lambda_{j_T} - \lambda_{k_T})| \to \infty \qquad \text{as } T \to \infty. \tag{1.7}$$

Thus, (1.7) is a *sufficient* condition for the asymptotic independence of $d_T(\lambda_{j_T})$ and $d_T(\lambda_{k_T})$. The necessity of this condition depends on the specific taper function $h(\cdot)$ as well as on the value of the dependence parameter d. For $d = 0$, if *all* Fourier coefficients of the function $h^2(\cdot)$ are nonzero, then (1.7) is also a necessary condition for the asymptotic independence of $d_T(\lambda_{j_T})$ and $d_T(\lambda_{k_T})$, and for $d \neq 0$, (1.7) becomes necessary if, in addition, $\int_{-\infty}^{\infty} \hat{h}(y - 2\pi\ell)\hat{h}(-[y + 2\pi m])|y|^{-2d}\,dy \neq 0$ for all $\ell, m \in \mathbb{Z}$, where $\hat{h}(y) = \int h(x)\exp(\sqrt{-1}xy)\,dx$, $y \in \mathbb{R}$. However, there are some taper functions for which many of these constants are zero (e.g., the cosine-bell taper; see Section 2.3). For such taper functions, a characterization of the asymptotic independence of the DFTs $d_T(\lambda_{j_T})$ and $d_T(\lambda_{k_T})$ depends on the form of their asymptotic distribution. In Section 2, we establish asymptotic joint normality of centered and scaled DFTs $d_T(\lambda_{j_T})$ and $d_T(\lambda_{k_T})$, and obtain a complete description of the covariance structure of their limit distribution. This enables us to formulate a necessary and sufficient condition for asymptotic independence of $d_T(\lambda_{j_T})$ and $d_T(\lambda_{k_T})$ for a *given* taper h. We illustrate our results with a few commonly used taper functions in Section 2.3, including the nontapered case where $h(x) \equiv 1$, $x \in [0, 1]$.

In Section 2, we also establish joint asymptotic normality of the DFTs for a *finite* set of ordinates $\lambda_{j_{1T}}, \ldots, \lambda_{j_{kT}}$, $1 \leq k < \infty$, and obtain expressions for the limiting covariance matrix. Specializing this to the case where $L(\cdot)$ is bounded, we easily obtain the known results on the DFTs under SRD [see Brillinger (1981)] with $d = 0$, and the results of Yajima (1989) and Pham and Guégan (1994) for the particular case of LRD with $0 < d < 1/2$. Further, for the low ordinates of the form $2\pi j/T$ with *fixed* $j \in \mathbb{Z}$ (not depending on T), this also yields results similar to those of Künsch (1986), Hurvich and Beltrao (1993) and Robinson (1995) on the asymptotic distribution of the DFTs under a somewhat more general setup, by allowing data tapering and a (possibly unbounded) slowly varying function $L(\cdot)$ in the spectral density. Proofs of our main results require a careful analysis of the integrals that represent the cumulants of the DFTs in conjunction with some elementary arguments from Fourier analysis and some inequalities developed by Dahlhaus (1983, 1985).

One of the implications of the characterization result of this paper is that quite *different* conclusions regarding asymptotic independence of the DFTs can be obtained under two different data tapers. This difference is noticeable when the nontapered case is compared with the tapered case. Note that by Robinson's (1995) result [cf. (1.5)], the correlation between the *nontapered* DFTs $d_{1T}(\lambda_{j_T})$ and $d_{1T}(\lambda_{k_T})$ is of $O(\log j_T/k_T)$, which goes to *zero* as $T \to \infty$ for any sequences $\{j_T\}$ and $\{k_T\}$ satisfying $k_T - j_T = O(1)$ and $j_T^{-1} + T^{-1} j_T = o(1)$ as $T \to \infty$. Hence, in the nontapered case, the pair of *consecutive* DFTs $d_{1T}(\lambda_{j_T})$ and $d_{1T}(\lambda_{j_T-1})$ with $k_T = j_T - 1$ are asymptotically independent whenever $j_T^{-1} + T^{-1} j_T = o(1)$ as $T \to \infty$. However, this is no longer true in the tapered case if $\int h^2(x)\exp(\pm i 2\pi x)\,dx \neq 0$. For example, the DFTs at such ordinates are asymptotically dependent if the cosine-bell taper or Kolmogorov's tapers are used (see Section 2.3). As a consequence, many of the commonly used time series methods and results based on asymptotic independence of the nontapered DFTs may not be valid in the tapered case. In Section 2.4, we briefly discuss some simple modifications to deal with the lack of independence of the DFTs in the tapered case based on a recent work of Velasco (1999b). The characterization results of this paper may be helpful in formulating similar modifications in other problems.

The rest of this paper is organized as follows. In Section 2, we state the assumptions and the main results. In Section 2, we also consider some specific examples of the taper function and discuss some of the implications of the characterization when the data are tapered. Proofs of the main results are presented in Section 3.

2. Assumptions and main results.

2.1. *Assumptions.* Assume that $\{X_t\}$ is a (strictly) stationary process with spectral density $f(\cdot)$ given by (1.1). Then there exists a zero mean and unit variance process $\{\varepsilon_t\}$ of uncorrelated r.v.s such that $\{X_t\}$ has the moving average representation [see Doob (1953), Chapter 10]

$$X_t = \mu + \sum_{j=-\infty}^{\infty} b_j \varepsilon_{t-j}, \qquad t \in \mathbb{Z}, \tag{2.1}$$

where $\mathbb{Z} = \{0, \pm 1, \dots\}$, $\{b_t\}$ is a sequence of constants satisfying $\sum_{t\in\mathbb{Z}} b_t^2 < \infty$ and $f(\lambda) = |b(\lambda)|^2/(2\pi)$, $\lambda \in \Pi$ with $b(\lambda) = \sum_{j\in\mathbb{Z}} b_j e^{i\lambda j}$, $\lambda \in \Pi$. We suppose that $\{\varepsilon_t\}$ is (strictly) stationary. Define the α-mixing coefficient of $\{\varepsilon_t\}$ by

$$\alpha(n) = \sup\bigl\{|P(A\cap B) - P(A)P(B)| : A \in \sigma\langle\{\varepsilon_t : t \le 0\}\rangle, B \in \sigma\langle\{\varepsilon_t : t \ge n\}\rangle\bigr\},$$

$n \in \mathbb{N} \equiv \{1, 2, \dots\}$. Let $\mathbb{1}(A)$ and $\mathbb{1}_A$ both denote the indicator function of a set A. Let $c_n = E\varepsilon_0^2 \mathbb{1}(|\varepsilon_0| > n)$, $n \in \mathbb{N}$. Denote the Fourier transform of a function $g(\cdot)$ by $\hat{g}(y) = \int_{\mathbb{R}} g(x) \exp(ixy)\, dx$, $y \in \mathbb{R}$. Also, for any two real numbers x, y, let $x \wedge y \equiv \min\{x, y\}$ and let $\lceil x \rceil$ denote the largest integer not exceeding x.

We shall use the following regularity conditions to prove the results.

ASSUMPTIONS

(A.1) There exists a sequence $\{l_n\} \subset \mathbb{N}$ such that $c_n l_n + \sum_{k=l_n}^{\infty} [n\alpha(k)^{1/2} + n^2 \alpha(k)] \to 0$ as $n \to \infty$. Further, $\sum_{n=1}^{\infty} n^r \alpha(n) < \infty$ for all $r \in \mathbb{N}$.

(A.2) $h(\cdot)$ is of bounded variation and $\int_{-\infty}^{\infty} |\hat{h}(y)|^2 |y|^{-2d}\, dy < \infty$ for all $d \in (-\frac{1}{2}, 0)$.

(A.3) $L(\cdot)$ is even, strictly positive and continuous on $(0, \pi]$, and it is slowly varying at zero in the sense of (1.2).

Assumption (A.1) specifies a set of weak dependence and moment conditions on $\{\varepsilon_t\}$ that are used for proving asymptotic normality of the DFTs. If $\{\varepsilon_t\}$ is m-dependent with $E\varepsilon_0^2 < \infty$, then (A.1) holds. In this case, $\alpha(k) = 0$ for $k > m$ and hence (A.1) holds if we set $l_n \equiv m + 1$ for all $n \in \mathbb{N}$. Similarly, if $\alpha(n) \le C_1 \exp(-C_2 n)$, $n \in \mathbb{N}$, for some $C_1, C_2 \in (0, \infty)$, then (A.1) holds, provided that $E|\varepsilon_0|^2 \log(1 + |\varepsilon_0|) < \infty$. This follows by choosing $l_n = [\log n] c_{1n}$, where $c_{1n}^{-2} \equiv [\log n] c_n \to 0$ as $n \to \infty$. In general, (A.1) holds if $E|\varepsilon_0|^{2+\delta} < \infty$ for some $\delta \in (0, \infty)$ and $\alpha(n) = O(n^{-a_n})$ for some $a_n^{-1} = o(1)$, as $n \to \infty$.

As for Assumption (A.2), the bounded variation of $h(\cdot)$ is a very standard condition [Brillinger (1981); Dahlhaus (1983, 1985); Yajima (1989)]. The other part of (A.2) turns out to be a *necessary* condition for the validity of the results. Indeed, the limiting variance of the DFTs near the origin involves the integral $\int |\hat{h}(y)|^2 |y|^{-2d}\, dy$ (see Lemma 3.4, Section 3) and we need to assume its finiteness for $d \in (-\frac{1}{2}, 0)$. When $0 \le d < 1/2$, by Parseval's identity,

$$
\begin{aligned}
\int_{-\infty}^{\infty} |\hat{h}(y)|^2 |y|^{-2d}\, dy &\le \int_{-\infty}^{\infty} |\hat{h}(y)|^2\, dy + \sup_{|y|<1} |\hat{h}(y)|^2 \int_{-1}^{1} |y|^{-2d}\, dy \\
&\le 2\pi \int_0^1 |h(w)|^2\, dw + \frac{2}{1-2d} \sup_{|w|<1} |h(w)|^2 < \infty.
\end{aligned}
\tag{2.2}
$$

In the special case of no data tapering, that is, for $h(w) \equiv 1$, it is easy to see that $|\hat{h}(y)|^2 = O(|y|^{-2})$ as $|y| \to \infty$ and hence (A.2) holds. By a similar argument, it follows that Assumption (A.2) is satisfied by the other data tapers considered in Section 2.3.

In Assumption (A.3), we assume strict positivity of $L(\cdot)$ only for simplicity of exposition. Since the normalizing constants [see (2.3) below] for the DFTs are defined using $L(\cdot)$ at *all* frequencies, we need to assume that it is positive everywhere to avoid division by zero. For frequencies near the origin, the rate of convergence of the DFTs depends on the slowly varying function $L(\cdot)$ and it must be included in the normalizing constant. However, at frequencies away from the origin, $L(\cdot)$ can be dropped from the normalizing constant.

2.2. *Main results.* First we establish the limit distributions of the DFTs and then obtain a characterization of the asymptotic independence by considering independence of the joint limit distributions. To that end, let

$$S_T(\lambda) = \sum_{t=0}^{T-1} h(t/T) X_t \sin(t\lambda),$$

$$C_T(\lambda) = \sum_{t=0}^{T-1} h(t/T) X_t \cos(t\lambda), \qquad w \in \Pi,$$

denote the sine and the cosine transforms of the data. Then $d_T(\lambda) = C_T(\lambda) - iS_T(\lambda)$. We describe the asymptotic behavior of the $d_T(\lambda)$ in terms of the real valued r.v.s $S_T(\lambda)$ and $C_T(\lambda)$. Note that because of the odd and even properties of the sine and the cosine functions, it is enough to study the properties of $S_T(\lambda)$ and $C_T(\lambda)$ only for *nonnegative* values of λ. To ensure that proper limit distributions for $S_T(\lambda_{jT})$ and $C_T(\lambda_{jT})$ exist at the discrete ordinates $\lambda_{j_T} = 2\pi j_T/T$, we need to further restrict attention to a suitable class of $\{\lambda_{j_T}\}$ sequences. Let $\Lambda_T = \{\frac{2\pi j}{T} : 1 \le j < T/2,\ j \in \mathbb{Z}\}$. For each $\lambda \in (0, \pi]$, we define the admissible class of sequences of discrete ordinates converging to λ as $\mathcal{C}_\lambda = \{\{\lambda_T\} : \lambda_T \in \Lambda_T$ for all T and $\lambda_T \to \lambda$ as $T \to \infty\}$. For $\lambda = 0$, some extra care is needed, since the limit distribution does *not* exist for every sequence $\{\lambda_T\}$ that converges to $\lambda = 0$. For $\lambda = 0$, we define the class of admissible sequences as $\mathcal{C}_0 = \mathcal{C}_{01} \cup \mathcal{C}_{02} \cup \mathcal{C}_{03}$ with $\mathcal{C}_{01} = \{\{\lambda_T\} : \lambda_T \in \Lambda_T$ for all T and $\lambda_T + |T\lambda_T|^{-1} \to 0$ as $T \to \infty\}$, $\mathcal{C}_{02} = \{\{\lambda_T\} : \lambda_T \in \Lambda_T$ for all T and $T\lambda_T \to 2\pi\ell$ for some $\ell \in \mathbb{Z}$ as $T \to \infty\}$ and $\mathcal{C}_{03} = \{\{\lambda_T\} : \lambda_T \equiv 0$ for all $T\}$. Then the class of *admissible* sequences $\{\lambda_T\}$ is given by

$$\mathcal{C} = \bigcup_{\lambda \in [0,\pi]} \mathcal{C}_\lambda.$$

Thus, $\mathcal{C}$ contains *all* sequences $\{\lambda_T\}$ of discrete ordinates that converge to a limit in the interval $(0, \pi)$. However, for sequences converging to $\lambda = 0$, it does not allow sequences $\{\lambda_T\}$ of discrete ordinates that alternate between the sets Λ_T and $\{0\}$. Note that if λ_T belongs to both Λ_T and $\{0\}$ *infinitely often*, then the sine transform $S_T(\lambda_T)$ has a nondegenerate normal limit through a subsequence, whereas $S_T(0)$ is zero with probability 1, and hence $S_T(\lambda_T)$ does not have a limit for such sequences. Similarly, for sequences $\{\lambda_T\}$ with $|T\lambda_T| = O(1)$ as $T \to \infty$, limit distributions for the sine and the cosine transforms do not exist if $T\lambda_T$ does not converge to a limit. This is why we required the existence of a finite limit for $T\lambda_T$ in the definition of the class $\mathcal{C}_{02}$. Indeed, in this case, the limit ℓ of $T\lambda_T$ shows up in the asymptotic variance of the transforms $S_T(\lambda_T)$ and $C_T(\lambda_T)$. It should be pointed out that the admissibility condition on a sequence $\{\lambda_T\}$ only guarantees a limit distribution of $\{(S_T(\lambda_T), C_T(\lambda_T))'\}$ for the *individual* sequence $\{\lambda_T\}$; for the existence of *joint* limit distributions of the transforms at a *pair* of sequences $\{\lambda_{1T}\}$ and $\{\lambda_{2T}\}$ in $\mathcal{C}_\lambda$ for $\lambda \in (0, \pi]$, some additional conditions are necessary [see (2.4) and (2.5) below].

Next define the scaling function $a_T(\cdot)$ by

$$a_T^2(\lambda) = \begin{cases} |\lambda|^{-2d} L(\lambda) T, & \text{if } \lambda \in \Pi \setminus \{0\}, \\ T^{1+2d} L(T^{-1}), & \text{if } \lambda = 0. \end{cases} \tag{2.3}$$

Then we have the following result on the joint limit distributions of the sine and the cosine transforms.

THEOREM 2.1. *Suppose that Assumptions* (A.1)–(A.3) *hold. Let* $\{\lambda_{1T}\}, \dots, \{\lambda_{kT}\} \in \mathcal{C}$ *be any* k $(k \geq 2)$ *sequences of ordinates such that for any* $1 \leq i < j \leq k$, *either*

$$|T(\lambda_{jT} - \lambda_{iT})| \to \infty \qquad \textit{as } T \to \infty \tag{2.4}$$

or

$$T(\lambda_{jT} - \lambda_{iT}) \to 2\pi\ell \qquad \textit{as } T \to \infty \tag{2.5}$$

for some nonzero integer ℓ, *not depending on* T. *Define the variables* $Y_{1jT} = [C_T(\lambda_{jT}) - EC_T(\lambda_{jT})]/a_T(\lambda_{jT})$ *and* $Y_{2jT} = [S_T(\lambda_{jT}) - ES_T(\lambda_{2jT})]/a_T(\lambda_{jT})$, $1 \leq j \leq k$. *Then the vector of* $2k$ *r.v.s* $Y_T \equiv (Y_{11T} \dots Y_{1kT}; Y_{21T} \dots Y_{2kT})'$ *converges in distribution to a multivariate normal distribution.*

Theorem 2.1 asserts asymptotic normality of the sine and cosine transforms of the data for finite sets of sequences of discrete ordinates under both SRD and LRD, and serves as the first step toward characterizing the asymptotic independence of the DFTs. Since the limiting joint distribution of the sine and the cosine transforms is Gaussian, asymptotic independence of these variables is determined by the covariance structure of the limit law. In Theorems 2.2 and 2.3 below, we consider the collection of sequences $\{\lambda_{jT}\}$ that satisfy (2.4) and (2.5), respectively. These results can be used to verify asymptotic independence not only of the DFTs, but, more generally, of any two finite *collections* of sine and cosine transforms.

To state Theorems 2.2 and 2.3, we need to introduce some notation. For any complex number z, let $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively, denote the real and the imaginary parts of z. Also, for a $k \times k$ matrix A, let $(A)_{ij}$ denote the (i, j)th element of A, $1 \leq i, j \leq k$, and let A' denote the transpose of A. Let $\xrightarrow{d}$ denote convergence in distribution of random vectors. Next define

$$\sigma_0(\ell; m) = c(\ell, m) \int_{-\infty}^{\infty} \hat{h}(y - 2\pi\ell)\hat{h}(-[y + 2\pi m])|y|^{-2d}\, dy, \tag{2.6}$$

$$\sigma_1(\ell) = \int_{-\infty}^{\infty} \hat{h}(y)\hat{h}(2\pi\ell - y)\, dy \tag{2.7}$$

for $\ell, m \in \mathbb{Z}$, where $c(m_1, m_2) = (2\pi)^{2d}|m_1 m_2|^d$ if $m_1, m_2 \in \mathbb{Z} \setminus \{0\}$, $c(0, m) = c(m, 0) = |2\pi m|^d$ if $m \in \mathbb{Z} \setminus \{0\}$ and $c(0, 0) = 1$.

The next result concerns sequences that satisfy (2.4).

THEOREM 2.2. *Suppose that Assumptions* (A.1)–(A.3) *hold. Let* $\{\lambda_{1T}\}, \ldots, \{\lambda_{kT}\} \in \mathcal{C}$, $2 \le k < \infty$, *be sequences of discrete ordinates such that* $\lambda_{jT} \to w_j \in [0, \pi]$ *and* $|T(\lambda_{jT} - \lambda_{iT})| \to \infty$, $1 \le i \ne j \le k$. *Define* $Y_{jT} = (Y_{1jT}, Y_{2jT})'$, $1 \le j \le k$.

(a) *The bivariate random vectors* $Y_{1T}, \ldots, Y_{kT}$ *are asymptotically independent*.

(b) *If* $T|\lambda_{jT}| \to \infty$ *as* $T \to \infty$ *for some* j, *then the components* Y_{1jT} *and* Y_{2jT} *of* Y_{jT} *are also asymptotically independent*, *and for* $r = 1, 2$, $Y_{rjT} \xrightarrow{d} N(0, 2^{-1}\sigma_1(0))$.

(c) *If for some* $1 \le j \le k$, $T|\lambda_{jT}| \to \ell$ *as* $T \to \infty$ *for an integer* $\ell \in \mathbb{Z}$, *then*

$$Y_{jT} \xrightarrow{d} N(0, \Xi_j),$$

where the elements of the 2×2 *matrix* Ξ_j *are given by* $(\Xi_j)_{11} = \mathrm{Re}[\sigma_0(\ell; \ell) + \sigma_0(\ell; -\ell)]/2$, $(\Xi_j)_{22} = \mathrm{Re}[\sigma_0(\ell; -\ell) - \sigma_0(\ell; \ell)]/2$ *and* $(\Xi_j)_{12} = -\mathrm{Im}[\sigma_0(\ell; \ell) + \sigma_0(\ell; -\ell)]/2$.

Theorem 2.2 shows that for any two sequences of $\{\lambda_{1T}\}$ and $\{\lambda_{2T}\}$ of *asymptotically distant* ordinates [i.e., satisfying $|T(\lambda_{1T} - \lambda_{2T})| \to \infty$ as $T \to \infty$], the DFTs $d_T(\lambda_{1T})$ and $d_T(\lambda_{2T})$ are asymptotically independent under *both* SRD and LRD. This extends the results of Yajima (1989), who proved asymptotic independence of the tapered DFTs at a finite number of *fixed* ordinates, assuming that $d \in (0, 1/2)$ and the function $L(\cdot)$ in (1.1) is *bounded*. Theorem 2.2 also extends the results of Pham and Guégan (1994) for time varying ordinates λ_{jT} by allowing the spectral density $f(\cdot)$ to have a more general form [cf. (1.1)]. For DFTs *without* data tapering, Theorem 2.2 supplements the results of Robinson (1995) by establishing asymptotic independence of DFTs for sequences of ordinates $\{2\pi j_T/T\}$ and $\{2\pi k_T/T\}$ for which the bound $O([\log j_T]/k_T)$ in (1.5) does not go to zero with T (e.g., consider $k_T \le \log T$ and $j_T \sim T^{\alpha}[\log T]^{\beta}$ as $T \to \infty$ for some $\alpha \in (0, 1)$ and $\beta \in \mathbb{R}$).

Theorem 2.2 also shows that for ordinates λ_{jT} converging to a *nonzero* frequency w_j, the corresponding sine and cosine transforms are also asymptotically independent for all values of the dependence parameter $d \in (-1/2, 1/2)$.

For λ_{jT}'s converging to zero, the independence property continues to hold for all $d \in (-1/2, 1/2)$, provided that $|T\lambda_{jT}| \to \infty$ as $T \to \infty$. Thus, under both SRD and LRD, a *sufficient* condition for asymptotic independence of *all* $2k$ *r.v.s* $\{Y_{11T}, \dots, Y_{1kT}; Y_{21T}, \dots, Y_{2kT}\}$ is that $|T\lambda_{jT}| \to \infty$ *and* $|T(\lambda_{jT} - \lambda_{iT})| \to \infty$ as $T \to \infty$ for all $1 \le i \ne j \le k$. In comparison, by Theorem 2.2(a), the corresponding DFTs [being linear functions of $S_T(\cdot)$'s and $C_T(\cdot)$'s] are asymptotically independent in both the SRD and the LRD cases under the *weaker* condition that $|T(\lambda_{jT} - \lambda_{iT})| \to \infty$ as $T \to \infty$ for all $1 \le i \ne j \le k$.

In the next result, we establish the asymptotic distribution of the sine and cosine transforms when the ordinates are "asymptotically close" and converge to a given frequency $2\pi\gamma \in [0, \pi]$.

THEOREM 2.3. *Suppose that Assumptions* (A.1)–(A.3) *hold. Let* $\lambda_{jT} = 2\pi(\lceil \gamma T \rceil + m_T + \ell_j)/T$, $1 \le j \le k$ $(k \in \mathbb{N})$, *for some* $\gamma \in [0, 1/2]$ *and for some* $m_T, \ell_1, \dots, \ell_k \in \mathbb{Z}$ *such that* γ *and* $\ell_1, \dots, \ell_k$ *do not depend on* T, *that* $\ell_1, \dots, \ell_k$ *are distinct and that* $m_T/T \to 0$ *as* $T \to \infty$.

(a) (The "asymptotically close to the zero frequency"/"low ordinate" case.) *Suppose that* $\gamma = 0$ *and* $m_T \to m$ *as* $T \to \infty$ *for some integer* $m \in \mathbb{Z}$. *Then*

$$(Y_{11T} \dots Y_{1kT}; Y_{21T} \dots Y_{2kT})' \xrightarrow{d} N\bigl(0, \Sigma_0^{(2k)}\bigr),$$

where for $1 \le i, j \le k$, *the* (i, j)*th element of* $\Sigma_0^{(2k)}$ *is given by*

$$\bigl(\Sigma_0^{(2k)}\bigr)_{ij} = \mathrm{Re}\bigl(\sigma_0(m + \ell_i; m + \ell_j) + \sigma_0(m + \ell_i; -[m + \ell_j])\bigr)/2,$$

$$\bigl(\Sigma_0^{(2k)}\bigr)_{k+i,\,k+j} = \mathrm{Re}\bigl(\sigma_0(m + \ell_i; -[m + \ell_j]) - \sigma_0(m + \ell_i; m + \ell_j)\bigr)/2,$$

$$\bigl(\Sigma_0^{(2k)}\bigr)_{i,\,k+j} = \mathrm{Im}\bigl(\sigma_0(m + \ell_i; -[m + \ell_j]) - \sigma_0(m + \ell_i; m + \ell_j)\bigr)/2.$$

(b) (The "asymptotically close but distant from the zero frequency" case.) *Suppose that either* $\gamma \ne 0$ *or* $\gamma = 0$ *and* $|m_T| \to \infty$ *as* $T \to \infty$. *Then*

$$(Y_{11T} \dots Y_{1kT}; Y_{21T} \dots Y_{2kT})' \xrightarrow{d} N\bigl(0, \Sigma_1^{(2k)}\bigr),$$

where for $1 \le i, j \le k$,

$$\bigl(\Sigma_1^{(2k)}\bigr)_{ij} = \mathrm{Re}\bigl(\sigma_1(\ell_j - \ell_i)\bigr)/2,$$

$$\bigl(\Sigma_1^{(2k)}\bigr)_{k+i,\,k+j} = \mathrm{Re}\bigl(\sigma_1(\ell_j - \ell_i)\bigr)/2,$$

$$\bigl(\Sigma_1^{(2k)}\bigr)_{i,\,k+j} = \mathrm{Im}\bigl(\sigma_1(\ell_j - \ell_i)\bigr)/2.$$

Theorem 2.3 shows that the sine and cosine transforms of the data with a general data taper may have nonzero asymptotic correlation for any two asymptotically close discrete ordinates in the neighborhood of a given frequency $2\pi\gamma \in [0, \pi]$. This asymptotic correlation depends on the dependence parameter d when the ordinates are of the form $2\pi j/T$ for integers j that do not depend on T. However, for discrete ordinates of the form $2\pi(m_T + \ell_i)/T$ with $m_T^{-1} + T^{-1}m_T \to 0$ as $T \to \infty$, which *also* converge to the zero frequency, the asymptotic covariance matrix of the sine and cosine transforms does *not* depend on d. In the nontapered case, Hurvich and Beltrao (1993) and Robinson (1995) demonstrated that the sine and cosine transforms are asymptotically correlated for low frequencies (i.e., for frequencies of the form $2\pi j/T$ for integers j not depending on T) under LRD. Theorem 2.3(a) extends their results by allowing a general data taper and by allowing the slowly varying function $L(\cdot)$ in the spectral density [cf. (1.1)].

Next we consider the problem of characterizing asymptotic independence of the DFTs. Note that by Theorem 2.1, asymptotic independence of a pair of DFTs would hold *if and only if* the cross-covariance terms of the relevant limiting normal distribution are zero. Thus, a characterization can be obtained by checking the expressions for the asymptotic covariance terms of the relevant sine and cosine transforms given in Theorems 2.2 and 2.3. Since these expressions depend on the data-taper function $h(\cdot)$, the set of asymptotically independent DFTs indeed depends on the specific function $h(\cdot)$. The following result characterizes asymptotic independence of the DFTs under a suitable condition on the data taper $h(\cdot)$. Let

$$D_{rT} = \big(d_T(\lambda_{rT}) - Ed_T(\lambda_{rT})\big)/a_T(\lambda_{rT}), \qquad r = 1, 2. \tag{2.8}$$

COROLLARY 2.4. *Suppose that the conditions of Theorem* 2.1 *hold with* $k = 2$ *for some sequences* $\{\lambda_{1T}\}$ *and* $\{\lambda_{2T}\}$ *in* $\mathcal{C}$. *Also suppose that*

$$\sigma_1(\ell) \neq 0; \qquad |\sigma_0(\ell_1; \ell_2) + \sigma_0(\ell_1; -\ell_2)| + |\sigma_0(\ell_1; \ell_2) - \sigma_0(\ell_1; -\ell_2)| \neq 0 \tag{2.9}$$

for all $\ell, \ell_1 \neq \ell_2 \in \mathbb{Z}$. *Then the normalized DFTs* $\{D_{1T}\}$ *and* $\{D_{2T}\}$ *are asymptotically independent if and only if* $|T(\lambda_{1T} - \lambda_{2T})| \to \infty$ *as* $T \to \infty$.

Corollary 2.4 provides a necessary and sufficient condition for the asymptotic independence of the DFTs for the class of data tapers $h(\cdot)$ satisfying (2.9). However, there are some commonly used taper functions, including the function $h(x) \equiv 1$, $x \in [0, 1]$, of the no-tapering case, that do not satisfy (2.9). For such functions, we have to check the asymptotic covariance terms of Theorem 2.3 to characterize the set of discrete ordinates for which the DFTs are asymptotically independent. In the next section, we consider some important special cases.

2.3. *Examples.* First we consider the case where the taper function is given by $h_0(x) \equiv 1$, $x \in [0, 1]$, that is, the case when no data tapering is used. To evaluate the constants $\sigma_1(\ell)$, note that by Parseval's identity, we may express $\sigma_1(\ell)$ as

$$\sigma_1(\ell) = 2\pi\left[\int_0^1 h(x)^2 \exp(i2\pi\ell x)\,dx\right], \qquad \ell \in \mathbb{Z}, \tag{2.10}$$

for any taper $h(\cdot)$. Hence for $h(\cdot) = h_0(\cdot)$, $\sigma_1(\ell) = 2\pi[\int_0^1 \exp(i2\pi\ell x)\,dx] = 0$ for all $\ell \in \mathbb{Z} \setminus \{0\}$. Also, $\hat{h}_0(y) = [\exp(iy) - 1]/(iy)$, $y \in \mathbb{R}$. Using these facts, we get:

COROLLARY 2.5. *Suppose that the conditions of Theorem* 2.1 *hold with* $k = 2$ *for some sequences* $\{\lambda_{1T}\}$ *and* $\{\lambda_{2T}\}$ *in* $\mathcal{C}$, *and that* $h(x) = h_0(x) = 1$ *for all* $x \in [0, 1]$.

(i) *If* $\max\{|T\lambda_{1T}|, |T\lambda_{2T}|\} \to \infty$, *then the normalized DFTs* $\{D_{1T}\}$ *and* $\{D_{2T}\}$ *defined by* (2.8) *are asymptotically independent.*

(ii) *If both* $\{|T\lambda_{1T}|\}$ *and* $\{|T\lambda_{2T}|\}$ *are bounded*, *and* $T\lambda_{jT} \to 2\pi\ell_j$ *for some* $\ell_j \in \mathbb{Z}$, $j = 1, 2$, *then*

$$\lim_{T\to\infty} E\{D_{1T}\bar{D}_{2T}\} = c(\ell_1, \ell_2) \int_{-\infty}^{\infty} \frac{2(1-\cos y)}{(y-2\pi\ell_1)(y+2\pi\ell_2)} |y|^{-2d}\, dy,$$

$$\lim_{T\to\infty} E\{D_{1T}D_{2T}\} = c(\ell_1, \ell_2) \int_{-\infty}^{\infty} \frac{2(1-\cos y)}{(y-2\pi\ell_1)(y-2\pi\ell_2)} |y|^{-2d}\, dy.$$

Thus, it follows from Corollary 2.5 that without data tapering, the DFTs are asymptotically independent for asymptotically distant ordinates [satisfying (2.4)] as well as for asymptotically close ordinates [satisfying (2.5)] such that either $\{\lambda_{1T}\}$ or $\{\lambda_{2T}\}$ is asymptotically distant from the sequence $\{0\}$. In particular, if $|T\lambda_{1T}| \to \infty$ as $T \to \infty$, for some sequence $\{\lambda_{1T}\} \in \mathcal{C}$, then the DFTs at the asymptotically close ordinates $\{\lambda_{1T}\}$ and $\{\lambda_{1T} + 2\pi j/T\}$ are asymptotically independent for any given integer $j \neq 0$.

Moulines and Soulier (1999) obtained a strong upper bound on the covariance of the *nontapered* DFTs in the context of broadband regression estimation of the dependence parameter d. For a stationary Gaussian process $\{X_t\}$ that has a spectral density of the form $f(\lambda) = |1 - \exp(i\lambda)|^{-2d} f^*(\lambda)$ with $-1/2 < d < 1/2$ and $|\frac{d}{d\lambda} f^*(\lambda)| < C|\lambda|^{-1}$, $\lambda \in \Pi \setminus \{0\}$, for some constant C, they showed [Moulines and Soulier (1999), Lemma 4] that $|E\{D_{1T}D_{2T}\}| + |E\{D_{1T}\bar{D}_{2T}\}| \leq C \log j_T k_T^{-|d|} |j_T|^{|d|-1}$ for $\lambda_{1T} = 2\pi j_T/T$ and $\lambda_{2T} = 2\pi k_T/T$ with $1 \leq k_T < j_T < T/2$. Under the Gaussianity assumption, this readily implies that the *nontapered* DFTs $d_{1T}(\lambda_{1T})$ and $d_{1T}(\lambda_{2T})$ are asymptotically independent whenever $k_T \to \infty$. Thus, in the important special case when no data taper is used, our results supplement their conclusion by allowing a more general form of the spectral density and by allowing the process $\{X_t\}$ to be non-Gaussian.

Next consider the "cosine-bell" taper $h_1 : [0, 1] \to \mathbb{R}$, given by

$$h_1(x) = \tfrac{1}{2}(1 - \cos 2\pi x), \qquad x \in [0, 1]. \tag{2.11}$$

Then it is easy to check that $\hat{h}_1(y) = [2\hat{h}_0(y) - \hat{h}_0(2\pi + y) - \hat{h}_0(y - 2\pi)]/4$, $y \in \mathbb{R}$, where $h_0(\cdot)$ is as in Corollary 2.5. In this case, using relationship (2.10), we can show that

$$\sigma_1(\ell) = [\pi/8] \sum_{k\in\{0,1,\ldots,4\}} (-1)^{4-k} \binom{4}{k} \mathbb{1}_{\{2-\ell\}}(k).$$

Hence, for $h(\cdot) = h_1(\cdot)$, $\sigma_1(\ell) = 0$ *if and only if* $|\ell| \geq 3$. Further, the constants $\sigma_0(\ell; m)$ may be found using (2.6) and the above expression for $\hat{h}_1(\cdot)$. We summarize the asymptotic behavior of the DFTs based on the cosine-bell taper in the following result.

COROLLARY 2.6. *Suppose that the conditions of Theorem* 2.1 *hold with* $k = 2$, *and that* D_{1T} *and* D_{2T} *are defined by* (2.8) *with the taper function* h_1 *of* (2.11).

(i) *If* $\max\{|T\lambda_{1T}|, |T\lambda_{2T}|\} \to \infty$ *and* $\lim_{T\to\infty} |T(\lambda_{1T} - \lambda_{2T})| \geq 6\pi$, *then* $\{D_{1T}\}$ *and* $\{D_{2T}\}$ *are asymptotically independent*.

(ii) *If* $\max\{|T\lambda_{1T}|, |T\lambda_{2T}|\} \to \infty$ *and* $\lim_{T\to\infty} |T(\lambda_{1T} - \lambda_{2T})| = 2\pi|\ell|$ *for some* $|\ell| \leq 2$, *then*

$$\lim_{T\to\infty} E\{D_{1T} D_{2T}\} = 0,$$

$$\lim_{T\to\infty} E\{D_{1T} \bar{D}_{2T}\} = \begin{cases} -\pi/2, & \text{if } \ell = \pm 1, \\ \pi/8, & \text{if } \ell = \pm 2. \end{cases}$$

(iii) *If* $\max\{|T\lambda_{1T}|, |T\lambda_{2T}|\}$ *is bounded and* $T\lambda_{jT} \to 2\pi\ell_j$ *for some* $\ell_j \in \mathbb{Z}$, $j = 1, 2$, *then*

$$\lim_{T\to\infty} E\{D_{1T} D_{2T}\} = \sigma_0(\ell_1; \ell_2),$$

$$\lim_{T\to\infty} E\{D_{1T} \bar{D}_{2T}\} = \sigma_0(\ell_1; -\ell_2).$$

Thus, for any $\{\lambda_{1T}\}$ and $\{\lambda_{2T}\}$ in $\mathcal{C}$ with $|T\lambda_{jT}| \to \infty$ for *at least one* $j \in \{1, 2\}$, the DFTs based on the cosine-bell taper $h_1(\cdot)$ are asymptotically independent *if and only if* $|T(\lambda_{1T} - \lambda_{2T})| \geq 6\pi$. This, in particular, allows asymptotic independence of the DFTs at the asymptotically close ordinates $\{\lambda_{1T}\}$ and $\{\lambda_{2T}\} \equiv \{\lambda_{1T} + 2\pi\ell/T\}$ for any fixed integer ℓ with $|\ell| \geq 3$, if $|T\lambda_{1T}| \to \infty$. In comparison, the corresponding nontapered DFTs $d_{1T}(\lambda_{1T})$ and $d_{1T}(\lambda_{2T})$ are asymptotically independent for all $\ell \neq 0$.

Next we consider a popular data taper of Kolmogorov [see Velasco (1999a, b); Zurbenko (1979)] that is useful in the context of efficient estimation of the spectral density under SRD and also in removing polynomial trends in nonstationary time series. Let $m \geq 2$ be a given integer and let h_2 denote the probability density function of the average of m independent and identically distributed Uniform[0, 1] r.v.s. Then, the weights generated by *Kolmogorov's taper of order m* are asymptotically equivalent to those given by the function $h_2(x)$. A closed form expression for $h_2(\cdot)$ is given by [see Field and Ronchetti (1990), page 36]

$$h_2(x) = \frac{m^m}{(m-1)!} \sum_{i=1}^{m-1} (-1)^i \binom{m}{i} \left[\left(1 - \frac{i}{m}\right) - x\right] \mathbb{1}_{[0,\, 1-i/m]}(x), \qquad x \in [0, 1]. \tag{2.12}$$

The Fourier transform of $h_2(\cdot)$ is $\hat{h}_2(y) = [\hat{h}_0(y/m)]^m$, $y \in \mathbb{R}$. It turns out that for Kolmogorov's taper of order m, the limiting covariance constants $\sigma_0(\cdot;\cdot)$ and $\sigma_1(\cdot)$ do not vanish and, therefore, the DFTs at asymptotically close ordinates cannot be asymptotically independent. For a specific example, suppose $m = 2$. Then, using (2.10), we get

$$\sigma_1(\ell) = \begin{cases} \dfrac{2^4}{\pi \ell^2}, & \text{if } \ell = \pm 2, \pm 4, \dots, \\ -\dfrac{2^4}{\pi \ell^2}, & \text{if } \ell = \pm 1, \pm 3, \dots. \end{cases} \tag{2.13}$$

Thus, from Theorems 2.1–2.3 it follows that in this case, for any sequences $\{\lambda_{1T}\}$ and $\{\lambda_{2T}\}$ in $\mathcal{C}$ with $\max\{|T\lambda_{1T}|, |T\lambda_{2T}|\} \to \infty$, the normalized DFTs $\{D_{1T}\}$ and $\{D_{2T}\}$ are asymptotically independent *if and only if* $\{\lambda_{1T}\}$ and $\{\lambda_{2T}\}$ are asymptotically distant, that is, $|T(\lambda_{1T} - \lambda_{2T})| \to \infty$ as $T \to \infty$. This property of the DFTs based on Kolmogorov's taper is in marked contrast with the nontapered case where $\sigma_1(\ell) = 0$ for all $\ell \neq 0$ and the DFTs at *distinct* ordinates can be asymptotically independent. As a result, various inference procedures designed under the asymptotic independence assumption on the nontapered DFTs may need some modification for their validity when a taper like Kolmogorov's taper is employed. In the next section, we consider such an example.

2.4. *Some implications.* Suppose that the process $\{X_t\}$ is Gaussian and that its spectral density is given by (1.1) with a bounded $L(\cdot)$. An important problem in this context is the estimation of the dependence parameter $d \in (-1/2, 1/2)$. Geweke and Porter-Hudak (1983) proposed an estimator of d using asymptotic independence of the DFTs $d_{1T}(\cdot)$. Let $I_{1T}(w) = T^{-1}|d_{1T}(\cdot)|^2$ denote the periodogram of the observations $\{X_0, \dots, X_{T-1}\}$ (*without* a data taper) and let $\{m_T\}$ be a sequence of integers such that $(\log T)^2/m_T + T^{-1}m_T \to 0$ as $T \to \infty$. Then, assuming that the variables $I_{1T}(2\pi k_T/T)/[(k_T/T)^{-2d}]$ and $I_{1T}(2\pi j_T/T)/[(j_T/T)^{-2d}]$ are asymptotically independent for any sequences of ordinates $\{2\pi k_T/T\}$ and $\{2\pi j_T/T\}$ with $1 \le k_T < j_T \le m_T$, they set up the "approximate regression model"

$$\log I_{1T}(\lambda_j) = \text{const} + a_j d + U_j, \qquad j = 1, \dots, m_T,$$

where $a_j = 2\log \lambda_j$, $\lambda_j = 2\pi j/T$ and the U_j's are approximately independent zero-mean random variables. The least squares estimator $\hat{d}$, say, of d in the above regression model is the Geweke and Porter-Hudak (1983) (GPH) estimator of d. It is known that under some regularity conditions on the function $f(\cdot)$, $\hat{d}$ is consistent and asymptotically normal [see Robinson (1995); Hurvich, Deo and Brodsky (1998)]. A crucial step in proving the validity of this result is the *uniform* bound on the covariance of the *nontapered* DFTs, obtained by Robinson (1995),

$$\left|\mathrm{Cov}\left(\frac{d_{1T}(\lambda_{j_T})}{\sqrt{Tf(\lambda_{j_T})}}, \frac{d_{1T}(\lambda_{k_T})}{\sqrt{Tf(\lambda_{k_t})}}\right)\right| + \left|\mathrm{Cov}\left(\frac{d_{1T}(\lambda_{j_T})}{\sqrt{Tf(\lambda_{j_T})}}, \frac{\bar{d}_{1T}(\lambda_{k_T})}{\sqrt{Tf(\lambda_{k_T})}}\right)\right| = O\left(\frac{\log j_T}{k_T}\right) \qquad \text{as } T \to \infty, \tag{2.14}$$

uniformly in $(\log T)^2 \le k_T < j_T \le m_T$. In particular, for any $\{k_T\} \subset [(\log T)^2, m_T] \cap \mathbb{Z}$, the covariance between $d_{1T}(\lambda_{k_T})$ and $d_{1T}(\pm\lambda_{k_T+\ell})$ is of $O((\log k_T)/k_T) = o(1)$ as $T \to \infty$ for any $\ell \in \mathbb{Z}$. However, as Theorem 2.3 shows, this is no longer true for a taper function $h(\cdot)$ with $\sigma_1(\ell) \neq 0$ for some $\ell \neq 0$. Consequently, consistency and asymptotic normality of the GPH estimator of d, defined using the *entire* collection of DFTs $\{d_T(\lambda_j) : j \in [(\log T)^2, m_T]\}$ need not hold in such tapered cases. Nonetheless, if $\sigma_1(\ell) = 0$ for all $\ell \ge \ell_0$ for some integer $\ell_0 \ge 1$, such as the cosine-bell taper considered above, we may use every ℓ_0th ordinate λ_j to define an estimator of d and, in the extreme case, where $\sigma_1(\ell) \neq 0$ for infinitely many $\ell \in \mathbb{Z}$ as in the case of Kolmogorov's tapers, Theorem 2.2 suggests that an estimator of d may be based on the DFTs $\{d_T(\lambda_{(j-1)p+1}) : j \in [1, m_T]\}$, where $p \equiv p_T \to \infty$ suitably with T. Indeed, consistency and asymptotic normality of such modified estimators of d have been established by Hurvich and Beltrao (1993) and Velasco (1999b) for the cosine-bell taper and by Velasco (1999b) for Kolmogorov's tapers; the latter paper further allows nonstationarity of the process $\{X_t\}$.

3. Proofs. We begin with some notation to be used in the rest of the paper. For any two sequences of positive real numbers $\{r_T\}$ and $\{s_T\}$, write $r_T \ll s_T$ if $r_T = o(s_T)$ as $T \to \infty$ and $r_T \sim s_T$ if $\lim_{T\to\infty} r_T/s_T = 1$. In the proofs below, we write $C, C(\cdot)$ to denote generic constants that depend only on their arguments (if any) and d, but do not depend on the variables T, n and w. Let $\|g\|_\infty$ denote the sup-norm of a function $g : A \to \mathbb{R}$ given by $\|g\|_\infty = \sup_{a\in A} |g(a)|$. Unless otherwise specified, all limits, including those in the order symbols, are taken by letting $T \to \infty$.

Next, let $L_0(a,b) = \max\{L(w) : w \in \Pi, |w - a| < b\}$, $a \in \mathbb{R}$, $b > 0$ and $L_1(a,b) = \max\{L(w) : a < w < b\}$, $a < b \in \mathbb{R}$. Also define the function Γ_T on Π by $\Gamma_T(w) = T\mathbb{1}(|w| < T^{-1}) + |w|^{-1}\mathbb{1}(T^{-1} < |w| \le \pi)$. In proving Theorems 2.1–2.3, we extensively make use of the inequality, due to Dahlhaus (1983),

$$|H_{k,T}(w)| \le C\Gamma_T(w), \qquad w \in \Pi, \tag{3.1}$$

where $H_{k,T}(w) = \sum_{t=0}^{T-1} h(t/T)^k \exp(-itw)$, $w \in \Pi$, $k \ge 1$.

Let $\chi_{k,\varepsilon}(t_1,\dots,t_{k-1})$ denote the kth order cumulant of $\varepsilon_t, \varepsilon_{t+t_1},\dots,\varepsilon_{t+t_{k-1}}$, defined by

$$\chi_{k,\varepsilon}(t_1,\dots,t_{k-1}) = (-i)^k \frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_k} E\exp\bigl(i(x_1\varepsilon_t + x_2\varepsilon_{t+t_1}+\cdots+x_k\varepsilon_{t+t_{k-1}})\bigr)\Big|_{x_1=\cdots=x_k=0}.$$

For complex r.v.s $Z_j = R_{1j} + iR_{2j}$, $j=1,\dots,k$, we define the cumulant of $Z_1,\dots,Z_k$ in terms of the cumulants of the real variables $\{R_{pj} : j=1,\dots,k, p=1,2\}$ as

$$\operatorname{cum}(Z_1,\dots,Z_k) \equiv \sum_{a_1,\dots,a_k\in\{1,2\}} (i)^{a_1+\cdots+a_k-k}\operatorname{cum}\bigl(R_{a_1 j},\dots,R_{a_k j}\bigr). \tag{3.2}$$

If $\Upsilon_k \equiv \sum_{t_1\cdots t_{k-1}} |\chi_{k,\varepsilon}(t_1,\dots,t_{k-1})| < \infty$ for some $k\ge 2$, then $\{\varepsilon_t\}$ has a kth order cumulant spectral density $f_{k,\varepsilon}(w_1,\dots,w_{k-1})$, defined by the inversion formula $f_{k,\varepsilon}(w_1,\dots,w_{k-1}) = (2\pi)^{1-k}\sum_{t_1,\dots,t_{k-1}}\exp(-i\sum_{j=1}^{k-1} t_j w_j)\chi_{k,\varepsilon}(t_1,\dots,t_{k-1})$, $w_1,\dots,w_{k-1}\in\Pi^{k-1}$. Further, if

$$\Upsilon_j < \infty \qquad \text{for all } 2\le j\le k, \tag{3.3}$$

for some $k\ge 2$, the kth order cumulant spectral density of $\{X_t\}$ exists and is given by [see Yajima (1989), Lemma 2; Hosoya and Taniguchi (1982), Lemma A2.1],

$$f_{k,X}(w_1,\dots,w_{k-1}) = b\Biggl(\sum_{j=1}^{k-1} w_j\Biggr)\Biggl[\prod_{j=1}^{k-1} b(-w_j)\Biggr] f_{k,\varepsilon}\Biggl(\sum_{j=1}^{k-1} w_j, w_2,\dots,w_{k-1}\Biggr), \tag{3.4}$$

$w_1,\dots,w_{k-1}\in\Pi^{k-1}$. The first result gives us an expression for the cumulants of $d_T(\lambda_j)$.

LEMMA 3.1. *For any* $\lambda_1,\dots,\lambda_k\in\Pi$, $k\ge 2$, *if* (3.3) *holds, then*

$$\begin{aligned}\operatorname{cum}\bigl(d_T(\lambda_1),\dots,d_T(\lambda_k)\bigr) &= \int_{\Pi^{k-1}}\Biggl\{H_{1,T}\Biggl(\lambda_k+\sum_{j=1}^{k-1} w_j\Biggr) b\Biggl(\sum_{j=1}^{k-1} w_j\Biggr)\Biggr\}\Biggl\{\prod_{j=1}^{k-1}\bigl(H_{1,T}(\lambda_j-w_j)b(-w_j)\bigr)\Biggr\}\\ &\qquad\times f_{\varepsilon,k}\Biggl(\sum_{j=1}^{k-1} w_j, w_2,\dots,w_{k-1}\Biggr)\,dw_1\cdots dw_{k-1}.\end{aligned}$$

PROOF. The proof follows by using (3.2). We omit the details. $\square$

LEMMA 3.2. *Suppose that Assumption* (A.3) *holds and that* $\varepsilon(\cdot):[0,\pi]\to(0,\infty)$ *is a function such that* $\varepsilon(x)\to 0+$ *as* $x\to 0+$. *Then, for any* $\delta>0$,

$$\limsup_{x\to 0+}\big[\{L_1(x\varepsilon(x);x)/[L(x)\varepsilon(x)^{-\delta}]\}+\{L(x)/[L(x\varepsilon(x))\varepsilon(x)^{-\delta}]\}\big]<\infty.$$

PROOF. It is known [Taqqu (1979), page 61] that under Assumption (A.3), for any $\beta,\delta\in(0,\infty)$, $y^{\delta}(\frac{L(xy)}{L(x)}-1)\to 0$ as $x\to 0+$ uniformly in $y\in(0,\beta]$ and $y^{-\delta}(\frac{L(xy)}{L(x)}-1)\to 0$ as $x\to 0+$ uniformly in $y\in[\beta,\infty)$. Lemma 3.2 follows from these facts. □

LEMMA 3.3. *Let* $\{X_t\}$ *admit representation* (2.1) *for some sequence of stationary zero-mean r.v.s* $\{\varepsilon_t\}$ *with* $\sum_{t\in\mathbb{Z}}|\chi_{2,\varepsilon}(t)|<\infty$. *Also suppose that Assumptions* (A.2) *and* (A.3) *hold and that* j_T, k_T *are integers satisfying* (i) $-T/2\le j_T<k_T\le T/2$, (ii) $k_T-j_T\to\infty$ *as* $T\to\infty$ *and* (iii) $2\pi j_T=\tilde\lambda_1 T+o(T)$ *and* $2\pi k_T=\tilde\lambda_2 T+o(T)$ *for some* $\tilde\lambda_1,\tilde\lambda_2\in[-\pi,\pi]$. *Then, with* $\lambda_{1T}=2\pi j_T/T$ *and* $\lambda_{2T}=2\pi k_T/T$, *and* $d_T(\cdot)$ *defined by* (1.3),

$$\lim_{T\to\infty}\frac{E\{[d_T(\lambda_{1T})-Ed_T(\lambda_{1T})][\bar d_T(\lambda_{2T})-E\bar d_T(\lambda_{2T})]\}}{a_T(\lambda_{1T})a_T(\lambda_{2T})}=0.$$

PROOF. Without loss of generality (w.l.o.g.), we may set $\mu=0$. (Otherwise, replace X_t by $X_t-\mu$ in all the steps below.) Then $Ed_T(\lambda_{1T})E\bar d_T(\lambda_{2T})=0$ for all T and, by (iii), $\lambda_{jT}\to\tilde\lambda_j$ as $T\to\infty$, for $j=1,2$. We now prove the lemma by considering several cases that arise from possible asymptotic behavior of the sequences $\{\lambda_{1T}\}$ and $\{\lambda_{2T}\}$. The primary cases are

(I) $\tilde\lambda_1=\tilde\lambda_2=\tilde\lambda$, say;
(II) $\tilde\lambda_1<\tilde\lambda_2$.

These are further subdivided into the following subcases:

(I.1) $j_T-\frac{\tilde\lambda T}{2\pi}\to\infty$, $k_T-\frac{\tilde\lambda T}{2\pi}\to\infty$, $k_T-j_T\to\infty$;
(I.2) $j_T=\frac{\tilde\lambda T}{2\pi}+O(1)$, $k_T-\frac{\tilde\lambda T}{2\pi}\to\infty$;
(I.3) $j_T-\frac{\tilde\lambda T}{2\pi}\to-\infty$, $k_T-\frac{\tilde\lambda T}{2\pi}\to-\infty$, $k_T-j_T\to\infty$;
(I.4) $j_T-\frac{\tilde\lambda T}{2\pi}\to-\infty$, $k_T=\frac{\tilde\lambda T}{2\pi}+O(1)$;
(I.5) $j_T-\frac{\tilde\lambda T}{2\pi}\to-\infty$, $k_T-\frac{\tilde\lambda T}{2\pi}\to\infty$;
(II.1) $\tilde\lambda_1=0<\tilde\lambda_2$;
(II.2) $\tilde\lambda_1<\tilde\lambda_2=0$;
(II.3) $\tilde\lambda_1<\tilde\lambda_2$, $(\tilde\lambda_1,\tilde\lambda_2)\notin\{(\{0\}\times(0,\pi])\cup([-\pi,0)\times\{0\})\}$.

We begin with case I.1 with $\tilde\lambda=0$. Then $\lambda_{jT}\to 0+$ as $T\to\infty$, for $j=1,2$. We establish case I.1 by showing that given any subsequence $\{T'\}$, there is a further subsequence $\{T''\}$ such that $\delta(T'') \equiv E\{[d_{T''}(\lambda_{1T''})][\bar{d}_{T''}(\lambda_{2T''})]\}/[a_{T''}(\lambda_{1T''}) \times a_T(\lambda_{2T''})] \to 0$ as $T'' \to \infty$. Since the set of limit points of the ratio $\{\lambda_{1T}/\lambda_{2T}\}$ is contained in $[0, 1]$, given any subsequence $\{T'\}$, we can extract a further subsequence $\{T''\}$ such that $\{\lambda_{1T}/\lambda_{2T}\}$ converges to a point in $[0, 1]$ along the subsequence $\{T''\}$. We show that $\delta(T'')$ goes to zero. The proof requires different arguments depending on whether $\{\lambda_{1T''}/\lambda_{2T''}\}$ converges to 0 or 1 or to a point in the set $(0, 1)$ and, therefore, the three cases will be treated separately. *For notational simplicity*, we also suppose that in place of the subsequence $\{\lambda_{1T''}/\lambda_{2T''}\}$, the *whole* sequence $\{\lambda_{1T}/\lambda_{2T}\}$ converges. Then we need to consider the three subcases, I.1.1: $\lambda_{1T} \sim \lambda_{2T}$ as $T \to \infty$, I.1.2: $\lambda_{1T} = o(\lambda_{2T})$ as $T \to \infty$ and I.1.3: $\lambda_{1T} \sim c\lambda_{2T}$ as $T \to \infty$ for some $c \in (0, 1)$.

Note that under I.1.1, $\lambda_{2T} - \lambda_{1T} = o(\lambda_{1T})$, whereas under I.1.2, $\lambda_{1T} = o(\lambda_{2T} - \lambda_{1T})$. Let $g_T(w) = |w - \lambda_{1T}|^{-1}|w - \lambda_{2T}|^{-1}|w|^{-2d} L(w)\mathbb{1}(w \in \Pi \setminus \{0, \lambda_{1T}, \lambda_{2T}\})$, $w \in \Pi$. We now consider subcase I.1.1. Let $q_{1T} = (\lambda_{2T} - \lambda_{1T})/2$, and $q_{2T} = \lambda_{1T}/2$. By (3.1) and Lemma 3.1 with $k = 2$,

$$
\begin{aligned}
&\Delta_T(\lambda_{1T}; \lambda_{2T}) \\
&\quad \equiv |E d_T(\lambda_{1T})\bar{d}_T(\lambda_{2T})| \\
&\quad = \left| \int_\Pi H_{1,T}(\lambda_{1T} - w) H_{1,T}(w - \lambda_{2T}) |b(w)|^2 f_{2,\varepsilon}(w)\, dw \right| \\
&\quad \le C\Bigg[\sum_{i=1}^2 \int_{|T(w-\lambda_{iT})|<1} T|w - \lambda_{iT}| g_T(w)\, dw \\
&\qquad + \sum_{i=1}^2 \int_{T^{-1}<|w-\lambda_{iT}|<q_{1T}} g_T(w)\, dw \\
&\qquad + \left\{ \int_{-q_{2T}}^{q_{2T}} + \int_{q_{2T}}^{\lambda_{1T}-q_{1T}} + \int_{\lambda_{2T}+q_{1T}}^{2\lambda_{2T}} + \int_{-\pi}^{-q_{2T}} + \int_{2\lambda_{2T}}^{\pi} \right\} g_T(w)\, dw \Bigg] \qquad (3.5) \\
&\quad \equiv I_{1T} + I_{2T} + I_{3T} + I_{4T} + I_{5T} + I_{6T} + I_{7T}, \qquad \text{say.}
\end{aligned}
$$

By Assumption (A.3) and the fact that $\lambda_{2T} - \lambda_{1T} \ll \lambda_{1T} \sim \lambda_{2T}$,

$$
\begin{aligned}
I_{1T} + I_{2T} &\le C \sum_{i=1}^2 \Bigg[\frac{L_0(\lambda_{iT}; T^{-1})}{|\lambda_{2T} - \lambda_{1T}||\lambda_{iT}|^{2d}} \\
&\qquad + \frac{2L_0(\lambda_{iT}; q_{1T})}{|\lambda_{2T} - \lambda_{1T}||\lambda_{iT}|^{2d}} \int_{T^{-1}}^{q_{1T}} |y|^{-1}\, dy \Bigg] \qquad (3.6) \\
&\le C|\lambda_{2T} - \lambda_{1T}|^{-1} \lambda_{1T}^{-2d} L(\lambda_{1T}) \log(T q_{1T}).
\end{aligned}
$$

Since $|\lambda_{iT} - w|^{-1} \le C\lambda_{1T}^{-1}$ for all $|w| < q_{2T}$, by (1.2) [Feller (1971), Chapter 8], we get

$$I_{3T} \le C\lambda_{1T}^{-2} \int_{|w|<q_{2T}} |w|^{-2d} L(w)\, dw \le C\lambda_{1T}^{-1-2d} L(\lambda_{1T}). \tag{3.7}$$

To estimate I_{4T} and I_{5T}, note that $|\lambda_{2T} - w| > |\lambda_{1T} - w|$ for $q_{2T} < w < \lambda_{1T} - q_{1T}$ and, similarly, $|\lambda_{1T} - w| > w - \lambda_{2T}$ for all $\lambda_{2T} + q_{1T} < w < 2\lambda_{2T}$. Since $\lambda_{1T} \sim \lambda_{2T}$, $|w|^{-2d} < C\lambda_{1T}^{-2d}$ for each $d \in (-\frac{1}{2}, \frac{1}{2})$ over both these w intervals. Hence, by (A.3),

$$\begin{aligned} I_{4T} + I_{5T} &\le C\lambda_{1T}^{-2d} L_1\left(\frac{\lambda_{1T}}{2}; 2\lambda_{2T}\right) \\ &\quad \times \left[\int_{q_{2T}}^{\lambda_{1T}-q_{1T}} \frac{1}{|w-\lambda_{1T}|^2}\, dw + \int_{\lambda_{2T}+q_{1T}}^{2\lambda_{2T}} \frac{1}{|w-\lambda_{2T}|^2}\, dw\right] \\ &\le C\lambda_{1T}^{-2d} |\lambda_{2T} - \lambda_{1T}|^{-1} L(\lambda_{1T}). \end{aligned} \tag{3.8}$$

Next, noting that for $i = 1, 2$, $|w - \lambda_{iT}| = -w + \lambda_{iT} > |w|$ for all $w < 0$, we have

$$I_{6T} \le C \int_{-\pi<w<-q_{2T}} |w|^{-2-2d} L(w)\, dw \le C\lambda_{1T}^{-1-2d} L(\lambda_{1T}). \tag{3.9}$$

To estimate I_{7T}, note that by (A.3)(ii), there exists $\eta > 0$ such that for all $|x| \le \eta$,

$$\sup\{L(y) : x/3 < y < 3x\} < CL(x). \tag{3.10}$$

Hence, for all $x \in (\lambda_{iT}/2, \eta)$, $i = 1, 2$,

$$L(x + \lambda_{iT}) \le \sup\{L(y) : x < y < 3x\} < CL(x). \tag{3.11}$$

Also, for all $w \in (2\lambda_{2T}, \pi)$,

$$|w - \lambda_{1T}| > |w - \lambda_{2T}|, \qquad |w| > w - \lambda_{2T} \quad \text{and} \quad |w| \le 2|w - \lambda_{2T}|. \tag{3.12}$$

Hence, $|w|^{-2d} \le C|w - \lambda_{2T}|^{-2d}$ for each $d \in (-\frac{1}{2}, \frac{1}{2})$ over the interval $(2\lambda_{2T}, \pi)$. So, by (3.11),

$$\begin{aligned} I_{7T} &\le C\left[\int_{2\lambda_{2T}<w<\eta} + \int_{\eta<w<\pi}\right] |\lambda_{2T} - w|^{-2-2d} L(w)\, dw \\ &\le C \int_{\lambda_{2T}}^{\eta-\lambda_{2T}} \frac{L(y)}{y^{2+2d}}\, dy + C(\eta) \le C(\eta)\lambda_{2T}^{-1-2d} L(\lambda_{2T}). \end{aligned} \tag{3.13}$$

Hence, by (3.5)–(3.9) and (3.13), it follows that

$$\begin{aligned} \Delta_T(\lambda_{1T}; \lambda_{2T}) &\le C(\eta)\big[|\lambda_{1T} - \lambda_{2T}|^{-1} \lambda_{1T}^{-2d} L(\lambda_{1T}) \log(Tq_{1T}) + \lambda_{1T}^{-1-2d} L(\lambda_{1T})\big] \\ &= C(\eta) a_T(\lambda_{1T}) a_T(\lambda_{2T}) \left[\frac{\log(k_T - j_T)}{|k_T - j_T|} + j_T^{-1}\right] \\ &= o\big(a_T(\lambda_{1T}) a_T(\lambda_{2T})\big). \end{aligned} \tag{3.14}$$

This proves subcase I.1.1.

Next, consider subcase I.1.2. In this case, $\lambda_{2T} - \lambda_{1T} \sim \lambda_{2T} \gg \lambda_{1T}$. Hence,

$$
\begin{aligned}
&\Delta_T(\lambda_{1T};\lambda_{2T})\\
&\quad\le C\Bigg[\sum_{i=1}^{2}\int_{|T(\lambda_{iT}-w)|<1} T|\lambda_{iT}-w| g_T(w)\,dw\\
&\qquad + \Bigg\{\int_{T^{-1}<|w-\lambda_{1T}|<\lambda_{1T}/2} + \int_{T^{-1}<|w-\lambda_{2T}|<q_{1T}} + \int_{-\lambda_{1T}/2}^{\lambda_{1T}/2}\\
&\qquad\quad + \int_{-q_{1T}}^{-\lambda_{1T}/2} + \int_{3\lambda_{1T}/2}^{(\lambda_{1T}+\lambda_{2T})/2} + \int_{-\pi}^{-q_{1T}} + \int_{\lambda_{2T}+q_{1T}}^{\pi}\Bigg\} g_T(w)\,dw\Bigg]\\
&\quad\equiv \tilde{I}_1(T)+\cdots+\tilde{I}_8(T), \qquad \text{say.}
\end{aligned}
\tag{3.15}
$$

Using arguments similar to (3.6) and (3.7), we have

$$
\begin{aligned}
&\tilde{I}_1(T)+\tilde{I}_2(T)+\tilde{I}_3(T)+\tilde{I}_4(T)\\
&\quad\le C\Bigg[\sum_{i=1}^{2}\lambda_{iT}^{-2d}L(\lambda_{iT})|\lambda_{2T}-\lambda_{1T}|^{-1}\\
&\qquad + \lambda_{1T}^{-2d}L(\lambda_{1T})|\lambda_{2T}-(3\lambda_{1T}/2)|^{-1}\log(T\lambda_{1T})\\
&\qquad + \lambda_{2T}^{-2d}L(\lambda_{2T})|\lambda_{2T}-\lambda_{1T}|^{-1}\log\{T(\lambda_{2T}-\lambda_{1T})\}\\
&\qquad + C\int_{|w|<\lambda_{1T}/2}|w|^{-2d}L(w)\,dw\,|\lambda_{1T}|^{-1}|\lambda_{2T}-(\lambda_{1T}/2)|^{-1}\Bigg]\\
&\quad\le C\big[\lambda_{2T}^{-1}\lambda_{1T}^{-2d}L(\lambda_{1T})\log(T\lambda_{1T})+\lambda_{2T}^{-1-2d}L(\lambda_{2T})\log(T\lambda_{2T})\big].
\end{aligned}
\tag{3.16}
$$

Next, consider $\tilde{I}_5(T)$ and $\tilde{I}_6(T)$. Note that $|w-\lambda_{2T}|^{-1}\le C\lambda_{2T}^{-1}$ over both intervals of w values. Also, as in (3.12), $|w|^{-2d} < C|w-\lambda_{1T}|^{-2d}$ for all $w > 3\lambda_{1T}/2$, $d\in(-\frac{1}{2},\frac{1}{2})$ and $|w-\lambda_{1T}|^{-1}\le|w|^{-1}$ for all $w<-\lambda_{1T}/2$. Hence, by (3.11) and the above inequalities,

$$
\begin{aligned}
&\tilde{I}_5(T)+\tilde{I}_6(T)\\
&\quad\le C\lambda_{2T}^{-1}\Bigg[\int_{-q_{1T}}^{-\lambda_{1T}/2}\frac{L(w)}{|w|^{1+2d}}\,dw+\int_{3\lambda_{1T}/2}^{(\lambda_{1T}+\lambda_{2T})/2}\frac{L(w)}{|w-\lambda_{1T}|^{1+2d}}\,dw\Bigg]\\
&\quad\le C\lambda_{2T}^{-1}\int_{\lambda_{1T}/2<y<q_{1T}} y^{-1-2d}L(y)\,dy\\
&\quad\le C\lambda_{2T}^{-1}\Bigg[\lambda_{1T}^{-2d}L(\lambda_{1T})+\lambda_{2T}^{-2d}L(\lambda_{2T})\\
&\qquad\qquad + L_1\bigg(\frac{\lambda_{1T}}{2};q_{1T}\bigg)\log\bigg(\frac{\lambda_{2T}}{\lambda_{1T}}\bigg)\mathbb{1}(d=0)\Bigg].
\end{aligned}
\tag{3.17}
$$

Finally, using arguments similar to (3.9) and (3.13), we get

$$\begin{aligned}\tilde{I}_7(T)+\tilde{I}_8(T) &\le C\biggl[\int_{-\pi<w<-q_{1T}} |w|^{-2-2d} L(w)\,dw \\ &\qquad + \int_{\lambda_{2T}+q_{1T}<w<\pi} |w-\lambda_{2T}|^{-2-2d} L(w)\,dw\biggr] \\ &\le C(\eta)\lambda_{2T}^{-1-2d} L(\lambda_{2T}).\end{aligned} \tag{3.18}$$

Next, define $\alpha = \min\{|d|, 1-2|d|\}/2$ if $d \neq 0$ and $\alpha = 1/8$ for $d = 0$. Then, by Lemma 3.2, there exist constants $C_1(\alpha) > 0$, $C_2(\alpha)$ and $C_3(\alpha) > 0$ such that for T large,

$$C_1(\alpha)(\lambda_{1T}/\lambda_{2T})^{+2\alpha} < [L(\lambda_{1T})/L(\lambda_{2T})] < C_2(\alpha)(\lambda_{1T}/\lambda_{2T})^{-2\alpha}, \tag{3.19}$$

$$L_1(\lambda_{1T}/2; \lambda_{2T}) \le C_3(\alpha)(\lambda_{2T}/\lambda_{1T})^{\alpha} L(\lambda_{2T}). \tag{3.20}$$

Hence, combining (3.15)–(3.20), we get

$$\begin{aligned}\Delta_T(\lambda_{1T}; \lambda_{2T}) &\le C(\eta)\lambda_{2T}^{-1}\bigl[\lambda_{1T}^{-2d} L(\lambda_{1T}) \log j_T + \lambda_{2T}^{-2d} L(\lambda_{2T}) \log k_T \\ &\qquad + L_1(\lambda_{1T}; \lambda_{2T}/2) \log(k_T/j_T)\mathbb{1}(d=0)\bigr] \\ &\le C(\eta,\alpha)\bigl(a_T(\lambda_{1T})a_T(\lambda_{2T})\bigr)(T\lambda_{2T})^{-1} \\ &\qquad \times \bigl[(\lambda_{1T}/\lambda_{2T})^{-d-\alpha} \log j_T + (\lambda_{1T}/\lambda_{2T})^{d-\alpha} \log k_T \\ &\qquad + (\lambda_{1T}/\lambda_{2T})^{-2\alpha} \log(k_T/j_T)\mathbb{1}(d=0)\bigr] \\ &= o\bigl(a_T(\lambda_{1T})a_T(\lambda_{2T})\bigr).\end{aligned} \tag{3.21}$$

This completes the proof of subcase I.1.2. [The proof of subcase I.1.3 is similar and is omitted.] Hence, the lemma holds for the case I.1, with $\tilde{\lambda} = 0$. For $\tilde{\lambda} \neq 0$, the spectral density $f(\cdot)$ is bounded in a neighborhood of $\tilde{\lambda}$, and both $a_T(\lambda_{1T})$ and $a_T(\lambda_{2T})$ are of the order $T^{-1/2}$. Hence, the case I.1, $\tilde{\lambda} \neq 0$, can be easily established by retracing the steps for the case I.1, $\tilde{\lambda} = 0$, with $d = 0$ and a *bounded* $L(\cdot)$. Thus, the case I.1 follows for all $\tilde{\lambda}$.

Next, consider the case I.2, $\tilde{\lambda} = 0$. In this case, $\lambda_{1T} = 2\pi j_T/T$ and $\lambda_{2T} = 2\pi k_T/T$ with $j_T = O(1)$ and $k_T \to \infty$. Let $M > 0$ be such that $M > 2|j_T| + 1$ for all T. Then, splitting the integral over Π into the subintervals $\{|w| < \frac{M}{T}\}$, $\{|w-\lambda_{2T}| < \frac{1}{T}\}$, $\{\frac{1}{T} < |w-\lambda_{2T}| < \lambda_{2T}/2\}$, $\{\frac{M}{T} < |w| < \lambda_{2T}/2\}$, $\{-\eta < w < -\lambda_{2T}/2\}$, $\{3\lambda_{2T}/2 < w < \eta\}$ and $\{\eta < |w| < \pi\}$ with $\eta \in (0,1)$ given by (3.10), and using (3.1), (3.11) and Lemmas 3.1 and 3.2, we can show that

$$\begin{aligned}\Delta_T(\lambda_{1T}; \lambda_{2T}) &\le C(\eta)\bigl[\lambda_{2T}^{-1} T^{2d} L(T^{-1}) + \lambda_{2T}^{-1-2d} \log(T\lambda_{2T}) L(\lambda_{2T}) \\ &\qquad + \lambda_{2T}^{-1} \log(T\lambda_{2T}) L_1(MT^{-1}; \lambda_{2T}/2)\mathbb{1}(d=0)\bigr]\end{aligned}$$

$$
\begin{aligned}
&\le C(\eta,\beta)a_T(\lambda_{1T})a_T(\lambda_{2T}) \\
&\quad \times \big[(\lambda_{2T}T)^{-(1-|d|)+\beta}\log(\lambda_{2T}T) \\
&\qquad + (T\lambda_{2T})^{-1+2\beta}\log(T\lambda_{2T})\mathbb{1}(d=0)\big] \\
&= o\big(a_T(\lambda_{1T})a_T(\lambda_{2T})\big),
\end{aligned}
$$

where $2\beta = (1-2|d|)/4$. Hence, the case I.2, $\tilde{\lambda}=0$, is proved. As indicated before, the case I.2, $\tilde{\lambda}\neq 0$, can be deduced from the case I.2, $\tilde{\lambda}=0$, by setting $d=0$ and $L(\cdot)$ to be a *bounded* function. Thus, the case I.2 also follows for all $\tilde{\lambda}$.

The rest of the cases, namely I.3–II.3 can be handled using simple modifications of the arguments above. We omit the details. □

LEMMA 3.4. *Let* $\{X_t\}$ *be given by* (2.1) *for some stationary zero-mean r.v.s* $\{\varepsilon_t\}$ *with* $\sum_{t\in\mathbb{Z}}|\chi_{2,\varepsilon}(t)|<\infty$. *Suppose that Assumptions* (A.2) *and* (A.3) *hold and that* $\mu=0$.

(a) *For any* $j\le k\in\mathbb{Z}$ *not depending on* T, *with* $c(j,k)$ *as in* (2.6),

$$
\begin{aligned}
&\lim_{T\to\infty} Ed_T(2\pi j/T)d_T(2\pi k/T)/[a_T(2\pi j/T)a_T(2\pi k/T)] \\
&\qquad = c(j,k)(2\pi)^{1-2d}\int_{-\infty}^{\infty}\hat{h}\big(2\pi(y-j)\big)\hat{h}\big(-2\pi(y+k)\big)|y|^{-2d}\,dy\, f_{2,\varepsilon}(0).
\end{aligned}
$$

(b) *Let*

$$
\lambda_{1T}=\frac{2\pi(\lceil\gamma T\rceil+m_T)}{T},\qquad \lambda_{2T}=\frac{2\pi(\lceil\gamma T\rceil+m_T+\ell)}{T},
$$

where $\gamma\in(0,\frac{1}{2}]$, $m_T,\ell\in\mathbb{Z}$, γ *and* ℓ *do not depend on* T, *and* $m_T=O(1)$. *Assume that* $\lambda_{1T},\lambda_{2T}\in[0,\pi]$ *for all large* T. *Then*

$$
\lim_{T\to\infty} E\frac{d_T(\lambda_{1T})\bar{d}_T(\lambda_{2T})}{[a_T(\lambda_{1T})a_T(\lambda_{2T})]} = 2\pi\int_{-\infty}^{\infty}\hat{h}(2\pi y)\hat{h}\big(-2\pi(y-\ell)\big)\,dy\, f_{2,\varepsilon}(0). \tag{3.22}
$$

(c) *Let*

$$
\lambda_{1T}=\frac{2\pi(\lceil\gamma T\rceil+m_T)}{T},\qquad \lambda_{2T}=\frac{2\pi(\lceil\gamma T\rceil+m_T+\ell)}{T},
$$

where $\gamma\in(0,\frac{1}{2}]$, $m_T,\ell\in\mathbb{Z}$, *and* γ *and* ℓ *do not depend on* T. *Assume that* $|m_T|^{-1}+m_T/T=o(1)$ *as* $T\to\infty$ *and that* $\lambda_{1T},\lambda_{2T}\in[0,\pi]$ *for all large* T. *Then* (3.22) *holds.*

PROOF. First we consider part (a). Let $\lambda_{1T} = 2\pi j/T$ and $\lambda_{2T} = 2\pi k/T$. Then, by Lemma 3.1, for any $M > 2(|j| + |k| + 1)$,

$$
\begin{aligned}
&E d_T(2\pi j/T) d_T(2\pi k/T) \\
&\quad = \int_{|w|<2\pi M/T} \\
&\qquad + \int_{2\pi M/T<|w|<\pi} [H_{1,T}(\lambda_{1T} - w) H_{1,T}(\lambda_{2T} + w)|w|^{-2d} L(w)]\, dw \qquad (3.23)\\
&\quad \equiv I_{1,T}(M) + I_{2,T}(M), \qquad \text{say.}
\end{aligned}
$$

By (3.1) and the fact that $|w - a| > |w| - |a| > |w|/2$ for all $|w| > 2|a|$, we have

$$
\begin{aligned}
&\lim_{T\to\infty} \frac{I_{2,T}(M)}{[a_T(\lambda_{1T}) a_T(\lambda_{2T})]} \\
&\qquad \le \lim_{T\to\infty} \frac{C}{(T^{1+2d} L(T^{-1}))} \int_{2\pi M/T<|w|<\pi} \frac{L(w)}{|w|^{2+2d}}\, dw \qquad (3.24)\\
&\qquad \le \frac{C}{M^{(1+2d)}} \lim_{T\to\infty} \frac{L(2\pi M/T)}{L(T^{-1})} = \frac{C}{M^{(1+2d)}},
\end{aligned}
$$

which tends to zero as $M \to \infty$. Next, setting $w = 2\pi y/T$ in $I_{1,T}(M)$, we get

$$
\begin{aligned}
&I_{1,T}(M)/\{T^{1+2d} L(T^{-1})\} \\
&\quad = (2\pi)^{1-2d} \int_{-M}^{M} \biggl[\frac{1}{T} H_{1,T}\biggl(\frac{2\pi(j-y)}{T}\biggr)\biggr]\biggl[\frac{1}{T} H_{1,T}\biggl(\frac{2\pi(k+y)}{T}\biggr)\biggr] \\
&\qquad\qquad \times \frac{L(2\pi y/T) f_{2,\varepsilon}(2\pi y/T)}{|y|^{2d} L(T^{-1})}\, dy.
\end{aligned}
$$

Note that by definition, $T^{-1} H_{1,T}(a/T) \to \int_0^1 e^{-iax} h(x)\, dx$ and $|T^{-1} H_{1,T}(a/T)| \le T^{-1} \sum_{i=1}^{T-1} |h(t/T)| \to \int_0^1 |h(x)|\, dx < \infty$, for any $a \in \mathbb{R}$. Hence, by Assumption (A.3) and by the bounded convergence theorem (BCT),

$$
\begin{aligned}
&\lim_{T\to\infty} I_{1,T}(M)/[a_T(\lambda_{1T}) a_T(\lambda_{2T})] \\
&\quad = c(j,k)(2\pi)^{1-2d} \qquad (3.25)\\
&\qquad \times \int_{-M}^{M} \int_0^1 \int_0^1 h(u) h(v) e^{-2\pi i(j-y)u} e^{-2\pi i(k+y)v} |y|^{-2d}\, du\, dv\, dy\, f_{2,\varepsilon}(0)
\end{aligned}
$$

for any $M > 2(|j| + |k| + 1)$. Next, using Assumption (A.2) and letting $M \to \infty$, by (3.23)–(3.25) part (a) of Lemma 3.3 follows.

Proofs of parts (b) and (c) are similar and hence are omitted. □

LEMMA 3.5. *Let $\{X_t\}$ be given by* (2.1) *for some stationary zero-mean r.v.s $\{\varepsilon_t\}$ that satisfy* (3.3) *with a given $k \ge 3$. For $j = 1, \ldots, k$, let $\ell_{jT} \in [-\frac{T}{2}, \frac{T}{2}]$ be integers such that $\lambda_{jT} = 2\pi \ell_{jT}/T \to \tilde{\lambda}_j$ for some $\tilde{\lambda}_j \in \Pi$ as $T \to \infty$. If*

Assumptions (A.2) *and* (A.3) *hold*, *then*

$$\left|\mathrm{cum}\big(d_T(\lambda_{1T}),\dots,d_T(\lambda_{kT})\big)\right| = o\left(\prod_{j=1}^{k} a_T(\lambda_{jT})\right).$$

PROOF. Let $\eta \in (0,1)$ be given. Define the sets $A(1,\eta) = \{(w_2,\dots,w_{k-1})' \in \Pi^{k-2} : |\tilde{\lambda}_1 - \langle -[\tilde{\lambda}_2 + \sum_{j=2}^{k-1} w_j]\rangle_{2\pi}| > \eta\}$ and $A(2,\eta) = \Pi^{k-2} \setminus A(1,\eta)$, where for any real number x, $\langle x\rangle_{2\pi} = x$ (modulo 2π) with values in $(-\pi,\pi]$. Also, define the function $G_T(\cdot)$ on Π^{k-2} by $G_T(w_2,\dots,w_{k-1}) = \int_\Pi |H_{1,T}(\lambda_{kT} + \sum_{j=1}^{k-1} w_j) b(\sum_{j=1}^{k-1} w_j) H_{1,T}(\lambda_{1T} - w_1) b(-w_1)|\, dw_1$, $w_2,\dots,w_{k-1} \in \Pi$. Then, by Lemma 3.1,

$$\begin{aligned}
&\left|\mathrm{cum}\big(d_T(\lambda_{1T}),\dots,d_T(\lambda_{kT})\big)\right| \\
&\quad \le \|f_{\varepsilon,k}\|_\infty \sum_{p=1}^{2} \int_{A(p,\eta)} G_T(w_2,\dots,w_{k-1}) \\
&\qquad\qquad \times \left[\prod_{j=2}^{k-1} \left|H_{1,T}(\lambda_{j,T} - w_j) b(-w_j)\right|\right] dw_2 \cdots dw_{k-1} \qquad (3.26)\\
&\quad \equiv \sum_{p=1}^{2} J_{pT}, \qquad \text{say.}
\end{aligned}$$

Next, using Lemma 3.4, the Cauchy–Schwarz inequality, the change of variable $y = -\sum_{j=1}^{k-1} w_j$ and the periodicity of the functions $H_{1,T}(\cdot)$ and $b(\cdot)$, we get

$$\begin{aligned}
G_T(w_2,\dots,w_{k-1}) &\le \left[\int_\Pi \left|H_{1,T}(\lambda_{1T} - w_1) b(-w_1)\right|^2 dw_1\right]^{1/2} \\
&\qquad \times \left[\int_\Pi \left|H_{1,T}(\lambda_{kT} - y) b(-y)\right|^2 dy\right]^{1/2} \qquad (3.27)\\
&\le C a_T(\lambda_{1T}) a_T(\lambda_{kT})
\end{aligned}$$

for all $w_2,\dots,w_{k-1} \in \Pi$ (where the constant C does not depend on $w_2,\dots,w_{k-1}$ and T). Also, by arguments similar to the proof of Lemma 3.3, it follows that

$$\lim_{T\to\infty} G_T(w_2,\dots,w_{k-1}) = 0 \qquad \text{for all } (w_2,\dots,w_{k-1})' \in A(1,\eta). \tag{3.28}$$

Next, let $Q_T \equiv \int_{\Pi^{k-2}} \{\prod_{j=2}^{k-1} |H_{1,T}(\lambda_{jT} - w_j) b(-w_j)|^2\}\, dw_2,\dots,dw_{k-1}$. Then, by Lemma 3.4, $Q_T = O(\prod_{j=2}^{k-1} a_T(\lambda_{jT}))$. Hence, by (3.27), (3.28), the Cauchy–Schwarz inequality and the BCT,

$$\begin{aligned}
J_{1T} &\le \left[\int_{A(1,\eta)} G_T(w_2,\dots,w_{k-1})^2\, dw_2 \cdots dw_{k-1}\right]^{1/2} [Q_T]^{1/2} \qquad (3.29)\\
&= o\left(\prod_{j=1}^{k-1} a_T(\lambda_{jT})\right).
\end{aligned}$$

Next, using (3.27), Lemma 3.4 and the Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
J_{2T} &\leq C[a_T(\lambda_{1T})a_T(\lambda_{kT})] \\
&\qquad \times \int_{A(2,\eta)} \left\{ \prod_{j=2}^{k-1} |H_{1,T}(\lambda_{jT} - w_j)b(-w_j)| \right\} dw_2 \cdots dw_{k-1} \\
&\leq C[a_T(\lambda_{1T})a_T(\lambda_{kT})] \left[\int_{A(2,\eta)} dw_2 \cdots dw_{k-1} \right]^{1/2} [Q_T]^{1/2} \\
&\leq C \left(\prod_{j=1}^{k} a_T(\lambda_{jT}) \right) \left[\int_{A(2,\eta)} dw_2 \cdots dw_{k-1} \right]^{1/2}
\end{aligned}
\tag{3.30}
$$

for all large T. Note that the (Lebesgue) measure of $A(2, \eta)$ tends to zero as $\eta \to 0+$. Hence, letting $T \to \infty$ first and then $\eta \downarrow 0$, the lemma follows from (3.26), (3.29) and (3.30). □

PROOF OF THEOREM 2.1. First suppose that ε_0 is a bounded r.v. Then, under (A.1), (3.3) holds for all $k \geq 3$. Since $C_T(\lambda) = [d_T(\lambda) + d_T(-\lambda)]/2$ and $S_T(\lambda) = [d_T(\lambda) - d_T(-\lambda)]/(2i)$, using (3.2), we can express the cumulants of $C_T(\lambda_{jT})$ and $S_T(\lambda_{jT})$, $1 \leq j \leq k$, as linear combinations of those of $d_T(\pm\lambda_{jT})$. Then, by Lemma 3.5, it follows that cumulants of the normalized DFTs of order 3 or higher go to zero with T. Further, by Lemmas 3.3 and 3.4, the covariances of the DFTs $d_T(\pm\lambda_{jT})$, $1 \leq j \leq k$, have a limit for sequences $\{\lambda_{jT}\}$ of discrete ordinates in the class $\mathcal{C}$. Hence, by the Fréchet–Shohat theorem [see Chow and Teicher (1988), Chapter 8], asymptotic normality of $C_T(\lambda_{jT})$ and $S_T(\lambda_{jT})$, $1 \leq j \leq k$, follows.

Next consider the general case where ε_0 is not necessarily a bounded r.v. For each $n \in \mathbb{N}$, define the variables $\tilde{\varepsilon}_{t,n} = \varepsilon_t \mathbb{1}(|\varepsilon_t| \leq n) - E\varepsilon_t \mathbb{1}(|\varepsilon_t| \leq n)$, $\check{\varepsilon}_{t,n} = \varepsilon_t - \tilde{\varepsilon}_{t,n}$, $\tilde{X}_{t,n} = \sum_{j \in \mathbb{Z}} b_j \tilde{\varepsilon}_{t-j,n}$ and $\check{X}_{t,n} = X_t - \tilde{X}_{t,n}$, $t \in \mathbb{Z}$. Also define the variables $\{\tilde{d}_{t,n}(\cdot), \tilde{Y}_{T,n}\}$ and $\{\check{d}_{t,n}(\cdot), \check{Y}_{T,n}\}$ by replacing $\{X_t\}$ with $\{\tilde{X}_{t,n}\}$ and $\{\check{X}_{t,n}\}$, respectively. Let $a_t = \alpha(|t|)^{-1/2}$, $t \in \mathbb{Z}$. Then, for every $n \in \mathbb{N}$ and $t \in \mathbb{Z}$, by the mixing property of $\{\varepsilon_t\}$,

$$
\begin{aligned}
|E\varepsilon_0 \tilde{\varepsilon}_{t,n}| &= |\operatorname{Cov}(\varepsilon_0, \tilde{\varepsilon}_{t,n})| \\
&\leq \left|\operatorname{Cov}\big(\varepsilon_0 \mathbb{1}(|\varepsilon_0| \leq a_t), \tilde{\varepsilon}_{t,n}\big)\right| + \left|\operatorname{Cov}\big(\varepsilon_0 \mathbb{1}(|\varepsilon_0| > a_t), \tilde{\varepsilon}_{t,n}\big)\right| \\
&\leq 16 a_t n \alpha(|t|) + 2nE|\varepsilon_0| \mathbb{1}(|\varepsilon_0| > a_t) \\
&\leq (16 + 2E\varepsilon_0^2) n \alpha(|t|)^{1/2}.
\end{aligned}
\tag{3.31}
$$

Hence, by (3.31), (A.1), the Cauchy–Schwarz inequality and the fact that $E\varepsilon_0\varepsilon_t = 0$ for $t \neq 0$,

$$
\begin{aligned}
&\sum_{t\in\mathbb{Z}} |\operatorname{Cov}(\check{\varepsilon}_{0,n}, \check{\varepsilon}_{t,n})| \\
&\quad = \sum_{|t|<l_n} |\operatorname{Cov}(\check{\varepsilon}_{0,n}, \check{\varepsilon}_{t,n})| + \sum_{|t|\geq l_n} \big|0 - E\varepsilon_0\tilde{\varepsilon}_{t,n} - E\tilde{\varepsilon}_{0,n}\varepsilon_t + E\tilde{\varepsilon}_{0,n}\tilde{\varepsilon}_{t,n}\big| \qquad (3.32)\\
&\quad \leq 2l_n E\check{\varepsilon}_{0,n}^2 + C\sum_{t=l_n}^{\infty}\big[n\alpha(t)^{1/2} + n^2\alpha(t)\big] = o(1) \qquad \text{as } n\to\infty.
\end{aligned}
$$

By (3.32), the stationary sequence $\{\check{\varepsilon}_{t,n}\}_{t\in\mathbb{Z}}$ has a spectral density $\check{f}_{n,\varepsilon}(\cdot)$ (say) for each $n \in \mathbb{N}$ and

$$
\lim_{n\to\infty}\sup_{w\in\Pi}\check{f}_{n,\varepsilon}(w) = 0. \qquad (3.33)
$$

Further, for each $n \in \mathbb{N}$, $\{\check{X}_{t,n}\}_{t\in\mathbb{Z}}$ admits the representation $\check{X}_{t,n} = \sum_{j\in\mathbb{Z}} b_j\check{\varepsilon}_{t-j,n}$, $t \in \mathbb{Z}$. Hence, by Lemma 3.1, for any sequence $\{\lambda_{1T}\} \subset \mathcal{C}$, $\Delta_n(\lambda_{1T}) \equiv E|d_T(\lambda_{1T}) - \tilde{d}_{T,n}(\lambda_{1T})|^2 = E|\check{d}_{T,n}(\lambda_{1T})|^2 = \int_\Pi |H_{1,T}(\lambda_{1T} - w)|^2|b(w)|^2 \times \check{f}_{n,\varepsilon}(w)\,dw \leq 2\pi\int_\Pi |H_{1,T}(\lambda_{1T} - w)|^2|w|^{-2d}L(w)\,dw\|\check{f}_{n,\varepsilon}\|_\infty$, so that by (3.33) and Lemma 3.4,

$$
\lim_{n\to\infty}\limsup_{T\to\infty}\check{\Delta}_n(\lambda_{1T})/[a_T(\lambda_{1T})^2] = 0. \qquad (3.34)
$$

Next, note that $|E\tilde{\varepsilon}_{0,n}\check{\varepsilon}_{t,n}| \leq 4nE|\varepsilon_t|\mathbb{1}(|\varepsilon_t| > n) \leq 4E\varepsilon_t^2\mathbb{1}(|\varepsilon_t| > n) = 4c_n$ for all $n \in \mathbb{N}$ and $t \in \mathbb{Z}$. Hence, by arguments similar to (3.32),

$$
\begin{aligned}
&\sum_{t\in\mathbb{Z}} |\operatorname{Cov}(\tilde{\varepsilon}_{0,n}, \tilde{\varepsilon}_{t,n}) - \operatorname{Cov}(\varepsilon_0, \varepsilon_t)| \\
&\quad \leq \sum_{|t|<l_n} |E\tilde{\varepsilon}_{0,n}\tilde{\varepsilon}_{t,n} - E\varepsilon_0\varepsilon_t| + 2\sum_{t\geq l_n} |\operatorname{Cov}(\tilde{\varepsilon}_{0,n}, \tilde{\varepsilon}_{t,n})| \\
&\quad \leq 2\sum_{|t|<l_n} \{|E\tilde{\varepsilon}_{0,n}\check{\varepsilon}_{t,n}| + |E\check{\varepsilon}_{0,n}\check{\varepsilon}_{t,n}|\} + 32\sum_{t\geq l_n} n^2\alpha(t) \\
&\quad \leq C\left[l_n c_n + \sum_{t\in\mathbb{Z}} |E\check{\varepsilon}_{0,n}\check{\varepsilon}_{t,n}|\right] + 32\sum_{t\geq l_n} n^2\alpha(t) = o(1) \qquad \text{as } n\to\infty.
\end{aligned}
$$

Thus, the spectral density $\tilde{f}_{n,\varepsilon}$ of $\{\tilde{\varepsilon}_{t,n}\}_{t\in\mathbb{Z}}$ exists for all $n \in \mathbb{N}$ and satisfies

$$
\lim_{n\to\infty}\|\tilde{f}_{n,\varepsilon} - f_\varepsilon\|_\infty = 0. \qquad (3.35)
$$

Note that for each $n \in \mathbb{N}$, $\varepsilon_{t,n}$ is bounded. Hence, by the argument for the bounded r.v. case above, there exists a zero-mean Gaussian random vector $\tilde{Y}_{\infty,n}$

such that $\tilde{Y}_{T,n} \xrightarrow{d} \tilde{Y}_{\infty,n}$. Write $\tilde{\Sigma}_{\infty,n}$ for the covariance matrix of $\tilde{Y}_{\infty,n}$. Then, by (3.35) and Lemma 3.4, $\Sigma_\infty = \lim_{n\to\infty} \tilde{\Sigma}_{\infty,n}$ exists [and is given by replacing $\tilde{f}_{n,\varepsilon}(\cdot)$ in $\tilde{\Sigma}_{\infty,n}$ by $f_\varepsilon(\cdot)$]. Let Y_∞ be a Gaussian random vector with mean zero and covariance matrix Σ_∞. Also, let ρ be a metric metricizing the topology of weak convergence on the set of all Borel probability measures on $\mathbb{R}^{2k}$. By an abuse of notation, for any two random vectors Y and Z, we write $\rho(Y, Z)$ for the ρ-distance between the probability laws of Y and Z. Then, it follows that $\lim_{T\to\infty} \rho(Y_{T,n}, \tilde{Y}_{\infty,n}) = 0$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \rho(\tilde{Y}_{\infty,n}, Y_\infty) = 0$. Also, by (3.34), we have $\lim_{n\to\infty} \limsup_{T\to\infty} \rho(Y_T, Y_{T,n}) = 0$. As a consequence, $\lim_{T\to\infty} \rho(Y_T, Y_\infty) = 0$. This completes the proof of Theorem 2.1. $\square$

PROOF OF THEOREM 2.2. W.l.o.g., assume that $\mu = 0$. Let A_1, A_2, B_1, B_2 be real valued random variables with finite second moment. Define $I_1 \equiv E([A_1 + \sqrt{-1}B_1][A_2 + \sqrt{-1}B_2])$ and $I_2 \equiv E([A_1 + \sqrt{-1}B_1][A_2 - \sqrt{-1}B_2])$. Then it is easy to verify that $EA_1A_2 = \mathrm{Re}(I_1 + I_2)/2$, $EB_1B_2 = \mathrm{Re}(I_2 - I_1)/2$ and $EA_1B_2 = \mathrm{Im}(I_1 - I_2)/2$. Next, set $A_1 = C_T(\lambda_{iT})/a_T(\lambda_{iT})$, $A_2 = C_T(\lambda_{jT})/a_T(\lambda_{jT})$, $B_1 = S_T(\lambda_{iT})/a_T(\lambda_{iT})$ and $A_2 = S_T(\lambda_{jT})/a_T(\lambda_{jT})$, $1 \le i, j \le k$. Then $I_1 = Ed_T(-\lambda_{iT})d_T(-\lambda_{jT})$ and $I_2 = Ed_T(-\lambda_{iT})d_T(\lambda_{jT})$. Now using the above identities and Lemmas 3.3 and 3.4, after some algebra, we can find the limiting covariance structure of $C_T(\lambda_{jT})$ and $S_T(\lambda_{jT})$. Hence, Theorem 2.2 now follows from Theorem 2.1. $\square$

PROOF OF THEOREM 2.3. The proof follows from Lemmas 3.3 and 3.4 and Theorem 2.1. $\square$

Acknowledgments. I thank the referees and the Associate Editor for a number of constructive suggestions that significantly improved an earlier draft of this paper. I also thank Professor H. Künsch for bringing the papers by Kawata (1966) and Thomson (1982) to my attention.

REFERENCES

ADENSTEDT, R. K. (1974). On large-sample estimation for the mean of a stationary random sequence. *Ann. Statist.* **2** 1095–1107.

BERAN, J. (1994). *Statistics for Long-Memory Processes*. Chapman and Hall, London.

BRILLINGER, D. (1981). *Time Series*: *Data Analysis and Theory*, 2nd ed. Holden–Day, San Francisco.

BROCKWELL, P. J. and DAVIS, R. A. (1991). *Time Series*: *Theory and Methods*, 2nd ed. Springer, New York.

CHOW, Y. S. and TEICHER, H. (1988). *Probability Theory*, 2nd ed. Springer, New York.

DAHLHAUS, R. (1983). Spectral analysis with tapered data. *J. Time Ser. Anal.* **4** 163–175.

DAHLHAUS, R. (1985). Asymptotic normality of spectral estimates. *J. Multivariate Anal.* **16** 412–431.

DAHLHAUS, R. and JANAS, D. (1996). A frequency domain bootstrap for ratio statistics in time series analysis. *Ann. Statist.* **24** 1934–1963.

DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**, 2nd ed. Wiley, New York.

FIELD, C. A. and RONCHETTI, E. (1990). *Small Sample Asymptotics.* IMS, Hayward, CA.

FRANKE, J. and HÄRDLE, W. (1992). On bootstrapping kernel spectral estimates. *Ann. Statist.* **20** 121–145.

FULLER, W. (1976). *Introduction to Statistical Time Series.* Wiley, New York.

GEWEKE, J. and PORTER-HUDAK, S. (1983). The estimation and application of long-memory time-series models. *J. Time Ser. Anal.* **4** 221–238.

GRANGER, C. W. J. and JOYEUX, R. (1980). An introduction to long-memory time series models and fractional differencing. *J. Time Ser. Anal.* **1** 15–29.

HANNAN, E. J. (1970). *Multiple Time Series*. Wiley, New York.

HANNAN, E. J. and THOMSON, P. J. (1971). Spectral inference over narrow bands. *J. Appl.Probab.* **8** 157–169.

HOSKING, J. R. M. (1981). Fractional differencing. *Biometrika* **68** 165–176.

HOSOYA, Y. and TANIGUCHI, M. (1982). A central limit theorem for stationary processes and the parameter estimation of linear processes. *Ann. Statist.* **10** 132–153.

HURVICH, C. M. and BELTRAO, K. I. (1993). Asymptotics for the low-frequency ordinates of the periodogram of a long memory time series. *J. Time Ser. Anal.* **14** 455–472.

HURVICH, C. M., DEO, R. and BRODSKY, J. (1998). The mean squared error of Geweke and Porter-Hudak's estimator of the memory parameter of a long memory time series. *J. Time Ser. Anal.* **19** 19–46.

HURVICH, C. M. and ZEGER, S. (1987). Frequency domain bootstrap methods for time series. Research working paper, Dept. Statistics and Operations Research, New York Univ.

KAWATA, T. (1966). On the Fourier series of a stationary stochastic process. *Z. Wahrsch. Verw. Gebiete* **6** 224–245.

KAWATA, T. (1969). On the Fourier series of a stationary process. II. *Z. Wahrsch. Verw. Gebiete* **13** 25–38.

KÜNSCH, H. R. (1986). Discrimination between monotonic trends and long range dependence. *J. Appl. Probab.* **23** 1025–1030.

MANDELBROT, B. B. and VAN NESS, J. W. (1968). Fractional Brownian motions, fractional noise and applications. *SIAM Rev.* **10** 422–437.

MOULINES, E. and SOULIER, P. (1999). Broadband log-periodogram regression of times series with long-range dependence. *Ann. Statist.* **27** 1415–1439.

PHAM, D. T. and GUÉGAN, D. (1994). Asymptotic normality of the discrete Fourier transform of long memory time series. *Statist. Probab. Lett.* **21** 299–309.

ROBINSON, P. M. (1986). On the errors-in-variables problem for time series. *J. Multivariate Anal.* **19** 240–250.

ROBINSON, P. M. (1995). Log-periodogram regression of time-series with long-range dependence. *Ann. Statist.* **23** 1048–1072.

TAQQU, M. S. (1979). Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. Verw. Gebiete* **50** 53–83.

THOMSON, D. J. (1982). Spectrum estimation and harmonic analysis. *Proc. IEEE* **70** 1055–1096.

VELASCO, C. (1999a). Gaussian semiparametric estimation of non-stationary time series. *J. Time Ser. Anal.* **20** 87–127.

VELASCO, C. (1999b). Non-stationary log-periodogram regression. *J. Econometrics* **91** 325–371.

YAJIMA, Y. (1989). A central limit theorem of Fourier transforms of strongly dependent stationary processes. *J. Time Ser. Anal.* **10** 375–383.

ZURBENKO, I. G. (1979). On the efficiency of estimates of a spectral density. *Scand. J. Statist.* **6** 49–56.

ZURBENKO, I. G. (1986). *The Spectral Analysis of Time Series*. North-Holland, Amsterdam.

DEPARTMENT OF STATISTICS
SNEDECOR HALL
IOWA STATE UNIVERSITY
AMES, IOWA 50011-1210
E-MAIL: snlahiri@iastate.edu