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ON AN EMPIRICAL BAYES TEST FOR A NORMAL MEAN

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We exhibit an empirical Bayes test δ_n^* for the normal mean testing problem using a linear error loss. Under the condition that the critical point of a Bayes test is within some known compact interval, δ_n^* is shown to be asymptotically optimal and its associated regret Bayes risk converges to zero at a rate $O(n^{-1}(\ln\ n)^{1.5})$, where n is the number of past experiences available when the current component decision problem is considered. Under the same condition this rate is faster than the optimal rate of convergence claimed by Karunamuni.

1. Introduction. Let X denote a random variable arising from an $N(\theta, 1)$-distribution. We consider the problem of testing the hypotheses H_0: $\theta \leq \theta_0$ against H_1: $\theta > \theta_0$, with the linear error loss function $l(\theta, i) = i(\theta_0 - \theta)I(\theta_0 - \theta) + (1 - i)(\theta - \theta_0)I(\theta - \theta_0)$, where i denotes the action in favor of $H_i, i = 0, 1$ and $I(x) = 1(0)$ if $x > 0(x \leq 0)$. It is assumed that θ is a realization of a random parameter Θ having an unknown prior distribution G over the parameter space $\Omega = (-\infty, \infty)$. Then, X follows a marginal pdf $f_G(x) = \int f(x|\theta)\, dG(\theta)$, where $f(x|\theta) = \exp(-(x-\theta)^2/2)/\sqrt{2\pi}$.

We study the preceding decision problem via the empirical Bayes approach of Robbins (1956, 1964) when a sequence of past data is available. Interest in this problem is raised by Karunamuni (1996) where an empirical Bayes test for a normal mean is proposed and claimed to achieve the optimal rate of convergence $n^{-2(r-1)/(2r+1)}$ under certain regularities, where n is the number of past data available and r is a positive integer pertaining to some conditions. As we shall see, however, the optimal rate of convergence is faster than this rate.

This paper is organized as follows. In Section 2, we first derive a Bayes test δ_G for the underlying decision problem. Then, by mimicking the behavior of the Bayes test δ_G, we construct an empirical Bayes test δ_n^*, based on Fourier integral estimates of $f_G(x)$ and its derivative $f_G^{(1)}(x)$. We study the asymptotic optimality of δ_n^* in Section 3. For each prior distribution G in ϑ_A, which will be defined later, it is shown that δ_n^* is asymptotically optimal and its associated regret Bayes risk converges to zero at a rate $O(n^{-1}(\ln\ n)^{1.5})$, which is an improvement on $n^{-2(r-1)/(2r+1)}$. This improved rate holds uniformly over a subclass of priors $\vartheta_A(a^*, B^*, \phi^*)$, a subset of ϑ_A.

Finally, we note that Johns and Van Ryzin (1972) and Van Houwelingen (1976) have studied empirical Bayes tests δ_n^{JV} and δ_n^{VH}, respectively, for the continuous one-parameter exponential family and their results can be applied

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to the normal mean testing problem considered in this paper. Under certain regularity conditions, δ_n^{JV} and δ_n^{VH} are shown to be asymptotically optimal at rates $O(n^{-\delta r/(2r+1)})$ and $O(n^{-2r/(2r+3)}(\ln n)^2)$, where $0 < \delta < 1$ and r is a positive integer. For details, interested readers are referred to Johns and Van Ryzin (1972) and Van Houwelingen (1976).

2. Construction of the empirical Bayes test.

2.1. *A Bayes test.* Let χ be the sample space of the random variable X. A test δ is defined to be a mapping from χ into the interval [0,1], so that $\delta(x) = P\{\text{ accepting } H_1|X = x\}$, the probability of accepting H_1 when $X = x$ is observed. Since $X - \theta_0 \sim N(\theta - \theta_0, 1)$, without loss of generality, we may assume $\theta_0 = 0$. Let $R(G, \delta)$ denote the Bayes risk of the test δ. Thus,

$$
\begin{aligned}
R(G,\delta) &= -\int_\chi \delta(x)H_G(x)\,dx + C \\
&= -\int_\chi \delta(x)f_G(x)\phi_G(x)\,dx \ + C,
\end{aligned}
\tag{2.1}
$$

where $C = \int \theta I(\theta)\,dG(\theta)$,

$$
\begin{aligned}
H_G(x) &= \int \theta f(x|\theta)\,dG(\theta) = f_G^{(1)}(x) + xf_G(x), \\
\phi_G(x) &= E[\Theta|X = x] = H_G(x)/f_G(x).
\end{aligned}
\tag{2.2}
$$

Note that $\phi_G(x)$ is continuous and nondecreasing in x. A Bayes test, say δ_G, which minimizes the Bayes risk among a class of tests, is clearly given by the following: for each x in χ,

$$
\delta_G(x) = \begin{cases} 1, & \text{if } H_G(x) > 0, \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} 1, & \text{if } \phi_G(x) > 0, \\ 0, & \text{otherwise.} \end{cases}
\tag{2.3}
$$

The minimum Bayes risk is

$$
R(G, \delta_G) = -\int_\chi \delta_G(x)H_G(x)\,dx + C.
\tag{2.4}
$$

It is assumed that

$$
\lim_{x\to-\infty} \phi_G(x) < 0 < \lim_{x\to\infty} \phi_G(x).
\tag{2.5}
$$

Under the assumption of (2.5), the prior G is nondegenerate, and therefore $\phi_G(x)$ is strictly increasing in x, and there exists a point C_G such that $\phi_G(C_G) = 0$, $\phi_G(x) < 0$ for $x < C_G$ and $\phi_G(x) > 0$ for $x > C_G$. Hence, the Bayes test δ_G can be expressed as

$$
\delta_G(x) = 1 \quad \text{if } x > C_G \text{ and 0 otherwise.}
\tag{2.6}
$$

2.2. *The proposed empirical Bayes test.* Note that the Bayes test δ_G heavily depends on the unknown prior distribution G. We study empirical Bayes tests for this decision problem when a sequence of past data is available.

In the following, we investigate an empirical Bayes test for the normal mean based on Fourier integral estimates of $f_G(x)$ and $f_G^{(1)}(x)$ with kernels $K(x)$ and $K^{(1)}(x)$, respectively, where

$$K(x) = \frac{\sin x}{\pi x},\ K^{(1)}(x) = \frac{x \cos x - \sin x}{\pi x^2}. \tag{2.7}$$

The Fourier transformation of the kernel $K(\cdot)$ is $\psi_K(t) = I_{[-1,1]}(t)$. For the two kernels, $k_0 = \int |K(x)|^2\, dx < \infty$ and $k_1 = \int |K^{(1)}(x)|^2\, dx < \infty$.

Let $\psi_X(t)$ denote the characteristic function of the random variable X with respect to the marginal pdf $f_G(x)$. Then, $\psi_X(t) = \psi_G(t)\psi_N(t)$, where $\psi_G(t)$ is the characteristic function of the prior distribution G, and $\psi_N(t) = \exp(-t^2/2)$, the characteristic function of an $N(0, 1)$ distribution. By the Fourier inversion formula,

$$f_G(x) = \frac{1}{2\pi}\int \exp(-itx)\psi_X(t)\, dt, \tag{2.8}$$

$$f_G^{(1)}(x) = \frac{1}{2\pi}\int (-it)\exp(-itx)\psi_X(t)\, dt. \tag{2.9}$$

Equations (2.8) and (2.9) will be used for constructing estimates for $f_G(x)$ and $f_G^{(1)}(x)$, respectively.

Let $X_1, X_2, \ldots, X_n$ and X_{n+1} be iid random variables, having marginal pdf $f_G(x)$, where $\mathbf{X}(\mathrm{n}) = (X_1, \ldots, X_n)$ denote the n past data and $X_{n+1} = X$ is the present random observation. For each n, let $b \equiv b_n = (2 \ln[n/(\ln n)^t])^{-1/2}$, where $0 < t < 1.5$. Define

$$\hat{\psi}_n(t) = \frac{1}{n}\sum_{j=1}^{n} \exp(itX_j), \tag{2.10}$$

$$f_n(x) = \frac{1}{2\pi}\int \exp(-itx)\psi_K(tb)\hat{\psi}_n(t)\, dt, \tag{2.11}$$

$$f_n^{(1)}(x) = \frac{1}{2\pi}\int (-it)\exp(-itx)\psi_K(tb)\hat{\psi}_n(t)\, dt. \tag{2.12}$$

The definitions of $f_n(x)$ and $f_n^{(1)}(x)$ are motivated by the form of $f_G(x)$ and $f_G^{(1)}(x)$, respectively, see (2.8) and (2.9). This type of estimator has been used by Pensky (1997) for an empirical Bayes estimation problem dealing with the location-parameter model. Similar estimators have also been used by Fan

(1991) in the deconvolution problem. Note that $f_n(x)$and $f_n^{(1)}(x)$ can also be expressed as follows:

$$f_n(x) = \frac{1}{nb}\sum_{j=1}^{n} K\left(\frac{x - X_j}{b}\right),$$
$$f_n^{(1)}(x) = \frac{1}{nb^2}\sum_{j=1}^{n} K^{(1)}\left(\frac{x - X_j}{b}\right). \tag{2.13}$$

Define

$$H_n(x) = f_n^{(1)}(x) + x f_n(x). \tag{2.14}$$

We consider those prior distributions G for which $G \in \vartheta_A \equiv \{G \mid |C_G| \le A\}$. By the fact that $G \in \vartheta_A$ and by mimicking the behavior of the Bayes test given in (2.3), we propose an empirical Bayes test δ_n^* as follows. For each x in χ, define

$$\delta_n^*(x) = \begin{cases} 1, & \text{if either } x \ge A \text{ or } (|x| < A \text{ and } H_n(x) > 0), \\ 0, & \text{if either } x \le -A \text{ or } (|x| < A \text{ and } H_n(x) \le 0). \end{cases} \tag{2.15}$$

The Bayes risk of δ_n^* is

$$R(G, \delta_n^*) = -\int_\chi E_n[\delta_n^*(x)] H_G(x)\, dx + C, \tag{2.16}$$

where the expectation E_n is taken with respect to the probability measure P_n generated by $\mathbf{X}(n)$.

3. Asymptotic optimality of $\boldsymbol{\delta_n^*}$. For an empirical Bayes test δ_n, let $R(G, \delta_n)$ denote its associated Bayes risk. Since δ_G is a Bayes test, $R(G, \delta_n) - R(G, \delta_G) \ge 0$ for all n. The nonnegative regret Bayes risk, $R(G, \delta_n) - R(G, \delta_G)$, is often used as a measure of performance of the empirical Bayes test δ_n [see Johns and Van Ryzin (1972), Van Houwelingen (1976) and Karunamuni (1996)]. An empirical Bayes test δ_n is said to be asymptotically optimal, relative to the prior distribution G, at a rate of convergence $O(\alpha_n)$ if $R(G, \delta_n) - R(G, \delta_G) = O(\alpha_n)$, where $\{\alpha_n\}$ is a sequence of decreasing, positive numbers such that $\lim_{n\to\infty} \alpha_n = 0$.

Define $B_G = \sup_{|x|\le A}[f_G(x)/\phi_G^{(1)}(x)]$. From Lemma 4.3, $0 \le B_G < \infty$. Define $a_G = \min_{|x|\le A} f_G(x)$ and $\rho(b) = (1/\pi)(1 + bA)\exp(-1/(2b^2))$. Note that the function $\rho(b)$ is increasing in b for $b > 0$ and $\lim_{b\to 0} \rho(b) = 0$. We let $C_1 \equiv C_1(b) < C_G < C_2 \equiv C_2(b)$ be the point such that $-\phi_G(C_1) = \phi_G(C_2) = 2\rho(b)/a_G$. Since $\phi_G(x)$ is continuous and strictly increasing in x and $\phi_G(C_G) = 0$, one can see that C_1 and C_2 are such that $\lim_{b\to 0} C_1 = C_G = \lim_{b\to 0} C_2$. Suppose the value of A is large enough so that $-A < C_1 < C_2 < A$.

For the empirical Bayes test δ_n^*, we claim the following asymptotic optimality.

THEOREM 3.1. *Let δ_n^* be the empirical Bayes test constructed in Section 2, with $b = (2\ln[n/(\ln\ n)^t])^{-1/2}$, $0 < t < 1.5$. Then, for each G belonging to ϑ_A,*

$$\begin{aligned} R(G, \delta_n^*) - R(G, \delta_G) &\le \frac{B_G}{2nb^3}\left[\frac{1}{\tau(b, a_G, -\phi_G(-A))} + \frac{1}{\tau(b, a_G, \phi_G(A))}\right] + \frac{4\rho(b)}{a_G} \\ &= O\left(\frac{(\ln\ n)^t}{n}\right) + O\left(\frac{(\ln[n/(\ln\ n)^t])^{1.5}}{n}\right) \\ &= O\left(\frac{(\ln\ n)^{1.5}}{n}\right), \end{aligned}$$

where $\tau(b, a, x) = \dfrac{a^2}{8\{2k_1 + 2A^2k_0b^2 + b(1+bA)x\}}$.

PROOF. By the assumption that $G \epsilon \vartheta_A$, and from (2.4), (2.15) and (2.16), the regret Bayes risk of δ_n^* can be written as

$$\begin{aligned} &R(G, \delta_n^*) - R(G, \delta_G) \\ &\quad = \int_{-A}^{C_1} P_n\{H_n(x) > 0\}[-H_G(x)]\,dx + \int_{C_1}^{C_G} P_n\{H_n(x) > 0\}[-H_G(x)]\,dx \\ &\quad\quad + \int_{C_G}^{C_2} P_n\{H_n(x) \le 0\}H_G(x)\,dx + \int_{C_2}^{A} P_n\{H_n(x) \le 0\}H_G(x)\,dx \\ &\quad = I_n + II_n + III_n + IV_n. \end{aligned} \tag{3.1}$$

Here, $\int_a^b \equiv 0$ if $a = b$. Therefore, to investigate the asymptotic behavior of the regret Bayes risk, it suffices to study the asymptotic behavior for each of the four terms on the rhs of (3.1).

For $-A < x < C_1$, by the increasing property of $\phi_G(x)$ and the definition of the point C_1, we have

$$H_G(x) \le \phi_G(C_1) f_G(x) \le \phi_G(C_1) a_G = -2\rho(b). \tag{3.2}$$

Combining (3.2) and Lemma 4.1(c) yields that for $-A < x < C_1$,

$$E_n[H_n(x)] < H_G(x)/2 < -\rho(b). \tag{3.3}$$

Also, we note that $H_n(x) = 1/n \sum_{j=1}^n V(x, X_j, b)$ where

$$V(x, X_j, b) = \frac{1}{b^2}K^{(1)}\left(\frac{x - X_j}{b}\right) + \frac{x}{b}K\left(\frac{x - X_j}{b}\right). \tag{3.4}$$

$V(x, X_j, b)$, $j = 1, \dots, n$, are iid, bounded random variables such that

$$\left|V(x, X_j, b) - E_n V(x,\ X_j,\ b)\right| \le 2\left(\frac{1}{b^2} + \frac{A}{b}\right), \tag{3.5}$$

for $|x| \le A$ since $|K(t)| \le 1$ and $|K^{(1)}(t)| \le 1$ for all t. Also, from Lemma 4.2(c),

$$Var\,(V(x,\ X_j,\ b)) \le \big(2k_1 + 2A^2 k_0 b^2\big) b^{-3}. \tag{3.6}$$

Now, by Bernstein inequality and (3.3)–(3.6), it follows that for $-A < x < C_1$,

$$\begin{aligned}
P_n\{H_n(x) > 0\} &\le P_n\{H_n(x) - E_n H_n(x) > -H_G(x)/2\} \\
&\le \exp\left\{ -\frac{n[H_G(x)/2]^2/2}{\mathrm{Var}(V(x,\ X_1, b)) + [2(1/b^2 + A/b)/3]|H_G(x)/2|} \right\} \\
&\le \exp\left\{ -\frac{n}{8} \times \frac{H_G^2(x)}{[2k_1 + 2A^2 k_0 b^2]/b^3 + (1+bA)|H_G(x)|/(3b^2)} \right\} \\
&= \exp\left\{ -\frac{nb^3}{8} \times \frac{f_G^2(x)\phi_G^2(x)}{2k_1 + 2A^2k_0 b^2 + b(1+bA) f_G(x)|\phi_G(x)|/3} \right\} \\
&\le \exp\left\{ -\frac{nb^3}{8} \times \frac{a_G^2\phi_G^2(x)}{2k_1 + 2A^2k_0 b^2 + b(1+bA)[-\phi_G(-A)]} \right\} \\
&= \exp\big\{ -nb^3\tau(b,\ a_G, -\phi_G(-A))\phi_G^2(x)\big\}.
\end{aligned} \tag{3.7}$$

Replacing (3.7) into I_n, and from Lemma 4.3, it follows that

$$\begin{aligned}
I_n &\le \int_{-A}^{C_1} \exp\{-nb^3\tau(b, a_G, -\phi_G(-A))\phi_G^2(x)\}[-\phi_G(x)] f_G(x)\,dx \\
&\le B_G \int_{-A}^{C_1} \exp\{-nb^3\tau(b, a_G, -\phi_G(-A))\phi_G^2(x)\}[-\phi_G(x)]\phi_G^{(1)}(x)\,dx \\
&\le \frac{B_G}{2nb^3\tau(b, a_G, -\phi_G(-A))} \\
&= O\left(\frac{1}{nb^3}\right).
\end{aligned} \tag{3.8}$$

Following a proof analogous to the preceding discussion, we can obtain

$$IV_n \le \frac{B_G}{2nb^3\tau(b, a_G, \phi_G(A))} = O\left(\frac{1}{nb^3}\right). \tag{3.9}$$

Next, by the strictly increasing property of the function $\phi_G(x)$, and by the definition of the points C_1 and C_2, we have $-2\rho(b)/a_G = \phi_G(C_1) \le \phi_G(x) \le \phi_G(C_G) = 0$ for $C_1 \le x \le C_G$, and $2\rho(b)/a_G = \phi_G(C_2) \ge \phi_G(x) \ge \phi_G(C_G) = 0$ for $C_G \le x \le C_2$. Therefore,

$$II_n \le \int_{C_1}^{C_G} [-\phi_G(x)] f_G(x)\,dx \le \frac{2\rho(b)}{a_G} = O(\rho(b)). \tag{3.10}$$

Similarly,

$$III_n \le \frac{2\rho(b)}{a_G} = O(\rho(b)). \tag{3.11}$$

Combining (3.1) and (3.8)–(3.11) together, we conclude that

$$\begin{aligned}R(G,\delta_n^*)-R(G,\delta_G)&\le \frac{B_G}{2nb^3}\Big[\frac{1}{\tau(b,a_G,-\phi_G(-A))}\\&\qquad+\frac{1}{\tau(b,a_G,\phi_G(A))}\Big]+\frac{4\rho(b)}{a_G}\\&=O\Big(\frac{1}{nb^3}\Big)+O(\rho(b)).\end{aligned}\tag{3.12}$$

When $b=(2\ln[n/(\ln\ n)^t])^{-1/2}$, where $0<t<1.5$, a straightforward computation shows that

$$R(G,\delta_n^*)-R(G,\delta_G)=O\Big(\frac{(\ln\ n)^t}{n}\Big)+O\Big(\frac{(\ln[n/(\ln\ n)^t])^{1.5}}{n}\Big).$$

Note that this rate is close to and a little faster than the rate $O((\ln\ n)^{1.5}/n)$ for any t between 0 and 1.5. Hence, the proof of the theorem is complete. □

For two finite, positive numbers a^* and B^*, define a class of prior distributions $\vartheta_A(a^*,B^*,\phi^*)=\{G\epsilon\vartheta_A|a_G\ge a^*,B_G\le B^*,\phi_G(A)\le\phi^*,|\phi_G(-A)|\le\phi^*\}$. We consider the performance of the empirical Bayes test δ_n^* over $\vartheta_A(a^*,B^*,\phi^*)$. The following corollary is a direct consequence of Theorem 3.1.

COROLLARY 3.1. *Let δ_n^* be the empirical Bayes test constructed in Section* 2. *Then, under the conditions of Theorem* 3.1, *we have*

$$\sup_{G\epsilon\vartheta_A(a^*,B^*,\phi^*)}[R(G,\delta_n^*)-R(G,\delta_G)]\le\frac{B_G}{nb^3\tau(b,a^*,\phi^*)}+\frac{4\rho(b)}{a^*}=O\Big(\frac{(\ln\ n)^{1.5}}{n}\Big)$$

REMARK 3.1. Let G_0 be the prior distribution having the probability density $g_0(\theta)=C_\ell(1+\theta^2)^{-\ell}$, $-\infty<\theta<\infty$, where $1<\ell<1.5$. Define $w(\theta)=1/\sqrt{2\pi}\exp(-\theta^2/2)$, $-\infty<\theta<\infty$ and let $H(\theta)=C_\ell[w(\theta)-w(\theta+1)]$. Let $\varepsilon_n=n^{-1/(2k-4)}$, where $k>4$ is an integer. Define $g_n(\theta)=g_0(\theta)+\varepsilon_n^kH(\theta\varepsilon_n)$. For n being sufficiently large, say $n\ge N(k)$, $g_n(\theta)$ is a probability density. Denote its corresponding distribution by G_n. Let $\vartheta_n=\{G_0,G_n\}$. We also denote the associated critical point C_G by C_{G_0} and C_{G_n}, respectively. Since $g_0(\theta)$ is an even function, $C_{G_0}=0$. Also, for n sufficiently large, C_{G_n} is close to $C_{G_0}=0$. Therefore, $|C_{G_n}|\le A$ for some $A>0$ for all $n\ge N(k)$. Let $(a_{G_n},B_{G_n},\phi_{G_n}(-A),\phi_{G_n}(A))$ be the values of the pair $(a_G,B_G,\phi_G(-A),\phi_G(A))$ associated with the distribution G_n. Define $a_0=\inf_{n\ge N(k)}a_{G_n}$, $B_0=\sup_{n\ge N(k)}B_{G_n}$ and $\phi_0=\sup_{n\ge N(k)}\max(|\phi_{G_n}(-A)|,\phi_{G_n}(A))$. A straightforward computation shows that $a_0>0$, $B_0<\infty$ and $\phi_0<\infty$. Therefore, for n being sufficiently large, $\vartheta_n\subset\vartheta_A(a_0,B_0,\phi_0)$. Then, by Corollary 3.1, we have

$$\sup_{G\epsilon\vartheta_n}[R(G,\delta_n^*)-R(G,\delta_G)]=O(n^{-1}(\ln\ n)^{1.5}).\tag{3.13}$$

Note that this rate is faster than the lower bound convergence rate of $dn^{-2(k-4)/(2k-4)}$ that was asserted to hold for this example in Karunamuni (1996).

4. Auxiliary results. The following lemmas are helpful for presenting a concise proof for Theorem 3.1. Since the proofs are simple algebraic computations, the details are omitted here.

LEMMA 4.1. (a) $|E_n f_n(x) - f_G(x)| \le b/\pi \exp(-1/(2b^2))$.

(b) $|E_n f_n^{(1)}(x) - f_G^{(1)}(x)| \le 1/\pi \exp(-1/(2b^2))$.

(c) *For* $|x| \le A$, $|EH_n(x) - H_G(x)| \le \rho(b)$, *where* $\rho(b) = 1/\pi(1 + bA) \times \exp(-1/2b^2)$.

LEMMA 4.2. (a) $\mathrm{Var}(K((x - X_j)/b)) \le bk_0$.

(b) $\mathrm{Var}(K^{(1)}((x - X_j)/b)) \le bk_1$.

(c) *For* $|x| \le A$,

$$\begin{aligned}\mathrm{Var}(V(x, X_j, b)) &\le \frac{2}{b^4}\mathrm{Var}\left(K^{(1)}\left(\frac{x - X_j}{b}\right)\right) + \frac{2A^2}{b^2}\mathrm{Var}\left(K\left(\frac{x - X_j}{b}\right)\right).\\ &\le (2k_1 + 2A^2 k_0 b^2)b^{-3}.\end{aligned}$$

LEMMA 4.3. *Let* $B_G = \sup_{|x|\le A}[f_G(x)/\phi_G^{(1)}(x)]$. *Then*, $0 < B_G < \infty$.

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