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AN AGE-DEPENDENT MODEL WITH PARENTAL SURVIVAL

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We consider an age-dependent model which not only allows the generating function to be age-dependent, but also allows the parent to reproduce several times during its lifetime. By using the notion of $\vee$-space-time harmonic functions, we study the behavior of Z_t, the number of particles alive at time t, in the supercritical case. In particular we obtain results which are analogous to the classically known results for the Bellman-Harris model; in fact, we obtain convergence in probability.

0. Introduction. In this paper we consider an age-dependent model X which allows the parent to reproduce several times during its lifetime. The generating function for newborn progeny is also allowed to be dependent upon age. We shall show that for this process, all the classical results are valid. In particular, if m is the mean number of progeny produced during the lifetime of a particle, then the extinction probability $q = 1$ iff $m \leqq 1$. The rest of the paper is concerned with the supercritical case, $m > 1$. Then, under a finite second moment assumption, $W_t^* = Z_t/ce^{\lambda t}$ converges in mean square and a.s. to a random variable W^* satisfying $E_0(W^*) = 1$. Here Z_t is the number of particles alive at time t, λ is the Malthusian parameter and c is an appropriate normalizing constant. Using the method of space-time harmonic functions we exhibit a function ϕ such that $W_t = e^{-\lambda t}\check{\phi}(X_t)$ is a martingale and being nonnegative it converges a.s. to a nonnegative random variable W. If ϕ is bounded, we show that under a technical condition, $E_0(W) = 1$ iff $\sum (k \log k)\tilde{p}_k < \infty$. Lastly, we make use of a majorization lemma to infer something about the behavior of W_t^*. In particular, it will follow that if $0 < a \leqq \phi \leqq b < \infty$, then W_t^* converges in distribution to a random variable W^* satisfying $E_0(W^*) = 1$ iff $E_0(W) = 1$.

1. The model. We use the framework of Ikeda, Nagasawa and Watanabe to set our model up rigorously. Let G be a (right-continuous) probability distribution function on $[0, \infty)$ satisfying $G(0+) = G(0) = 0$. We define $T = \inf\{t \geqq 0: G(t) = 1\}$ ($\inf \emptyset = \infty$) and take $S = [0, T)$ as our base space. Let α and β be two nonnegative measurable functions on S such that $\alpha + \beta \equiv 1$. Also, let $\pi_i(x, s) = \sum_{k=0}^{\infty} p_{ik}(x)s^k$, $s \in [0, 1]$, $i = 1, 2$, be two age-dependent generating functions with p_{ik} measurable functions on S. If $T < \infty$, we allow α, β and p_{ik} to be defined on $[0, T]$ but assume that $\alpha(T) = 0$.

Then, according to [3], there exists a unique right-continuous strong Markov process X on $\hat{\mathbf{S}}$ possessing the branching property and satisfying the following

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S-equation. If f is a measurable function on S such that $\sup_{x\in S}|f(x)| \leqq 1$, then $u(x, t) = E_x[\hat{f}(X_t)]$ is a solution of

$$(1.1) \qquad u(x, t) = f(x+t)[1 - G_x(t)] + \int_{(0,t]} \beta(x+y)\pi_2[x+y, u(0, t-y)]\, dG_x(y) + \int_{(0,t]} \alpha(x+y)\pi_1[x+y, u(0, t-y)]u(x+y, t-y)\, dG_x(y)$$

($x \in S$, $t \geqq 0$). Furthermore, if $f \geqq 0$, u is the minimal solution in the class of all solutions U satisfying $0 \leqq U \leqq 1$. Here G_x is the distribution function given by

$$G_x(t) = [1 - G(x)]^{-1}[G(x+t) - G(x)] .$$

As usual, $\hat{\mathbf{S}} = \mathbf{S} \cup \{\Delta\}$ is the one point-compactification of $\mathbf{S}$, $\mathbf{S} = \bigcup_{n=0}^{\infty} S^n$, S^n is the quotient topological space of the n-fold Cartesian product of S under the equivalence relation of permutation for $n \geqq 1$, and S^0 is the set consisting of an isolated point ∂. (∂ corresponds to extiction and Δ corresponds to explosion.) Also

$$\begin{aligned} \hat{f}(\mathbf{x}) &= 1 && \text{if } \ \mathbf{x} = \partial \\ &= \textstyle\prod_{i=1}^{n} f(x_i) && \text{if } \ \mathbf{x} = [x_1, \cdots, x_n] \in S^n \\ &= 0 && \text{if } \ \mathbf{x} = \Delta . \end{aligned}$$

Recall that the branching property is the statement

$$T_t\hat{f} = \widehat{(T_t\hat{f})\,|\,S} ,$$

where T_t is the induced semigroup of X.

Intuitively this model can be described as follows. An object of age zero waits a random length of time τ_1 according to the distribution function G. At this time, it either continues living with probability $\alpha(\tau_1)$ or dies with probability $\beta(\tau_1)$. In the former case it gives birth according to the age-dependent generating function $\pi_1(\tau_1, s)$; in the latter case it gives birth according to the age-dependent generating function $\pi_2(\tau_1, s)$ before dying. If the original object has not died by time τ_1, it waits another random length of time τ_2 according to the distribution function G_{τ_1}. In this case, the relevant parameters are $\alpha(\tau_1 + \tau_2)$, $\beta(\tau_1 + \tau_2)$, and $\pi_i(\tau_1 + \tau_2, s)$. The parent object continues in this manner until it dies. All newborn progeny exhibit the same behavior independently of one another and of the parent. This model includes as special cases then the classical Bellman-Harris age-dependent model and the age-dependent birth-and-death model of Kendall.

Another integral equation of improtance is the following. Let g be any bounded (or nonnegative) measurable function on S. Then $v(x, t) = E_x[\check{g}(X_t)]$ satisfies

$$(1.2) \qquad v(x, t) = g(x+t)[1 - G_x(t)] + \int_{(0,t]} m(x+y)v(0, t-y)\, dG_x(y) + \int_{(0,t]} \alpha(x+y)v(x+y, t-y)\, dG_x(y) ,$$

where

$$m(y) = \alpha(y)m_1(y) + \beta(y)m_2(y) ,$$

$$m_i(y) = \pi_i'(y, 1) = \textstyle\sum_{k=0}^{\infty} kp_{ik}(y)$$

and

$$\begin{aligned}\check{g}(\mathbf{x}) &= 0 && \text{if} \quad \mathbf{x} = \partial \quad \text{or} \quad \Delta \\ &= \textstyle\sum_{i=1}^{n} g(x_i) && \text{if} \quad \mathbf{x} = [x_1, \cdots, x_n] \in S^n .\end{aligned}$$

Furthermore, if $g \geqq 0$, v is the minimal solution in the class of all solutions V satisfying $V \geqq 0$.

We note that (1.1) is nonlinear whereas (1.2) is linear.

It is convenient to introduce the following notation.

$$\gamma(x) = \int_{(0,x]} \alpha(z)[1 - G(z)]^{-1} \, dG(z) \tag{1.3}$$

$$m = \int_{(0,T]} m(y) e^{\gamma(y)} \, dG(y) . \tag{1.4}$$

We are now ready to state our main assumptions.

(1.6) ASSUMPTIONS.

(i) G is continuous
(ii) $\lim_{x \uparrow T} [1 - G(x)] e^{\gamma(x)} = 0$
(iii) $m < \infty$.

The first assumption is a technical assumption which facilitates computations. The second assures us that the parent eventually dies. The last gurantees no explosion (in finite time).

We conclude this section with several facts that we shall have occasion to use.

(1.7) If f is any bounded (or nonnegative) measurable function on S and $x \in S$, then

$$\int_0^T f(x + y) \, dG_x(y) = [1 - G(x)]^{-1} \int_x^T f(y) \, dG(y) .$$

(1.8) If F is any continuous monotone function on S and ϕ is any function absolutely continuous with respect to Lebesgue measure on the range of F, then

$$\int_a^b \phi'(F(x)) \, dF(x) = \phi(F(b)) - \phi(F(a))$$

for all $0 \leqq a \leqq b < T$.

$$\beta e^{\gamma} = -\frac{d}{dG} [(1 - G) e^{\gamma}] \qquad \text{a.s. } (dG) . \tag{1.9}$$

2. Reduction. It is well known that the behavior of branching processes is intimately connected with the behavior of the solutions to certain renewal equations. Classical renewal theory has been an indispensable aid in the study of such solutions. Although the renewal equations that we are led to, namely (1.1) and (1.2), are not of the classical type, this causes no difficulties as the next theorem shows.

Consider the following two integral equations.

$$\begin{aligned} U(t) = {} & [1 - G(t)] f(t) \exp\{\textstyle\int_0^t \alpha(z) \pi_1[z, U(t - z)][1 - G(z)]^{-1} \, dG(z)\} \\ & + \textstyle\int_0^t \beta(y) \pi_2[y, U(t - y)] \exp\{\int_0^y \alpha(z) \pi_1[z, U(t - z)] \\ & \times [1 - G(z)]^{-1} \, dG(z)\} \, dG(y) \end{aligned} \tag{2.1}$$

and

$$V(t) = [1 - G(t)]e^{\gamma(t)}g(t) + \int_0^t e^{\gamma(y)}m(y)V(t-y)\,dG(y) \tag{2.2}$$

where $0 \leqq f \leqq 1$, $0 \leqq g$.

(2.3) THEOREM. *If* $\underline{u}$ *and* $\underline{U}$ ($\underline{v}$ *and* $\underline{V}$) *are the minimal solutions of* (1.1) *and* (2.1) ((1.2) *and* (2.2)) *respectively, then* $\underline{u} = \hat{E}\underline{U}$ ($\underline{v} = \check{E}\underline{V}$) *where* $\hat{E}(\check{E})$ *is the operator defined by* (2.4) ((2.6)).

PROOF. Let $0 \leqq U(t) \leqq 1$ be a solution of (2.1). We define in extension operator $\hat{E}$ by

$$\begin{aligned}(\hat{E}U)(x, t) = {} & [1 - G_x(t)]f(x+t) \\ & \times \exp\{\textstyle\int_x^{x+t} \alpha(z)\pi_1[z, U(x+t-z)][1-G(z)]^{-1}\,dG(z)\} \\ & + [1-G(x)]^{-1}\textstyle\int_x^{x+t} \beta(y)\pi_2[y, U(x+t-y)] \\ & \times \exp\{\textstyle\int_x^y \alpha(z)\pi_1[z, U(x+t-z)][1-G(z)]^{-1}\,dG(z)\}\,dG(y)\end{aligned} \tag{2.4}$$

for $x \in S$, $t \geqq 0$. It is easy to see that $\hat{E}U$ is a solution of (1.1).

Now suppose that $0 \leqq u(x, t) \leqq 1$ is a solution of (1.1). We will show that $u(0, t)$ is a supersolution of (2.1); by this we mean that $u(0, t)$ satisfies (2.1) if we replace the equal sign ($=$) by the inequality sign ($\geqq$). Since (1.1) is valid for all $x \in S$, it is valid for $x = 0$. In the resulting expression there is one term in the integrand, namely $u(y, t-y)$, which depends on $x \neq 0$. For this term we substitute the right-hand side of (1.1). Continuing in this manner, we obtain after n substitutions the expression

$$\begin{aligned}u(0, t) = {} & f(t)[1 - G(t)] \\ & \times \left\{\textstyle\sum_{k=0}^n \frac{1}{k!}\left(\int_0^t \alpha(z)\pi_1[z, u(0, t-z)][1-G(z)]^{-1}\,dG(z)\right)^k\right\} \\ & + \textstyle\int_0^t \beta(y)\pi_2[y, u(0, t-y)] \\ & \times \left\{\textstyle\sum_{k=0}^n \frac{1}{k!}\left(\int_0^y \alpha(z)\pi_1[z, u(0, t-z)][1-G(z)]^{-1}\,dG(z)\right)^k\right\} dG(y) \\ & + R_n(t)\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}R_n(t) = {} & \textstyle\int_0^t \alpha(y)\pi_1[y, u(0, t-y)]u(y, t-y) \\ & \times \frac{1}{n!}\left(\textstyle\int_0^y \alpha(z)\pi_1[z, u(0, t-z)][1-G(z)]^{-1}\,dG(z)\right)^n dG(y)\,.\end{aligned}$$

If we could show that for each fixed t, $R_n(t) \to 0$ as $n \to \infty$, we could conclude that $u(0, t)$ was a solution of (2.1). This can be done for each t such that $\gamma(t) < \infty$, but it is not clear otherwise. Instead we observe that since $R_n \geqq 0$, we have the inequality $\geqq$ in (2.5) if we throw that term away. Letting $n \to \infty$ we deduce that $u(0, t)$ is a supersolution of (2.1). Actually, in the same manner, we can conclude that $u(x, t) \geqq [\hat{E}u(0, \cdot)](x, t)$ for all $x \in S$, $t \geqq 0$.

So now let $\underline{u}(x, t)$ be the minimal solution of (1.1) and let $\underline{U}(t)$ be the minimal solution of (2.1). Then we have $\underline{u} \leqq \hat{E}\underline{U}$. On the other hand, since the minimal solutions are obtained by the usual iteration procedure starting with $U^{(0)} = 0$, it follows that $\underline{u}(0, t) \geqq U(t)$. Consequently, $\underline{u} \geqq \hat{E}(\underline{u}(0, \cdot)) \geqq \hat{E}\underline{U}$ and we are done.

The proof of the corresponding statement for $\underline{v}$ and $\underline{V}$ is similar except now the extension operator $\check{E}$ is defined by

$$(2.6) \qquad (\check{E}V)(x, t) = [1 - G_x(t)]e^{\gamma(x+t)-\gamma(x)}g(x + t) + [1 - G(x)]^{-1}e^{-\gamma(x)} \int_x^{x+t} m(y)e^{\gamma(y)}V(x + t - y)\, dG(y) .$$

As an immediate consequence of the above we have the following result.

(2.7) THEOREM. *There is no explosion (in finite time) if $m < \infty$.*

PROOF. Let $Z_t = \check{1}(X_t)$ be the number of particles alive at time t and set $u(x, t) = E_x[\hat{1}(X_t)] = P_x(Z_t < \infty)$. According to Theorem (2.3), $u(t) = u(0, t)$ is the minimal solution of (2.1) with $f \equiv 1$; furthermore, it is easily seen that $U(t) \equiv 1$ is also a solution. Since we have uniqueness of solution when $m < \infty$, we deduce that $P_0(Z_t < \infty) = 1$ for all $t \geqq 0$, and hence $P_0(Z_t < \infty$, all $t \geqq 0) = 1$.

3. Extinction problem. If $Z_t = \check{1}(X_t)$ denotes the population size at time t, then we know that the probability of extinction before time t, $q(x, t) = E_x[\hat{0}(X_t)] = P_x(Z_t = 0)$, is the minimal solution of (1.1) with $f = 0$. Letting $t \to \infty$, we see that extinction probability $q(x) = P_x(Z_t = 0$ for some $t \geqq 0)$ satisfies

$$(3.1) \qquad q(x) = \int_0^T \beta(x + y)\pi_2[x + y, q(0)]\, dG_x(y) + \int_0^T \alpha(x + y)\pi_1[x + y, q(0)]q(x + y)\, dG_x(y) .$$

As usual we have the following characterization.

(3.2) THEOREM. *$q(x)$ is the minimal solution of* (3.1).

PROOF. Set $\tau_0 = 0$ and for $k \geqq 1$, let τ_k be the kth split time. Let $q^{(n)}(x) = P_x$ (there are at most n splits). Then, using the strong Markov property, we have the inequality

$$q^{(n+1)}(x) \leqq \int_0^T \beta(x + y)\pi_2[x + y, q^{(n)}(0)]\, dG_x(y) + \int_0^T \alpha(x, y)\pi_1[x + y, q^{(n)}(0)]q^{(n)}(x + y)\, dG_x(y) .$$

Now note that if $\tilde{q}(x)$ is any other nonnegative solution of (3.1), then $\tilde{q}(x) \geqq q^{(0)}(x) \equiv 0$. By induction it follows that $\tilde{q}(x) \geqq q^{(n)}(x)$ and hence $\tilde{q}(x) \geqq q(x)$ since $q^{(n)} \uparrow q$ clearly.

From the results of Section 2, we also know that $q(0, t)$ is the minimal solution of (2.1) with $f = 0$. So letting $t \uparrow \infty$ it follows that

$$(3.3) \qquad q = P_0(Z_t = 0 \text{ for some } t \geqq 0) = q(0)$$

is a solution of

$$(3.4) \qquad q = \int_0^T \beta(y)\pi_2[y, q] \exp\{\int_0^y \alpha(z)\pi_1[z, q][1 - G(z)]^{-1}\, dG(z)\}\, dG(y) .$$

Using the same techniques as in Section 2, we arrive at the following conclusion.

(3.5) THEOREM. *q given by* (3.3) *is the minimal solution of* (3.4). *Furthermore*

$$q(x) = [1 - G(x)]^{-1} \int_x^T \beta(y)\pi_2[y, q] \exp\{\int_x^y \alpha(z)\pi_1[z, q][1 - G(z)]^{-1} dG(z)\} dG(y) .$$

Consider now the function H defined by

$$H(s) = \int_0^T \beta(y)\pi_2[y, s] \exp\{\int_0^y \alpha(z)\pi_1[z, s][1 - G(z)]^{-1} dG(z)\} dG(y) \tag{3.6}$$

for $s \in [0, 1]$. It is clear that H is a power series with nonnegative coefficients. Furthermore, from (1.9),

$$H(1) = 1 - \lim_{x \uparrow T} [1 - G(x)]e^{r(x)} = 1 .$$

Consequently H is a generating function; in fact, it is the generating function of the underlying Galton–Watson process. By the standard arguments, we know that the smallest solution of $s = H(s)$ is determined by the value of $H'(1)$. Differentiating equation (3.6) and using Fubini it is easy to see that $H'(1) = m$.

(3.7) THEOREM. *The extinction probability q is the smallest root, $0 \leqq s \leqq 1$, of $s = H(s)$; moreover, $q = 1$ iff $m \leqq 1$ (except in the degenerate case $H(s) \equiv s$).*

In the remainder of this paper we shall only consider the supercritical case $m > 1$.

(3.8) ASSUMPTION. $m > 1$.

4. Behavior of Z_t. Let $M(t) = E_0[Z_t] = E_0[\check{1}(X_t)]$. Then M is the minimal solution of the renewal equation

$$M(t) = [1 - G(t)]e^{r(t)} + \int_0^t m(y)e^{r(y)}M(t - y)\, dG(y) . \tag{4.1}$$

Using the standard results from renewal theory we deduce

(4.2) THEOREM. *$M(t) \sim ce^{\lambda t}$ as $t \to \infty$, where*

$$c = \int_0^\infty[1 - G(t)]e^{r(t)}e^{-\lambda t}\, dt / \int_0^\infty te^{-\lambda t}e^{r(t)}m(t)\, dG(t) ,$$

and λ is the Malthusian parameter given by the unique positive root of

$$\int_0^T e^{-\lambda y}e^{r(y)}m(y)\, dG(y) = 1 .$$

In order to study mean square convergence, we consider the joint second moment of $Z(t)$ and $Z(t + \tau)$, $\tau \geqq 0$. Let

$$F_2(s_1, s_2; t, \tau) = E_0[s_1^{Z_t}s_2^{Z_{t+\tau}}] \qquad \text{for} \quad s_1, s_2 \in [0, 1] .$$

Using the Markov and branching property of X we rewrite F_2 as

$$F_2(s_1, s_2; t, \tau) = E_0[\hat{s}_1(X_t)\hat{s}_2(X_{t+\tau})] = E_0[\hat{s}_1(X_t)E_{X_t}[\hat{s}_2(X_\tau)]]$$
$$= E_0[\widehat{s_1F_1(s_2; \cdot, \tau)}(X_t)]$$

where

$$F_1(s; x, t) = E_x[\hat{s}(X_t)] = E_x[s^{Z_t}] .$$

Consequently, F_2 is the minimal solution of a renewal equation of the form (2.1). The same analysis as in Harris ([2], page 140) leads us to conclude that if $H''(1) < \infty$, $M_2(t, \tau) = E_0[Z_t Z_{t+\tau}]$ is the unique solution bounded on finite t-intervals of

$$\begin{aligned} M_2(t, \tau) = \textstyle\int_0^t m(y)e^{r(y)}M_2(t-y, \tau)\, dG(y) + [1 - G(t)]e^{r(t)}M_1(t, \tau) \\ + \textstyle\int_0^t e^{r(y)}\{\pi''(y, 1)M_1(0, t+\tau-y)M_1(0, t-y) \\ + \alpha(y)m_1(y)M_1(0, t-y)M_1(y, t+\tau-y) \\ + \alpha(y)m_1(y)M_1(y, t-y)M_1(0, t+\tau-y)\}\, dG(y) \end{aligned} \tag{4.3}$$

where $M_1(x, t) = E_x[Z_t]$ and $\pi''(y, 1) = \alpha(y)\pi_1''(y, 1) + \beta(y)\pi_2''(y, 1)$. Now it follows from (2.3) and (4.2) that

$$M_1(x, t) \sim c\phi(x)e^{\lambda t} \qquad \text{as } t \to \infty$$

where

$$\phi(x) = e^{-r(x)}e^{\lambda x}[1 - G(x)]^{-1} \textstyle\int_x^T m(y)e^{r(y)}e^{-\lambda y}\, dG(y)\,. \tag{4.4}$$

Applying renewal theory to (4.3) we conclude the following result.

(4.5) THEOREM. *If*

$$\begin{aligned} H''(1) = \textstyle\int_0^T \pi''[y, 1]e^{r(y)}\, dG(y) \\ + 2\textstyle\int_0^T m(y)e^{r(y)} \int_0^y \alpha(z)m_1(z)[1 - G(z)]^{-1}\, dG(z)\, dG(y) < \infty\,, \end{aligned}$$

then

$$M_2(t, \tau) \sim c^2Ke^{\lambda(2t+\tau)}/(1 - \bar{m}) \qquad \textit{as} \quad t \to \infty\,, \quad \textit{uniformly in} \quad \tau > 0\,,$$

where

$$\bar{m} = \textstyle\int_0^T m(y)e^{r(y)}e^{-2\lambda y}\, dG(y)$$

and

$$K = \textstyle\int_0^T e^{r(y)}e^{-2\lambda y}\{\pi''[y, 1] + 2\alpha(y)m_1(y)\phi(y)\}\, dG(y)\,.$$

Following Harris [2] and Jagers [4], we deduce

(4.6) THEOREM. *If* $1 < m < \infty$ and $H''(1) < \infty$, *then* $Z_t/ce^{\lambda t}$ *converges in mean square and* a.s. *to a random variable* W^* *satisfying* $E_0[W^*] = 1$, $\mathrm{Var}\,(W^*) = (K/(1 - \bar{m})) - 1 > 0$.

Many times it is also of interest to study $Z(x, t)$, the number of particles alive at time t and of age $\leq x$. Since $Z(x, t) = \textstyle\sum 1_{[0,x]}(X_t)$, a similar analysis leads us to

(4.7) THEOREM. *If* $1 < m < \infty$, $E_0[Z(x, t)] \sim c(x)e^{\lambda t}$ *as* $t \to \infty$ *where*

$$c(x) = \textstyle\int_0^x [1 - G(t)]e^{-\lambda t}e^{r(t)}\, dt / \int_0^T te^{-\lambda t}e^{r(t)}m(t)\, dG(t)$$

for $0 \leq x \leq T$. *In addition, if* $H''(1) < \infty$, $Z(x, t)/ce^{\lambda t}$ *converges in mean square to* $A(x)W^*$ *where* $A(x)$ *is the limiting age distribution given by* $A(x) = c(x)/c$.

(4.8) REMARK. Presumably one can also obtain a.s. convergence as in Harris ([2], page 154]).

Note that for $x = T$, $c(T) = c$.

5. $\vee$-space-time harmonic functions. Following Savits [5] we define a $\vee$-space-time harmonic function of X as any function h satisfying

$$h(x, \sigma) = E_x[\check{h}(X_t, t + \sigma)]$$

for all $x \in S$, $t \geqq 0$, $\sigma \geqq 0$. The study of such functions is of interest since they yield martingales.

In this section, we shall state our results without giving proofs since the proofs are essentially only slight modifications of the corresponding results found in [6].

(5.1) THEOREM. *$h(x, t) = e^{-\lambda t}\phi(x)$ is a $\vee$-space-time harmonic function, where*

$$\phi(x) = e^{\lambda x}e^{-\gamma(x)}[1 - G(x)]^{-1} \int_x^\infty m(y)e^{-\lambda y}e^{\gamma(y)}\, dG(y)\,.$$

Consequently, $W_t = e^{-\lambda t}\check{\phi}(X_t)$ is a nonnegative martingale (with respect to each P_x) and converges a.s. *to a nonnegative random variable W satisfying $E_x[W] \leqq \phi(x)$.*

(5.2) REMARKS.

(i) It is interesting to observe that $\int_0^\infty \phi(x)\, dA(x) = 1/c$; i.e., the average value of ϕ with respect to the age-distribution is the reciprocal of the constant c.

(ii) For the classical Galton–Watson process and for the age-dependent birth and death process of Kendall having constant birth and death rates, $\phi \equiv 1$. Consequently $e^{-\lambda t}Z_t$ is a martingale in these cases.

(5.3) THEOREM. *If $H''(1) < \infty$ and $e^{-2\lambda t}e^{\gamma(t)}[1 - G(t)]\phi^2(t)$ is bounded then $\{W_t\}_{t \geqq 0}$ is a square integrable matingale. Hence $W_t \to W$* a.s. *and in L^2; moreover, $E_x[W] = \phi(x)$ and $P_x(W = 0) = q(x)$.*

(5.4) REMARK. If in the above, $e^{-2\lambda t}e^{\gamma(t)}[1 - G(t)]\phi^2(t) \to 0$ as $t \to \infty$, then one can show $E_0[(W_t - W_t^*)^2] \to 0$ as $t \to \infty$ where $W_t^* = Z_t/ce^{\lambda t}$. Consequently $W = W^*$ a.s. in this case.

We now set $\pi(y, s) = \alpha(y)\pi_1(y, s) + \beta(y)\pi_2(y, s)$. Then

$$\pi(y, s) = \sum_{k=0}^\infty p_k(y)s^k$$

where $p_k(y) = \alpha(y)p_{1k}(y) + \beta(y)p_{2k}(y)$. Suppose

$$d = \int_0^T e^{-\lambda y}e^{\gamma(y)}\, dG(y) < \infty$$

and set

$$\tilde{p}_k = d^{-1} \int_0^T p_k(y)e^{-\lambda y}e^{\gamma(y)}\, dG(y)\,.$$

(5.5) THEOREM. *Suppose $d < \infty$. If $\sum_{k=2}^\infty (k \log k)\tilde{p}_k = \infty$, $W = 0$* a.s. *If (i) $\int_0^T e^{-2\lambda y}e^{\gamma(y)}\alpha(y)m_1(y)\phi(y)\, dG(y) < \infty$ and (ii) $e^{-2\lambda t}e^{\gamma(t)}[1 - G(t)]\phi^2(t)$ is bounded, then W is nontrivial provided $\sum_2^\infty (k \log k)\tilde{p}_k < \infty$; furthermore, $E_0[W] = 1$, $P_0(W = 0) = q$ and W has a continuous density on $(0, \infty)$.*

(5.6) REMARK. If ϕ is bounded, then (i) and (ii) above are automatically satisfied.

6. Deductions. In this section we shall see how much information about

$W_t^* = Z_t/ce^{\lambda t}$ can be inferred from W_t. Set

$$a = \inf_{x \in S} \phi(x)$$
$$b = \sup_{x \in S} \phi(x) .$$

Then we have the inequality

$$(ac) W_t^* \leqq W_t \leqq (bc) W_t^* . \tag{6.1}$$

This leads to the following

(6.2) PROPOSITION. $(bc)^{-1} W \leqq \liminf_{t \uparrow \infty} W_t^* \leqq \limsup_{t \uparrow \infty} W_t^* \leqq (ac)^{-1} W$.

In particular, then,

(i) if $W = 0$ a.s. and $a > 0$, $W_t^* \to 0$ a.s.
(ii) if $0 < a \leqq b < \infty$, $W_t^* \to 0$ a.s. iff $W_t \to 0$ a.s.

Now let $\psi(u, t) = E_0[e^{-uW_t}]$ be the Laplace transform of W_t and $\psi(u) = E_0[e^{-uW}]$ be the Laplace transform of W. Since $E_0[e^{-uW_t}] = E_0[[\exp - ue^{-\lambda t}\phi(\cdot)](X_t)]$ and $W_t \to W$ a.s., we see from Theorem (2.3) that $\psi(u)$ is a solution of the functional equation

$$\psi(u) = \int_0^T \beta(y)\pi_2[y, \psi(ue^{-\lambda y})] \times \exp\{\int_0^y \alpha(z)\pi_1[z, \psi(ue^{-\lambda z})][1 - G(z)]^{-1} dG(z)\} dG(y) \tag{6.3}$$

for all $u \geqq 0$.

This functional equation will play an important role in our subsequent discussion. Firstly we note the following uniqueness result.

(6.4) LEMMA. *For every $\theta \in [0, \infty)$, there is at most one solution ψ of* (6.3) *satisfying*

(i) $0 \leqq \psi \leqq 1$, $\psi(0) = 1$
(ii) $[1 - \psi(u)]/u \to \theta$ *as* $u \downarrow 0$.

PROOF. A slight modification of the proof given in Athreya ([1], Theorem 1, page 748) works.

For $0 \leqq x \leqq T$ we set $W^*(x, t) = Z(x, t)/ce^{\lambda t}$ and define

$$a(x) = \inf_{0 \leqq y \leqq x} \phi(y) .$$

Note that $Z(T, t) = Z(t)$ and $a(T) = a$. Let us now fix x and set $\theta(t) = E_0[W^*(x, t)]$, $\xi(u, t) = E_0[e^{-uW^*(x,t)}]$, $I(u, t) = u^{-1}E_0[p(uW^*(x, t))]$ and $K(u) = \limsup_{t \to \infty} \sup_{s \geqq 0} u^{-1}|\xi(u, t + s) - \xi(u, t)|$ for $u > 0$, where $p(x) = e^{-x} + x - 1$, $x \geqq 0$. If $a(x) > 0$, we have that $W^*(x, t) \leqq kW_t$ for $k = [ca(x)]^{-1}$. Since $p \geqq 0$ and is nondecreasing, $0 \leqq I(u, t) \leqq u^{-1}E_0[p(ukW_t)] = kH(uk, t)$, where $H(u, t) = u^{-1}E_0[p(uW_t)]$. Consequently, if $E_0(W) = 1$,

$$\lim_{u \downarrow 0} \limsup_{t \to \infty} I(u, t) = 0 . \tag{6.5}$$

Since

$$u^{-1}|\xi(u, t + s) - \xi(u, t)| \leqq I(u, t + s) + I(u, t) + |\theta(t + s) - \theta(t)| ,$$

it follows that

(6.6) $$\lim_{u \downarrow 0} K(u) = K(0+) = 0 .$$

Now, using (2.1), it is not hard to show as in Athreya ([1], page 756) that

(6.7) $$K(u) \leqq E[K(ue^{-\lambda X})]$$

where X has distribution function given by $P(X \leqq x) = \int_0^x m(y)e^{-\lambda y}e^{r(y)}\, dG(y)$. Iterating (6.7) and using the strong law of large numbers along with (6.6), we deduce that $K(u) = 0$ for all $u > 0$. Consequently, for each $u \geqq 0$, $\lim_{t\to\infty} \xi(u, t) = \xi(u)$ exists. Since $\{W^*(x, t)\}$ is tight, we deduce that $W^*(x, t)$ converges in distribution to a random variable having Laplace transform $\xi(u)$ satisfying (6.3). Furthermore, from (6.5) and (4.7), it has mean value $A(x)$. We now use the uniqueness result (6.4) to conclude the following theorem.

(6.8) THEOREM. *If $E_0(W) = 1$ and $a(x) > 0$, then $W^*(x, t) \to A(x)W$ in disbution.*

(6.9) REMARKS.

(i) If we know that $W_t^* \to W^*$ in distribution with $E_0(W^*) = 1$, then as above, $W^*(x, t) \to A(x)W^*$ in distribution for all $x \in S$. Furthermore, if $b < \infty$, then $E_0(W) = 1$.

(ii) If $0 < a \leqq b < \infty$, then $W_t^* \to W^*$ in distribution with $E_0(W^*) = 1$ iff $E_0(W) = 1$.

7. Convergence in probability. We conclude this paper by showing that without any additional assumptions, it is possible to replace convergence in distribution in Theorem (6.8) with convergence in probability.

(7.1) THEOREM. *Assume $E_0[W] = 1$. Then*

(i) *$W^*(x, t) \to A(x)W$ in probability and in L^1 if $a(x) > 0$.*

(ii) *$Z(x, t)/Z(t) \to A(x)$ in probability and in L^1 off $\{W = 0\}$ if $a > 0$.*

PROOF. We shall first show that the joint distribution $(W_t, W^*(x, t))$ converges in distribution to $(W, A(x)W)$. It suffices to show that for each $u, v \geqq 0$, $Y(t) = uW(t) + vW^*(x, t)$ converges to $(u + vA(x))W$ in distribution. Let $\eta(r, t) = E_0[\exp\{-rY(t)\}]$ be the Laplace transform of $Y(t)$ and set $J(r, t) = r^{-1}E_0[p(rY(t))]$. Since $Y(t)$ is majorized by a constant multiple of $W(t)$, the result follows from the methods of Section 6.

It now follows that $A(x)W(t) - W^*(x, t) \to 0$ in distribution and hence in probability. But $A(x)W(t) \to A(x)W$ w.p. 1 which implies that $W^*(x, t) \to A(x)W$ in probability. Since $E_0[W^*(x, t)] \to A(x) = E_0[A(x)W]$, we also get convergence in L^1. This proves part (i).

To prove part (ii), we note that for every sequence $\langle t_j \rangle$, there exists a subsequence $\langle t_j' \rangle$, such that w.p. 1 $W^*(x, t_j') \to A(x)W$ and $W^*(t_j') \to W$. Consequently

$$W^*(x, t_j')/W^*(t_j') = Z(x, t_j')/Z(t_j') \to A(x) \quad \text{w.p. } 1$$

off $\{W = 0\}$. Hence we have convergence in probability; moreover, since

$$|Z(x, t)/Z(t) - A(x)|I\{W \neq 0\} \leqq 1,$$

we have convergence in L^1.

(7.2) REMARK. Note that $J(r, t) \leqq K(u + v)^{-1}[uH(r, t) + vI(r, t)]$ where K is a constant depending upon u and v. This follows since the function $p(x) = e^{-x} + x - 1$ is convex and has the property that for every $k > 0$ there is a constant $K > 0$ such that $p(kx) \leqq Kp(x)$ for all $x \geqq 0$. It thus follows that $\lim_{r \downarrow 0} \limsup_{t \to \infty} J(r, t) = 0$ if the same is true of H and I. This was the main result we needed in order to prove (7.1). In particular, then, the results of Theorem (7.1) are valid if $W^*(t) \to W^*$ in distribution with $E_0[W^*] = E_0[W] = 1$.

(7.3) COROLLARY. *For the supercritical classical Bellman–Harris age-dependent model* ($\alpha = 0$ *and* π_2 *independent of age*), *the results of Theorem* (7.1) *are valid if* $\sum_{k=2}^{\infty} (k \log k)p_{2k} < \infty$.

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