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LOCALLY FINITE RANDOM SETS: FOUNDATIONS FOR POINT PROCESS THEORY[1]

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The foundations of point process theory are surveyed. An abstract theory motivated by applications in stochastic geometry is presented. It is shown that it is sufficient to know only which sets are measurable and which are bounded in the basic space, where we use countability hypotheses rather than topological assumptions. (The sole exception is in the construction of probabilities where pseudo-topological hypotheses are needed.) It is shown that there are close connections with the random set theories of Kendall and Matheron.

1. Introduction. This work was motivated by the desire to define point processes on the spaces which occur in stochastic geometry (Ripley (1976a)). These provide examples in which several topologies are equally natural but there is only one Borel σ-field. This led me to seek to build point process theory on purely measure-theoretic foundations (rather than the usual topological approach). This proved to be possible except for the construction of probabilities; many similar problems testify to the need here for pseudo-topological assumptions. This approach leads to simpler proofs of the basic results under conditions which are easily verified; for instance in R^n one may replace half-open rectangles by balls or open rectangles.

Another objective was to unify point process theory with the theories of general random sets given by Kendall and Matheron. This follows from an extension of Kallenberg's characterization theorem. Various applications are given in Ripley (1976b).

2. Bounded spaces. Throughout this paper X is a set and $\mathscr{A}$ is a σ-field on X containing all the singleton subsets. We will be interested in random collections of points from X, a finite number of which are from each member of a class $\mathscr{B}$ of bounded sets. We say $(X, \mathscr{A}, \mathscr{B})$ is a *bounded space* if $\mathscr{A}$ contains all singletons (so that points are measurably distinguishable) and $\mathscr{B}$ satisfies:

B(i). $\mathscr{B}$ is hereditary, i.e., if $E \in \mathscr{B}$ and $F \subset E$ then $F \in \mathscr{B}$,

B(ii). $\mathscr{B}$ is closed under finite unions,

B(iii). $\mathscr{B}$ covers X,

B(iv). $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$ is cofinal in $\mathscr{B}$ under inclusion, i.e., if $E \in \mathscr{B}$ there is $F \in \mathscr{C}$ with $E \subset F$.

Conditions B(i)—(iii) are obvious requirements, stating that $\mathscr{B}$ is a covering

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ideal in $\mathscr{P}(X)$, the class of all subsets of X. We often define $\mathscr{B}$ as the ideal generated by a class of subsets; in particular B(iv) is the assumption that $\mathscr{B}$ is generated by its measurable members.

The usual examples of bounded spaces are locally compact Hausdorff spaces with their Borel σ-fields and relatively compact subsets; metric spaces with the Borel σ-field and metrically bounded sets; and of course measurable spaces with $\mathscr{B} = \mathscr{P}(X)$ (we call these *totally bounded*). Other examples occur in stochastic geometry (Ripley (1976a)) which cannot be fitted into the existing theories.

We say a bounded space is *σ-bounded* if X has a countable cover from $\mathscr{B}$, and *countably bounded* if $\mathscr{B}$ has a countable cofinal subclass. A locally compact space is σ-bounded if and only if it is σ-compact, in which case it is countably bounded (by Bourbaki (1966) I, Section 9.9, Proposition 15). A metric space is always σ-bounded; it is countably bounded if it is separable.

Throughout this paper $(X, \mathscr{A}, \mathscr{B})$ will be a bounded space, and $\mathscr{C} = \mathscr{A} \cap \mathscr{B}$ will be the class of bounded measurable sets.

The class $\mathscr{C}$ is a conditional σ-ring (Segal (1951); called a semi-tribe by Dinculeanu (1967)), i.e., it is a ring and its trace on each member is a σ-field. There is a smallest conditional σ-ring, denoted by $\mathscr{C}(\mathscr{E})$ containing a class $\mathscr{E}$ of subsets. The class of (disjoint or increasing) countable unions from $\mathscr{C}(\mathscr{E})$ is the σ-ring $\mathscr{S}(\mathscr{E})$ generated by $\mathscr{E}$. (Dinculeanu (1967), Chapter I, Proposition 9). Thus $\mathscr{S}(\mathscr{C}) = \mathscr{A}$ only if the space is σ-bounded; the converse also holds. Thus a σ-bounded space may be specified by the set X and a conditional σ-ring of subsets countably covering X and containing all singletons. In general $\mathscr{C}$ does not determine $\mathscr{A}$ (take $\mathscr{B}$ to be the class of finite subsets in an uncountable space).

We say a class $\mathscr{T}$ of subsets of X separates points of a subset E if, given any finite subset of E, there are disjoint members of $\mathscr{T}$ such that each member of the finite set is contained in precisely one of these disjoint sets. We say $\mathscr{T}$ *(strictly) countably separates* E if there is a countable subclass of $\mathscr{T}$ ($\mathscr{T} \cap \mathscr{P}(E)$) which separates points of E. We say a bounded space is countably separated by $\mathscr{T}$ if each $E \in \mathscr{C}$ is countably separated by $\mathscr{T}$; if $\mathscr{T}$ is omitted we assume $\mathscr{T} = \mathscr{C}$. In all applications I know the space is countably separated.

3. Point processes. A *multiset* (Rado (1974), Lake (1976)) is a collection of points from X, distinct or not. We say a multiset is locally finite if it contains a finite number of points from each bounded set. A locally finite multiset may be viewed as a subset of $(X \times \{1, 2, 3, \cdots\})$, the index denoting the multiplicity of the point. Let N (N') denote the class of completely additive (σ-additive) functions $n: C \to Z_+$, the nonnegative integers, i.e., if $E \in \mathscr{C}$ is the union of an arbitrary (countable) disjoint subclass (E_α) of $\mathscr{C}$ then $n(E) = \sum n(E_\alpha)$. The following result is immediate.

PROPOSITION 1. *If $n \in N'$ then $D(n) = \{x: n(\{x\}) > 0\}$ is locally finite. If $n \in N$ then $n = \sum_{x \in D(n)} n(\{x\})\varepsilon_x$.*

Thus each $n \in N$ can be identified with the locally finite multiset $\{(x, n(\{x\})) : x \in D(n)\}$, and this map is a bijection (it is here that condition B(iv) is used.) We see we identify $\mathscr{L}\mathscr{F}$, the class of locally finite *subsets* of X with $N_0 = \{n : n \in N, n(\{x\}) \leqq 1 \ \forall\, x \in X\}$. We need to define a σ-field on N. In the totally bounded case it is well established that $\mathscr{N}$, the smallest σ-field making the evaluation maps e_A measurable for each $A \in \mathscr{C}$, is the only natural choice (Carter and Prenter (1972), Fortet (1968), Moyal (1962)). In the general locally finite case this σ-field is justified by the following procedure (the topological analogue of which fails in general).

For each $E \in \mathscr{C}$ let $(N_E, \mathscr{N}_E)$ denote $(N, \mathscr{N})$ for $(E, E \cap \mathscr{C}, \mathscr{P}(E))$. If $E, F \in \mathscr{C}$ and $E \subset F$ we can define the restriction map $r_{EF} : N_F \to N_E$. Then $(N_E, r_{EF})_{\mathscr{C}}$ is an inverse system with inverse limit

$$M = \{(n_E) : n_E \in N_E, n_E = r_{EF}(n_F) \text{ for } E \subset F\}$$

(Bourbaki (1968), III, Section 7). We give M the inverse limit σ-field M generated by the canonical maps $r_E : M \to N_E$.

Proposition 2. *The spaces $(M, \mathscr{N})$ and $(N, \mathscr{N})$ are isomorphic under the natural map.*

Proof. Let $u_E : N \to N_E$ be the restriction map. The family (u_E) separates points of N and so induces a measurable bijection $u : N \to M$ by $u_E = r_E \circ u$ (Bourbaki (1968), III, Section 7.2, Proposition 1). Define a map $v : M \to N$ by $v(m)(E) = r_E(m)(E)$. Then v is a measurable inverse of u.

This result is equally true if $\mathscr{C}$ is replaced by a cofinal subclass throughout. It shows that M is nonempty, and justifies the use of $\mathscr{N}$.

We define a *point process* to be a measurable map from a probability space to $(N, \mathscr{N})$, and its *distribution* to be the probability induced on $\mathscr{N}$. We say a point process is *simple* if N_0 is thick for its distribution P, i.e., the outer measure P^* gives N_0 measure 1. Let $\mathscr{N}_0$ be the trace of $\mathscr{N}$ on N_0.

Proposition 1 shows that elements of N are purely atomic and may be uniquely extended to (possibly infinite-valued) purely atomic measures on $\mathscr{P}(X)$. The following lemma shows that e_A is measurable for all $A \in \mathscr{S}(\mathscr{C})$ and that $\mathscr{N}$ is generated by $\{e_A : A \in \mathscr{T}\}$ for any π-system $\mathscr{T}$ with $\mathscr{T} \subset \mathscr{C} \subset \mathscr{S}(\mathscr{T})$.

Lemma. *Suppose $\mathscr{E}$ is a class of subsets of X closed under countable increasing (or disjoint) unions and $\mathscr{B} \cap \mathscr{E}$ is closed under finite disjoint unions and proper differences. Suppose $\mathscr{B} \cap \mathscr{E}$ contains a π-system $\mathscr{T}$. Then $\mathscr{E} \supset \mathscr{S}(\mathscr{T})$.*

Proof. The ideal generated by $\mathscr{T}$ is a conditional σ-ring, so each $A \in \mathscr{C}(\mathscr{T})$ is contained in some finite disjoint union T from $\mathscr{T}$. Let $\mathscr{E}' = \{F : F \in \mathscr{E}, F \subset T\}$ and $\mathscr{T}' = \{F : F \in \mathscr{T}, F \subset T\}$. Then $\mathscr{E}'$ is a λ-system containing the π-system $\mathscr{T}'$, so $\mathscr{E}' \supset \mathscr{P}(T) \cap \mathscr{S}(\mathscr{T})$ and hence $\mathscr{E} \supset \mathscr{C}(\mathscr{T})$ and $\mathscr{E} \supset \mathscr{S}(\mathscr{T})$.

The following result has been proved under compactness conditions (Carter and Prenter (1972), Harris (1968), Jagers (1974), Kerstan, Matthes and Mecke

(1974)), but that a pure countability hypothesis suffices seems to have been overlooked.

THEOREM 1. *If the bounded space is countably separated then* $N = N'$.

PROOF. From the definitions we may assume that the space is totally bounded. Suppose $\{A_m\}$ is a countable subfield of $\mathscr{A}$ separating points of X. Fix $n \in N'$ and $x \in X$. Let $B_m = \bigcap \{A_i : i \leqq m, x \in A_i\}$. Then $B_m \in \{A_i\}$ and $B_m \downarrow \{x\}$, so $n(B_m) = n(\{x\})$ for some m. Thus $\{A_m : n(A_m) = 0\}$ is a countable cover of $X \backslash D(n)$, so $n(X \backslash D(n)) = 0$ and $n = \sum_{x \in D(n)} n(\{x\}) \varepsilon_x \in N$.

Even in the totally bounded case we cannot replace σ-additivity by additivity unless the space is finite, for Ulam (1929) gives an example of a $\{0, 1\}$-valued function which is additive on the class of all subsets of a countable set but is not σ-additive.

This theorem shows that we have extended other point process theories (for instance those in Jagers (1974) and Kerstan, Matthes and Mecke (1974)). The following result enables us to transfer results from these theories. A Lusin space is the continuous injective metrizable image of a Polish space (Bourbaki (1966), IX, Section 6).

THEOREM 2. *Suppose X is a Lusin space with Borel σ-field $\mathscr{A}$ and $(X, \mathscr{A}, \mathscr{B})$ is a countably bounded space. There is a complete metric on X making it a locally compact space with a countable base, with $\mathscr{A}$ as its Borel σ-field and $\mathscr{B}$ both the class of relatively compact sets and the class of metrically bounded sets.*

PROOF. Let (K_r) be an increasing countable cofinal subclass of $\mathscr{C}$, with successive differences (E_r). Then $\{E_r\}$ is a Borel partition of X. Each E_r is a Lusin space and so a Borel set in its completion (Bourbaki (1966), IX, Section 6). It can be given a metric d_r bounded by $\frac{1}{2}$ making it a compact metric space with $E_r \cap \mathscr{A}$ as its Borel σ-field. (If E_r is countable let $d_r(x, y) = \frac{1}{2}$ if $x \neq y$. If E_r is uncountable it is Borel isomorphic to 2^{Z_+} (Parthasarathy (1967), Chapter I, Theorem 2.12) which can be metrized suitably). Let $d(x, y) = \max(r, s)$ for $x \in E_r$ and $y \in E_s$ unless $r = s$ when $d(x, y) = d_r(x, y)$. Then d is a complete metric inducing the sum topology on X which makes X a locally compact separable metric space. Each E_r is compact and open, so $\mathscr{B}$ is the class of relatively compact subsets.

We call bounded spaces *standard* if they satisfy the hypotheses of this theorem. If each E_r is uncountable it is also Borel isomorphic to $[0, 1)$, so we can take our standard bounded space to be the real line.

4. Construction of point processes. We have to show that point processes do exist. To do so we must specify a probability P on $\mathscr{N}$. Such a probability is determined by its restriction to

$$\mathscr{F}\mathscr{D}(\mathscr{T}) = \{\textstyle\bigcap_1^m \{n : n(A_i) = k_i, i = 1, \cdots, m\} : k_i \in Z_+, A_i \in \mathscr{T}, m = 1, 2, \cdots\}$$

for a π-system $\mathscr{T}$ with $\mathscr{T} \subset \mathscr{C} \subset \mathscr{S}(\mathscr{T})$. We call this restriction the *finite-dimensional distribution* which we view as a function $p\colon \bigcup_m (T^m \times (Z_+)^m) \to [0, 1]$. We ask what conditions p must satisfy to ensure that it is the finite-dimensional distribution of a probability on N. Obviously p satisfies:

P(i). $p(A_{\sigma(1)}, \cdots, A_{\sigma(m)}; r_{\sigma(1)}, \cdots, r_{\sigma(m)}) = p(A_1, \cdots, A_m, r_1, \cdots, r_m)$ for any permutation σ

P(ii). $\sum_r p(A, A_1, \cdots, A_m; r, r_1, \cdots, r_m) = p(A_1, \cdots, A_m; r_1, \cdots, r_m)$

P(iii). $\sum_r p(A; r) = 1$

P(iv). $p(A \cup B, A, B; r, s, t) = 0$ unless $r = s + t$ if $A \cap B = \emptyset$

and

P(v). $q(A) = 1 - p(A; 0)$ is continuous at $\emptyset$.

In general P(i)—(v) are not sufficient, for Dr. David Fremlin has constructed a totally bounded separable metric space for which P(i)—(v) are not sufficient to ensure that a function p, with $\mathscr{T} = \mathscr{C}$, defines a probability on $\mathscr{N}$. Thus we are forced to add pseudo-topological conditions. We suppose $\mathscr{K}$ is a compact class closed under finite unions (i.e., $\mathscr{K}$ possesses the finite-intersection property). We say an increasing finite function q on $\mathscr{T}$ is *tight* if for every $\varepsilon > 0$ and $A \in \mathscr{T}$ there is $B \in \mathscr{T}$ and $K \in \mathscr{K}$ with $B \subset K \subset A$ and $q(A) - q(B) < \varepsilon$.

PROPOSITION 3. *Suppose $\mathscr{T}$ is a ring and $z\colon \mathscr{T} \to Z_+$ is additive and $z(A) > 0$ implies $z(B) > 0$ for some $B \in \mathscr{T}$ with $K \in \mathscr{K}$ such that $B \subset K \subset A$. Then z is tight and may uniquely be extended to a member of N.*

PROOF. We show by induction on n that if $z(A) = n$ there is $B \in \mathscr{T}$ and $K \in \mathscr{K}$ such that $B \subset K \subset A$ and $z(B) = n$, so z is tight. This is true for $n = 0$ and $n = 1$ by hypothesis. Suppose it holds for all $m < n$ and $z(A) = n$. There is $A_1 \in \mathscr{T}$ and $K_1 \in \mathscr{K}$ with $A_1 \subset K_1 \subset K$ and $z(A_1) > 0$. Thus $z(A \backslash A_1) < n$ and there are $A_2 \in \mathscr{T}$ and $K_2 \in \mathscr{K}$ with $A_2 \subset K_2 \subset A \backslash A_1$ and $z(A_2) = z(A \backslash A_1)$. Let $B = A_1 \cup A_2$ and $K = K_1 \cup K_2$. This completes the inductive step. We now restrict attention to a member of $\mathscr{T}$ and so assume $\mathscr{T}$ is a field. A tight additive function $\mathscr{T} \to Z_+$ may be extended to a measure on $\sigma(\mathscr{T})$ tight with respect to the closure of $\mathscr{K}$ under countable intersections (which is still compact) (Pfanzagl and Pierlo (1966), Section 3.5). This measure must coincide with the outer measure defined by its restriction to $\mathscr{T}$ and so is Z_+-valued. If the space is countably separated then Theorem 1 shows that this extension belongs to N. In general there are $B_x \in \mathscr{C}$ and $K_x \in \mathscr{K}$ with $B_x \subset K_x \subset X \backslash \{x\}$ and $z(X \backslash (B_x \cup \{x\})) = 0$. Thus $\bigcap_{x \in X} K_x = \emptyset$ and $K_{x_1} \cap \cdots \cap K_{x_r} = \emptyset$ for some finite set $\{x_1, \cdots, x_r\}$. Thus $z(E) = z(\bigcup (E \backslash B_{x_i})) \leqq \sum z(E \backslash B_{x_i}) = \sum z(E \cap \{x_i\})$, for all $E \in \mathscr{C}$, so $z \in N$.

THEOREM 3. *Suppose p satisfies* P(i)—(iv), *$\mathscr{T}$ is a ring and q is tight. Then p is the finite-dimensional distribution of a unique probability on $\mathscr{N}$.*

PROOF. Suppose $\mathscr{T}$ is a ring. By P(i)—(iii) and the Daniel–Kolomogorov theorem there is a unique probability P on $(Z_+)^{\mathscr{T}}$ with restriction p to the cylinder sets. For each $A \in \mathscr{T}$ we can find sequences $(B(A, n)) \subset \mathscr{T}$ and $(K(A, n)) \subset \mathscr{K}$ with $B(A, n) \subset K(A, n) \subset A$ and $q(A) - q(B(A, n)) < 2^{-n}$. Define: $L = \{z: z$ is additive, $z(A) > 0 \Rightarrow z(B(A, n)) > 0$ for some $n\}$. We will show $P^*(L) = 1$. Suppose A is measurable in $(Z_+)^{\mathscr{T}}$ and $A \supset L$. Then A restricts z at a countable set $\mathscr{C}_0$ only; we enlarge $\mathscr{C}_0$ to a countable ring. Thus $A \supset L_0$ where $L_0 = \{z: z$ is additive on $\mathscr{C}_0$, $z(A) > 0$ for $A \in \mathscr{C}_0$ implies $z(B(A, n)) > 0$ for some $n\}$. From P(iv) and the definition of $B(A, n)$ it follows that $P(L_0) = 1$, hence $P(A) = 1$ and $P^*(L) = 1$. By Proposition 3 we can embed L in N. If we take the trace of P on L we obtain a probability on $\mathscr{N}$ with p as its finite-dimensional distribution.

PROPOSITION 4. *Suppose $(X, \mathscr{A}, \mathscr{B})$ is a standard bounded space. Then p is the finite-dimensional distribution of a probability on $\mathscr{N}$ if and only if it satisfies* P(i)—(v).

PROOF. We take X to be a *LCD* space with $\mathscr{B}$ the class of relatively compact subsets. From P(i)—(iv) q is increasing and strongly subadditive;

$$q(A \cup B) + q(A \cap B) \leqq q(A) + q(B) \qquad \text{for all} \quad A, B \in \mathscr{C}.$$

Thus there is an extension of q to a Choquet $\mathscr{C}$-capacity (Dellacherie and Meyer (1975), III, 32). This is a fortiori a $\mathscr{K}$-capacity where $\mathscr{K}$ is the class of compact sets, so q is tight by Choquet's theorem if P(i)—(v) hold.

5. Simple point processes. In this section we give a result which is peculiar to simple point processes. These are defined by $P^*(N_0) = 1$ for the distribution P, so it is of interest whether N_0 is measurable.

PROPOSITION 5. (i) *$N_0 \in \mathscr{N}$ only if the space is σ-bounded.*

(ii) *$N_0 \in \mathscr{N}$ if the space is countably separated and σ-bounded.*

PROOF. (i) Suppose the space is not σ-bounded. Then any $A \in \mathscr{N}$ is the trace on N of a subset of $(Z_+)^{\mathscr{C}}$ restricting only at a countable set $\mathscr{C}_0$. Thus A restricts the mass given to singletons only inside $\mathscr{C}_0$.

(ii) Suppose $E \in \mathscr{C}$; define $E_2 = \{(x, x): x \in E\}$. Then $E_2 \in \mathscr{C}(\mathscr{C} \times \mathscr{C})$ (Christensen (1974), Theorem 2.1). We can extend $n \times n$, for $n \in N$, as a finite measure to $\mathscr{C}(\mathscr{C} \times \mathscr{C})$. The argument of Krickeberg (1973), Theorem 3, shows that $\{n: n(\{x\}) \leqq 1\ \forall\, x \in E\} = \{n: (n \times n)(E_2) = n(E)\}$ which is measurable since $n \to (n \times n)(E_2)$ is measurable by our lemma.

For each $n \in N$ define $Sn = \sum_{D(n)} \varepsilon_x$. The map $S: N \to N_0$ correspond to regarding a multiset as a set. Let $\mathscr{S}\mathscr{N}$ be the smallest σ-field on N making S measurable. Of course $Sn = n$ if and only if $n \in N_0$, and $N_0 \cap \mathscr{S}\mathscr{N} = \mathscr{N}_0$.

THEOREM 4. *Suppose $\mathscr{T} \subset \mathscr{C}$ and $\mathscr{T}' = \{E: E \in \mathscr{C}, E$ is strictly countably separated by $\mathscr{T}\}$ contains a π-system $\mathscr{T}''$ with $\mathscr{S}(\mathscr{T}'') \supset \mathscr{C}$. Then*

(i) $\mathscr{N} = \sigma(e_A: A \in \mathscr{T})$

and

(ii) $\mathscr{S}\mathscr{N} = \sigma(\mathscr{E}_0)$ *where* $\mathscr{E}_0 = \{\{n: n(E) = 0\}: E \in \mathscr{T}\}$, *so S is measurable.*

PROOF. Suppose $n \in N$ and $E \in \mathscr{T}'$. Let Σ be a countable subclass of $\mathscr{T} \cap \mathscr{P}(E)$ separating points of E. Let $\mathscr{L}$ be the class of finite sets of disjoint members of Σ. Suppose $Sn(E) = r$, so $D(n) \cap E = \{x_1, \cdots, x_r\}$. There is $\{A_1, \cdots, A_r\} \in \mathscr{L}$ with $x_i \in A_i$, $i = 1, \cdots, r$, so

$$n(E) = \sup \{\sum n(A_i) : (A_i) \in \mathscr{L}\},$$

and

$$Sn(E) = \sup \{\sum 1_{(n(A_i)>0)} : (A_i) \in \mathscr{L}\}.$$

Since $\mathscr{L}$ is countable e_E is $\sigma(e_A : A \in \mathscr{T})$-measurable and $S \circ e_E$ is $\sigma(\mathscr{E}_0)$-measurable. The theorem follows from the lemma applied to $\mathscr{T}''$.

COROLLARY. *Suppose $\mathscr{T}$ is closed under finite unions. A probability P on $\mathscr{N}$ with $P^*(N_0) = 1$ is determined by its restriction to $\mathscr{E}_0$.*

The conditions on $\mathscr{T}$ seem complex; it suffices that $\mathscr{T}$ is itself a π-system countably separating the space and generating $\mathscr{E}(\mathscr{T})$. The complexity is necessary to cover the class of open balls in a separable metric space. In particular, S is measurable on any countably separated space.

Suppose X is a metric space. We say $\mathscr{T}$ is a *DC-class* of subsets if every bounded set has a finite cover from $\mathscr{T}$ of sets of arbitrarily small diameter.

PROPOSITION 6. *$(X, \mathscr{A}, \mathscr{B})$ is countably separated by any DC-class $\mathscr{T}$. If X is the countable union of bounded closed sets then $\mathscr{S}(\mathscr{T})$ contains the Borel σ-field.*

PROOF. Suppose $E \in \mathscr{E}$ and x and y are distinct points of E. Then $d(x, y) > 1/n$ for some n. Now E has a finite cover $\mathscr{T}_n$ from $\mathscr{T}$ of sets of diameter less than $1/2n$. Thus if $x \in A_1 \in \mathscr{T}_n$ and $y \in A_2 \in \mathscr{T}_n$ then $A_1 \cap A_2 = \emptyset$, so $\bigcup \mathscr{T}_n$ countably separates E.

Suppose E is closed. Let $E_n = \{A : A \in \mathscr{T}_n, A \cap E \neq \emptyset\}$. Then $E_n \downarrow E$, so $E \in \mathscr{S}(\mathscr{T})$. Under our hypotheses $\mathscr{S}(\mathscr{T})$ is a σ-field containing the bounded closed sets, hence all Borel sets.

Proposition 6 shows that Theorem 4(ii) and its corollary generalize the results of Kallenberg (1973) for *DC-rings* in a locally compact space. (They were discovered independently.) There is related work by Belayev (1969), Leadbetter (1972) and Mönch (1971). We shall see the natural setting for these results in Section 7. They are applied in Ripley (1976b).

The following simple example shows that the distribution of a point process is not in general characterized even by its restriction to

$$\mathscr{E}_1 = \{\{n : n(A) = k\} : A \in \mathscr{E}, k \in Z_+\}.$$

Let $(X, \mathscr{A}, \mathscr{B})$ be the two-point space. Define P and P' by the following probabilities:

$n(\{0\})$	$n(\{1\})$	0	1	2	0	1	2
0		$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0	$\frac{2}{9}$
1		$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	0
2		$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0	$\frac{2}{9}$	$\frac{1}{9}$

Then P and P' agree on $\mathscr{E}_1$ but differ. Of course both $P(N_0)$ and $P'(N_0)$ are less than one.

6. Marks. Suppose a mark is associated with each point of the random multiset (Matthes (1963)). A *marked point process* with marks in a measurable space $(K, \mathscr{K})$ is a point process on $(X \times K, \mathscr{A} \otimes \mathscr{K}, \{B \times K : B \in \mathscr{B}\})$. A simple point process will be one in which no point occurs twice with the same mark. To forbid points to occur more than once irrespective of mark we introduce:

$$N_{00} = \{n : n \in N, n(\{x\} \times K) \leqq 1 \ \forall\ x \in X\}\,.$$

The map $r : N(X \times K) \to N(X)$ given by $rn(A) = n(A \times K)$ which corresponds to ignoring marks is measurable and $N_{00} = r^{-1}(N_0)$, so $N_{00} \in \mathscr{N}$ if X is σ-bounded and countably separated.

There is a sense in which every point process can be converted to a simple one, as noted by Srinvasan (1962). We form a new process by replacing the multiset by a set with points marked by the number of occurrences. Let $h : N(X) \to N_{00}(X \times (Z_+\backslash\{0\}))$ be the map thus defined.

PROPOSITION 7. *Suppose the bounded space is countably separated. The map h is an isomorphism.*

PROOF. $h(\{n(A) = k\}) = \{n : \sum mn(A \times \{m\}) = k\}$, so h^{-1} is measurable. Let $T_m = S(I - S)^m$ for $m \geqq 1$. Then $h^{-1}(\{n : n(A \times \{m\}) = k\}) = \{n : T_m n(A) = k\}$ which is measurable by Theorem 4.

COROLLARY. *Under the hypotheses of Theorem 4 $\mathscr{N}$ is generated by $\bigcup_m \{(S(I - S)^m)^{-1}(\mathscr{E}_0) : m \geqq 1\}$.*

More generally for a marked point process we may replace the multiset of marked points by a set of points marked the collection of marks which occur at the point, defining a map $h_1 : N(X \times K) \to N_{00}(X \times (K_e\backslash\{\varnothing\}))$ where K_e is the exponential space of K, the class of all finite multisets from K (Carter and Prenter (1972)).

PROPOSITION 8. *Suppose both $(X, \mathscr{A}, \mathscr{B})$ and $(K, \mathscr{K})$ are countably separated. Then h_1 is an isomorphism.*

PROOF. For $B \subset K$ let B_m be the class of multisets from B of size m. Then $h_1(\{n(A \times B) = k\}) = \{n : \sum mn(A \times B_m) = k\}$, so h_1^{-1} is measurable. We know $h_2 : N(X \times K) \to N_{00}((X \times K) \times (Z_+\backslash\{0\}))$ is measurable. A basic measurable set in the range space is of the form $\{n : n(A \times B) = 0\}$ by Theorem 4, where $A \in \mathscr{C}$ and $B = \{k :$ no s-fold points of k are in $C\}$ for $C \in \mathscr{K}$ by the corollary of Proposition 7 (since we may identify K_e with $N(K)$). This set has inverse image $\{n : n(A \times C \times \{s\}) = 0\}$.

7. General random set theories. We suppose throughout this section that X is a locally compact Hausdorff space with a countable base, $\mathscr{A}$ is its Borel σ-field, and $\mathscr{B}$ contains all the compact subsets, so $(X, \mathscr{A}, \mathscr{B})$ is σ-bounded.

Each locally finite set meets each compact set in a finite set and so is a closed set (Bourbaki (1966), I, Section 9.7, Proposition 11). According to Theorem 4(ii) applied to $\mathscr{K}$, the class of compact sets, we have given $\mathscr{L}\mathscr{F}$ the smallest σ-field containing $\{\{F: F \cap K = \emptyset\}: K \in \mathscr{K}\}$. Let $\mathscr{F}$ denote the class of closed subsets of X and $\mathscr{V}$ the σ-field generated by $\{\{F: F \in \mathscr{F}, F \cap K = \emptyset\}: K \in \mathscr{K}\}$. Thus $\mathscr{L}\mathscr{F}$ is a subspace of $(\mathscr{F}, \mathscr{V})$. Matheron (1972, 1975) defines a *random closed set* to be a measurable map from a probability space to $(\mathscr{F}, \mathscr{V})$, so a random set with range in $\mathscr{L}\mathscr{F}$ a.s. is precisely a simple point process. (This special case of Theorem 4 was given by Matheron in an unpublished note dated February, 1968.)

Suppose $\mathscr{T}$ is a base for the topology of X. Then Kendall's theory (1974) gives $\mathscr{F}$ the σ-field $\mathscr{V}(\mathscr{T})$ generated by $\{\{F: F \cap E = \emptyset\}: E \in \mathscr{T}\}$. If $\mathscr{G}$ is the class of open sets it is easy to show that $\mathscr{V}(\mathscr{T}) = \mathscr{V}(\mathscr{G}) = \mathscr{V}(\mathscr{F}) = \mathscr{V}$, so the theories coincide. Christensen (1974, Theorem 3.8) shows that $\{F: F \cap A = \emptyset\} \in \mathscr{V}$ if and only if $A \in \mathscr{K}_\sigma$. Matheron shows that $\mathscr{V}$ is the Borel σ-field of a topology on $\mathscr{F}$; this is true for almost all topologies proposed on $\mathscr{F}$ (surveyed by Flachsmeyer (1964)).

PROPOSITION 9. *Suppose $\mathscr{B}$ has a countable cofinal subclass from K_σ. Then $\mathscr{L}\mathscr{F} \in \mathscr{V}$.*

PROOF. Exactly as in the proof of Theorem 4(ii), taking $\mathscr{T} = \mathscr{K}_\sigma$, we have $F \to \#(F \cap A)$ measurable on $\mathscr{F}$ for all $A \in \mathscr{K}_\sigma$. Let (E_m) be the countable cofinal class. Then

$$\mathscr{L}\mathscr{F} = \{F: F \in \mathscr{F}, \#(F \cap E_m) < \infty \text{ for all } m\} \in \mathscr{V}.$$

The random set theories characterize probabilities P on $\mathscr{V}$ by the avoidance function A on $\mathscr{K}$ (Matheron) or $\mathscr{T}_+$, the closure of $\mathscr{T}$ under finite unions (Kendall) defined by

$$A(E) = P(\{F: F \cap E = \emptyset\}).$$

An obvious problem is to express $P^*(\mathscr{L}\mathscr{F}) = 1$ in terms of A. Then we could construct a simple point process (on a standard space) from a specified function q by setting $A = 1 - q$ and taking an a.s. locally finite random closed set with this avoidance function. There is related work by Karbe (see Kerstan, Matthes and Mecke (1974), Theorem 1.3.8) and Kurtz (1974) for A specified on a ring and all Borel sets respectively.

Suppose $\mathscr{T}$ is a subclass of $\mathscr{K}_\sigma$ on which A is defined, $(E_i) \subset \mathscr{T}$ is a countable cofinal class in $\mathscr{B}$, and each E_i has a dissecting system (Leadbetter (1972)) from $\mathscr{T}$, i.e., there is $\{T_{nk}: 1 \leqq k \leqq k_n\} \subset T$ satisfying

(a) for each n (T_{nk}) are disjoint subsets of E_i,

(b) $E_i \backslash \bigcup_k T_{nk} \subset F_n \downarrow \emptyset$,

and

(c) $\{T_{nk}: n \geqq m\}$ separates points of E_i for each m.

PROPOSITION 10. *Let $\mathscr{L}_i$ denote the class of finite disjoint subclasses of $\mathscr{T} \cap \mathscr{P}(E_i)$. The following are equivalent*:

(i) $P(\mathscr{L}\mathscr{F}) = 1$

(ii) $\sup_{\phi \in \mathscr{L}_i} P(\{F \text{ hits at least } r \text{ members of } \phi\}) \to 0$ *as* $r \to \infty$

(iii) $\inf_{\phi \in \mathscr{L}_i} \sum_{\chi \subset \phi, |\phi| \leqq r} \sum_{\phi \backslash \psi \subset \chi \subset \phi} (-1)^{|\phi|+|\chi|+|\phi|} A(\bigcup \chi) \to 1$ *as* $r \to \infty$,

for each i.

PROOF. The equivalence of (ii) and (iii) follows from the equivalence of $P(\{F$ hits those members of ϕ in $\psi\})$ with the inner sum of (iii). For $\phi \in \mathscr{L}_i$ $P(\{F$ hits at least r members of $\phi\}) \leqq P(\#(F \cap E_i) \geqq r)$, so (i) implies (ii). Conversely, consider $\phi_n = \{T_{nk} \colon 1 \leqq k \leqq k_n\} \in \mathscr{L}_i$. Let Y_{nk} be the indicator of the event that T_{nk} is hit. Then $\#(F \cap E_i) = \lim_n \sum_k Y_{nk}$ by (b) and (c), so

$$\begin{aligned} P(\#(F \cap E_i) \geqq r) &= \lim_n P(\{F \colon \textstyle\sum_k Y_{nk}(F) \geqq r\}) \\ &= \sup_{\phi \in L_i} P(\{F \text{ hits at least } r \text{ members of } \phi\}) \,. \end{aligned}$$

The condition (iii) of this proposition seems to be too complicated to be useful. It would be very useful to have a simple sufficient condition for a random closed set to be a.s. locally finite. We give separate necessary and sufficient conditions for a stationary random closed set on the real line.

Suppose X is the real line, $\mathscr{B}$ is the class of relatively compact sets and $\mathscr{T}$ is the class of bounded open intervals. We say a probability P on $\mathscr{Y}$ is stationary if its avoidance function is invariant under shifts. (If P corresponds to a simple point process this is then strictly stationary, Ripley (1976b).) For each $t \geqq 0$ let $a(t) = A((0, t))$. Kendall (1974) showed that for a stationary probability there is a subprobability p on $(0, \infty]$ with

$$a(t) = \int_{(0,\infty]} (1 - t/x)_+ \, dp(x) \qquad \text{for} \quad t > 0 \,.$$

Thus a always has a right-hand limit $a(0+)$ at 0.

THEOREM 5. *Suppose a corresponds to a stationary probability P on $\mathscr{Y}$. Then* LF(i) *and* LF(ii) *are equivalent and necessary for $P(\mathscr{L}\mathscr{F}) = 1$ and* LF(iii)—(v) *are equivalent and sufficient for $P(\mathscr{L}\mathscr{F}) = 1$.*

LF(i). $a(0+) = 1$

LF(ii). *p is a probability*

LF(iii). *a has a finite right-hand derivative at* 0

LF(iv). $\lim 2^n(1 - a(2^{-n})) < \infty$

LF(v). *p is a probability and $x \to x^{-1}$ is integrable.*

PROOF. (a) $a(0+) = p((0, \infty])$. Suppose p is a probability. Then $t^{-1}(1 - a(t)) = \int \min(t^{-1}, x^{-1}) \, dp(x) \uparrow \int x^{-1} \, dp(x)$. This establishes the equivalences.

(b) $1 - a(0+) = \lim (1 - a(2^{-n})) = P(\{F \text{ hits } (0, 2^{-n}) \ \forall n\})$ which is 0 if $P(LF) = 1$, so (i) is necessary.

(c) Assume (iv). Then $b_n = A([0, 2^{-n})) = A(\{0\} \cup (0, 2^{-n})) \geqq a(2^{-n}) - A(\{0\}) = a(2^{-n}) - (1 - a(0+)) = a(2^{-n})$, so $\lim 2^n(1 - b_n) < \infty$.

(d) Let $I(m, n, r) = [-m, m] \cap [r2^{-n}, (r+1)2^{-n})$. Let $A(m, n, r) = \{F \text{ hits } I(m, n, r) \text{ and } I(m, n, r+1)\}$ and $A(m, n) = \bigcup_r A(m, n, r)$. Suppose $F \in \mathscr{F} \backslash \mathscr{LF}$. For some m $F \cap [-m, m]$ is infinite and has a cluster point, so $F \in \limsup_n A(m, n)$. Thus $P(\mathscr{LF}) = 1$ if $P(\limsup_n A(m, n)) = 0$ for all m for which it suffices that $\sum_n P(A(m, n)) < \infty$ for all m.

(e) $P(A(m, n)) \leqq 2m2^n P(A(m, n, 0)) = 2m2^n(1 - b_n + b_{n-1})$ by stationarity. Thus

$$\textstyle\sum_1^s P(A(m, n)) \leqq 4m(2^s(1 - b_s) + b_1 - 1)$$

which is bounded by part (c). Thus (iv) suffices for $P(\mathscr{LF}) = 1$.

The following examples show that the conditions of Theorem 5 are in a sense the best possible in a alone.

(1°) The mixtures of randomized arithmetic progressions discussed by Kendall (1974) are locally finite a.s. if and only if p is a probability. This class exhausts the class of functions a, so if $a(0+) = 1$ then there is another random closed set with the same a which is locally finite a.s. We may take p to be Lebesgue measure on (0, 1), violating our sufficient condition.

(2°) Suppose $a_m = 2^m(1 - a(2^{-m}))$ increases to infinity. We will construct a random closed set with $a'(0+) = 1$, $P'(LF) = 0$ and $2^m(1 - a'(2^{-m})) \leqq a_m$. We define (b_n) inductively by $b_0 = 0$, $b_n = \max(b_{n-1} + 1, \min\{m: a_m \geqq n + 3\})$, and set $c_n = 2^{-b_n}$. Then $c_n \downarrow 0$. Thus $\{n + c_m : n \in Z, m \geqq 1\}$ is a closed set which is not locally finite, and we may form a stationary random closed set by translating it by a random variable with uniform distribution on (0, 1). Then $2^m(1 - a'(2^{-m})) \leqq \min(a_m, m + 3)$, so this is the required random set.

(3°) Suppose $A(E) = \exp(-\lambda|E|)$ for $E \in \mathscr{T}_+$, so a satisfies LF(iii). The associated random set is the Poisson process of intensity λ. This provides a construction of the Poisson process and another proof of Rényi's (1967) theorem for Lebesgue measure $|\cdot|$. (The general case follows from Theorem 4.)

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