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SPEEDS OF CONVERGENCE FOR THE MULTIDIMENSIONAL CENTRAL LIMIT THEOREM

BY T. J. SWEETING

University of Surrey

Speeds of convergence to normality for sums of independent and identically distributed random vectors in $\mathbb{R}^k$, $k \geqq 1$, are investigated using the method of operators. Results obtained improve and extend existing results on speeds of convergence for the expectations of both bounded and certain unbounded Borel measurable functions, and nonuniform convergence rates.

1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of independent random vectors in $\mathbb{R}^k$ with common distribution F; $EX_1 = 0$, $DX_1 = I$, the identity matrix (where $D^\bullet$ denotes the dispersion of a random vector). Let G be the k-dimensional standard normal distribution, with density N_k; let $Y_1, Y_2, \cdots$ be a sequence of independent random vectors with common distribution G. For any distribution P in $\mathbb{R}^k$ define the scaled distribution $\bar{P}$ by $\bar{P}\{dx\} = P\{n^{\frac{1}{2}}\, dx\}$ where n is some positive integer. Thus $\bar{F}^{*n}$ is the distribution of $S_n = n^{-\frac{1}{2}} \sum_{i=1}^{n} X_i$ (where * denotes convolution). The central limit theorem in $\mathbb{R}^k$ asserts the weak convergence of $\bar{F}^{*n}$ to G.

Most proofs of results on rates of convergence for the multidimensional central limit theorem use expansions of the relevant characteristic functions. Bergström, however, keeps closer to the distribution functions and uses essentially the same expansions as are used here: in [1] he generalizes the Berry–Esseen theorem to cover the multidimensional case. More recently work has been devoted to obtaining rates of convergence for the probabilities of arbitrary Borel sets in $\mathbb{R}^k$. Ranga Rao (1961) proved that if fourth moments are finite then uniformly over all convex Borel sets C in $\mathbb{R}^k$

$$|\bar{F}^{*n}\{C\} - G\{C\}| \leqq C(k)n^{-\frac{1}{2}}(\log n)^b$$

where $b = (k-1)/2(k+1)$. Von Bahr (1967) and Bhattacharya (1970a), (1971) obtained results for general classes of sets. In particular, the logarithmic term was removed in the above rate of convergence for convex sets; Von Bahr assumed the existence of $(k+1)$th order moments (for $k \geqq 2$) while Bhattacharya (1970a) assumed that moments of order $3 + \delta$ were finite for some $\delta > 0$, which he later relaxed in [4] to third-order moments. Sazonov (1968) also obtained the result assuming finiteness of third-order moments by using a similar technique to Bergström.

In this paper distributions are treated as operators on bounded Borel measurable functions on $\mathbb{R}^k$. In Sections 2 through 5 some basic lemmas are proved

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and in Sections 6 and 7 the main results of the paper are presented. Theorem 1 of Section 6 improves and extends existing results on speeds of convergence for the expectations of bounded and certain unbounded measurable functions, while Theorem 2 of Section 7 generalizes the nonuniform result of Rotar (1970) when the latter is specialized to identically distributed summands.

For vectors $x = (x_1, \cdots, x_k)$, $y = (y_1, \cdots, y_k)$ in $\mathbb{R}^k$, (x, y) denotes the usual inner product between x and y, $|x| = (x, x)^{\frac{1}{2}}$. Throughout the paper, constants will usually be denoted $C(\cdot)$, with any argument(s) in parenthesis; however, arguments have been suppressed in the proofs of theorems and lemmas.

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2. Definitions and a preliminary lemma. Let M^k be the space of finite signed measures on the Borel σ-field of $\mathbb{R}^k$ and B^k be the class of real Borel measurable functions on $\mathbb{R}^k$. The norm of a bounded function $v\colon \mathbb{R}^k \to \mathbb{R}$ is given by $||v|| = \sup_{x \in \mathbb{R}^k} |v(x)|$. Let B_1^k be the class of bounded functions in B^k. We define the operator $\mathscr{P}$ associated with a finite signed measure P in M^k as the function $\mathscr{P}\colon B_1^k \to B_1^k$ whose values at $v \in B_1^k$ are given by

$$\mathscr{P}v(x) = \int v(x - y)P\{dy\} .$$

A discussion of these operators for probability measures (defined on a narrower class of functions) may be found in Feller (1966). In particular, $\mathscr{P}\mathscr{Q}$ (composition of operators) is the operator associated with the convolution $P * Q$ of the corresponding measures. We shall make use of the following identity (here $\mathscr{P}^0$ is the identity operator).

$$(1) \qquad (\mathscr{P}^l - \mathscr{Q}^l) = (\mathscr{P} - \mathscr{Q}) \sum_{i=0}^{l-1} \mathscr{P}^i \mathscr{Q}^{l-i-1}$$

for all positive integers l. Let C^k be the class of real functions on $\mathbb{R}^k$ for which partial derivatives of all orders exist and are bounded. D_i will denote the partial differential operator on C^k with respect to the ith variable. If $v \in C^k$, $t \in \mathbb{R}^k$, define

$$v_t^{(m)} = (\textstyle\sum_{i=1}^k t_i D_i)^m v$$

for all nonnegative integers m. Let N be a positive integer, $m_1, \cdots, m_N$ nonnegative integers and $t^{(1)}, \cdots, t^{(N)} \in \mathbb{R}^k$. Put $m = (m_1, \cdots, m_N)$, $T = (t^{(1)}, \cdots, t^{(N)})$ and define

$$v_T^{(m)} = \textstyle\prod_{j=1}^N (\sum_{i=1}^k t_i^{(j)} D_i)^{m_j} v .$$

Note that if $\mathscr{P}$ is the operator associated with a measure P in M^k and $v \in C^k$, then $(\mathscr{P}v)_T^{(m)} = \mathscr{P}v_T^{(m)}$. If $u \in C^k$, $x, h \in \mathbb{R}^k$, p is a positive integer, then Taylor's theorem in $\mathbb{R}^k$ asserts that

$$(2) \qquad u(x + h) - u(x) = \textstyle\sum_{l=1}^{p-1} (l!)^{-1} u_h^{(l)}(x) + (p!)^{-1} u_h^{(p)}(x + \theta h)$$

where $0 < \theta < 1$.

A real continuous even function v on $\mathbb{R}^k$ is positive definite (p.d.) if $\sum_{i,j} v(x_i - x_j)y_i y_j \geqq 0$ for every choice of finitely many vectors $x_1, \cdots, x_n$ and real numbers $y_1, \cdots, y_n$. Bochner's theorem (for $\mathbb{R}^k$) implies that v is p.d. iff it is the Fourier–Stieltjes transform of a finite measure A on $\mathbb{R}^k$; that is

$$v(x) = \int e^{i(x,y)} A\{dy\} . \tag{3}$$

It follows from (3) that

$$||v|| = v(0) = A\{\mathbb{R}^k\} . \tag{4}$$

Let $N \geqq 1$ and define m and T as above; put $M = \sum_{j=1}^N m_j$. From the theory of Fourier transforms one can deduce that if v is p.d. then $(-)^M v_T^{(2m)}$ is also p.d. with associated measure $\prod_{j=1}^N (t^{(j)}, y)^{2m_j} A\{dy\}$; thus, using property (4) for p.d. functions and applying Hölder's inequality, we find that

$$||v_T^{(2m)}|| \leqq k^{M-1}(\prod_{j=1}^N |t^{(j)}|^{2m_j}) \sum_{i=1}^k D_i^{2M} v(0) . \tag{5}$$

A p.d. function will be called h-smooth if its associated measure A vanishes outside the closed sphere centered at zero of radius $h > 0$. It is easy to see that if v is h-smooth then for all $t \in \mathbb{R}^k$, $m \geqq 0$

$$||v_t^{(2m+2)}|| \leqq |ht|^2 ||v_t^{(2m)}|| . \tag{6}$$

Note that if v is 1-smooth, then the function v_h defined by $v_h(x) = v(hx)$ is h-smooth ($h > 0$). We shall say that an operator is p.d. if its associated (finite signed) measure possesses a p.d. density. It is easy to show using Fourier transforms that if the operator $\mathscr{P}$ is p.d. and v is a p.d. function, then $\mathscr{P}v$ is p.d.; furthermore, if v is h-smooth then $\mathscr{P}v$ is also h-smooth.

We also need to estimate the norms of odd-order derivatives of a p.d. function. We use the following multidimensional form of the Landau–Hadamard inequality; if $v \in C^k$, $t \in \mathbb{R}^k$, $m \geqq 1$ then

$$||v_t^{(m)}|| \leqq 2[||v_t^{(m-1)}|| \, ||v_t^{(m+1)}||]^{\frac{1}{2}} . \tag{7}$$

It follows that if v is a p.d. and h-smooth function then for all $m \geqq 1$

$$||v_t^{(m)}|| \leqq 4h|t| \, ||v_t^{(m-1)}|| . \tag{8}$$

For all real $r > 0$, define the absolute moments $\beta_r = E|X_1|^r$ (possibly infinite), $\alpha_r = E|Y_1|^r$, and let $\mathscr{F}$, $\mathscr{G}$ be the operators associated with the probability measures F, G defined in Section 1.

LEMMA 1. *Suppose $\beta_3 < \infty$ and put $h = 0.1\beta_3^{-1}$. If v is a* p.d. *h-smooth function then for $0 \leqq i \leqq n$*

$$||\mathscr{F}^i \mathscr{G}^{n-i} v|| \leqq ||\mathscr{H}^i \mathscr{G}^{n-i} v|| \tag{9}$$

where $\mathscr{H}$ is the operator associated with the normal distribution having dispersion matrix $\frac{1}{2}I$.

PROOF. For $1 \leqq i \leqq l$ we have the inequality

$$(10) \qquad \|\mathscr{F}^i\mathscr{G}^{l-i}v\| - \|\mathscr{H}^i\mathscr{G}^{l-i}v\| \leqq \|(\mathscr{F}^i - \mathscr{G}^i)\mathscr{G}^{l-i}v\| + [\|\mathscr{G}^l v\| - \|\mathscr{H}^i\mathscr{G}^{l-i}\|] .$$

Since $\mathscr{G}$, $\mathscr{H}$ are p.d. and v is h-smooth it follows from (4), (1), the expansion (2) with $p = 3$, (5) with $m = 1$ and (7) that

$$(11) \qquad \begin{aligned} \|\mathscr{G}^l v\| - \|\mathscr{H}^i\mathscr{G}^{l-i}v\| &= \textstyle\sum_{j=0}^{i-1} (\mathscr{G} - \mathscr{H})\mathscr{H}^j\mathscr{G}^{l-j-1}v(0) \\ &\leqq \textstyle\sum_{j=0}^{i-1} (Ah - \frac{1}{4}) \sum_{r=1}^{k} \|D_r^2\mathscr{H}^j\mathscr{G}^{l-j-1}v\| \end{aligned}$$

where $A = \frac{2}{3}\alpha_3(1 + 2^{-\frac{3}{2}})$. We establish (9) by induction on n; assume (9) holds for $n = l - 1 \geqq 0$. From (1), (2)

$$(12) \qquad \|(\mathscr{F}^i - \mathscr{G}^i)\mathscr{G}^{l-i}v\| \leqq \tfrac{1}{6} \textstyle\sum_{j=0}^{i-1} \int \|\mathscr{F}^j\mathscr{G}^{l-j-1}v_y^{(3)}\|(F + G)\{dy\} .$$

But $-v_y^{(2)}$, $v_y^{(4)}$ are p.d. h-smooth, so from (7), the inductive hypothesis, (6) and (5) with $m = 1$ we find that

$$\|\mathscr{F}^j\mathscr{G}^{l-j-1}v_y^{(3)}\| \leqq 2h|y|^3 \textstyle\sum_{r=1}^{k} \|D_r^2\mathscr{H}^j\mathscr{G}^{l-j-1}v\| .$$

Using this in (12) we have from (10) and (11)

$$\|\mathscr{F}^i\mathscr{G}^{l-i}v\| - \|\mathscr{H}^i\mathscr{G}^{l-i}v\| \leqq \textstyle\sum_{j=0}^{i-1} (Bh\beta_3 - \frac{1}{4}) \sum_{r=1}^{k} \|D_r^2\mathscr{H}^j\mathscr{G}^{l-j-1}v\|$$

where $B = A + \frac{1}{3}(1 + \alpha_3)$ (using $\beta_3 > 1$). The r.h.s. is $\leqq 0$ whenever $h \leqq (4B\beta_3)^{-1}$. Thus (9) holds for $n = l$, since $0.1 < (4B)^{-1}$; the result is trivial when $n = 0$.

If $\mathscr{P}$ is the operator associated with a measure P in M^k, $\bar{\mathscr{P}}$ will denote the operator associated with $\bar{P}$; if P has density p, then $\bar{p}$ will denote the density of $\bar{P}$. Put $\mathscr{L}_{i,n} = \mathscr{F}^i\mathscr{G}^{n-i}$.

COROLLARY 1. *Let $h = 0.1\beta_3^{-1}$ and suppose v is a* p.d. *h-smooth probability density. Let $m = (m_1, \cdots, m_N)$, $T = (t^{(1)}, \cdots, t^{(N)})$ where the m_j are nonnegative integers, $t^{(j)} \in \mathbb{R}^k$ and $N \geqq 1$. Then*

$$\|(\bar{\mathscr{L}}_{i,n}\bar{v})_T^{(m)}\| \leqq C(k, M) \textstyle\prod_{j=1}^{N} |t^{(j)}|^{m_j}$$

where $M = m_1 + \cdots + m_N$.

PROOF. If all the m_j are even, the result follows easily from Lemma 1 and (5). If some of the m_j are odd, we may use (7) repeatedly to express $\|(\bar{\mathscr{L}}_{i,n}\bar{v})_T^{(m)}\|$ as a product of terms with even indices, and again the result holds.

3. Truncation. The truncation used here is the same as that used by Bikyalis (1968) and Bhattacharya (1970b). Put $X_{i,n} = X_i I\{|X_i| \leqq n^{\frac{1}{2}}\}$ and let $\mu = EX_{i,n}$, $\Sigma = DX_{i,n}$. We use the same symbol to denote a linear operator and its associated matrix. Since Σ is symmetric there exists an orthogonal matrix P such that $P'\Sigma P = \Lambda$, where Λ is the diagonal matrix with the eigenvalues $\lambda_1, \cdots, \lambda_k$ of Σ displayed on its diagonal. Thus we may write $\Sigma = \sum_{i=1}^{k} \lambda_i E_i$. If Σ is nonsingular, all its eigenvalues are positive and the matrix $T = \sum_{i=1}^{k} \lambda_i^{-\frac{1}{2}} E_i$ is symmetric, positive definite and satisfies $T'T = T^2 = \Sigma^{-1}$. We may then define $\tilde{X}_i$ by $\tilde{X}_i = T(X_{i,n} - \mu)$; $E\tilde{X}_i = 0$, $D\tilde{X}_i = I$, $\tilde{\beta}_j = E|\tilde{X}_i|^j < \infty$ for all j. If P is a probability measure with finite second absolute moment, define

$$\Lambda_n(P) = \int_{|y|>n^{\frac{1}{2}}} |y|^2 P\{dy\} .$$

Let x be a unit $k \times 1$ column vector; it is not hard to obtain the following estimates:

$$n^{\frac{1}{2}}|\mu| \leqq \Lambda_n(F) \tag{13}$$

$$0 \leqq x'(I - \Sigma)x \leqq 2\Lambda_n(F) . \tag{14}$$

If A is the linear operator associated with a symmetric matrix, then $||A|| = M$, the maximum absolute eigenvalue of A. We shall assume that $\Lambda_n(F) \leqq \rho < \frac{1}{2}$, so that $x'\Sigma x \geqq 1 - 2\rho > 0$ for all unit column vectors x. It follows that $||T|| \leqq (1 - 2\rho)^{-\frac{1}{2}}$ and we have the following bounds for $|\tilde{X}_i|$, $\tilde{\beta}_j$:

$$|\tilde{X}_1| \leqq ||T||(|X_{1,n}| + |\mu|) \leqq 2(1 - 2\rho)^{-\frac{1}{2}}n^{\frac{1}{2}} \tag{15}$$

$$\tilde{\beta}_j = E|T(X_{1,n} - \mu)|^j \leqq 2^j(1 - 2\rho)^{-j/2}E|X_{1,n}|^j , \tag{16}$$

using moment inequalities. The distributions of $X_{1,n}$, $\tilde{X}_1$ will be denoted by F_n, $\tilde{F}$ respectively with associated operators $\mathscr{F}_n$, $\tilde{\mathscr{F}}$. We have

$$\begin{aligned} \tilde{\mathscr{F}}^n v(x) - \tilde{\mathscr{F}}_n^n v(x) &= \textstyle\sum_{i=0}^{n-1} (\tilde{\mathscr{F}} - \tilde{\mathscr{F}}_n)\tilde{\mathscr{F}}^i \tilde{\mathscr{F}}_n^{n-i-1} v(x) \\ &= \textstyle\sum_{i=0}^{n-1} \int_{|y|>1} [\mathscr{M}_i v(x - y) - \mathscr{M}_i v(x)]\tilde{F}\{dy\} \end{aligned} \tag{17}$$

where $\mathscr{M}_i = \tilde{\mathscr{F}}^i \tilde{\mathscr{F}}_n^{n-i-1}$. We also truncate the normal variables $Y_1, Y_2, \cdots$. Put $Y_{i,n} = Y_i I\{|Y_i| \leqq n^{\frac{1}{2}}\}$; then $EY_{i,n} = 0$ (by symmetry). Let $\Sigma_1 = DY_{i,n}$; it is easily shown that $\Lambda_n(G) \leqq \rho_1 < 1$ for all $n \geqq 1$ and that if x is a $k \times 1$ unit column vector

$$0 \leqq x'(I - \Sigma_1)x \leqq \Lambda_n(G) .$$

This implies that Σ_1 is nonsingular, so there exists a symmetric positive definite matrix T_1 satisfying $T_1^2 = \Sigma_1^{-1}$ and we may define $\tilde{Y}_i = T_1 Y_{i,n}$. Similar inequalities to (15), (16) hold with $(1 - 2\rho)$ replaced by $(1 - \rho_1)$, and evidently a similar expression to (17) holds with the obvious notation.

4. The operator $[\cdot]$ and the main lemma. In this section we obtain nonuniform bounds for the differentials of the smoothed convolution density and use these to get a nonuniform speed of convergence for the smoothed convolution density of the truncated variables. Let $v \in B_1^k$ and s be a nonnegative integer; if the function $|x|^s v(x)$ is bounded, define $v^{[s]} \in B_1^k$ by $v^{[s]}(x) = (c, x)^s v(x)$ where c is some fixed unit vector. Evidently $v^{[a+b]} = (v^{[a]})^{[b]}$. Define a similar operation on measures; if $P \in M^k$ and if $\int |x|^s |P|\{dx\} < \infty$ put $P^{[s]}\{dx\} = (c, x)^s P\{dx\}$. Note that when P has a bounded density p, $p^{[s]}$ is the density of $P^{[s]}$. It is a routine matter to show by induction that for positive integers ν, a

$$(\mathscr{P}_1 \cdots \mathscr{P}_\nu)^{[a]} = \textstyle\sum_{\Omega_a} a! \, [\prod_{i=1}^{a} a_i!]^{-1} \mathscr{P}_1^{[a_1]} \cdots \mathscr{P}_\nu^{[a_\nu]} \tag{18}$$

where $\Omega_a = \{(a_1, \cdots, a_\nu): a_i$ nonnegative integers, $\sum_{i=1}^{\nu} a_i = a\}$. If $v \in C^k$, m, s are positive integers, $t \in \mathbb{R}^k$ the following formula for $(v_t^{(m)})^{[s]}$ may be established by induction on m.

$$(19) \qquad (v_t^{(m)})^{[s]} = \sum_{i=0}^{\min(m,s)} (-c, t)^i \binom{m}{i} [s!/(s-i)!] (v^{[s-i]})_t^{(m-i)} .$$

V will denote a probability distribution in $\mathbb{R}^k$ with moments of all orders, possessing a p.d. 1-smooth density (it is easy to construct such a distribution); let $\gamma_r = E|V|^r$. Define $V_h(x) = V(hx)$; when $\beta_r < \infty$ $(r \geqq 3)$, the smoothing distribution will be $U = W^{*(r+1)}$ where $W = V_h$ and $h = 0.1\beta_3^{-1}$. Then both W and U have moments of all orders and possess p.d. h-smooth densities w, u respectively. If $r \geqq 3$, $B > 0$ it is not hard to show that the condition $n \geqq B\beta_r^{2/(r-2)}$ implies $n \geqq 2B\beta_i^{2/(i-2)}$ for all $3 \leqq i \leqq r$. For simplicity we treat $\mathscr{F}^n$ rather than $\mathscr{L}_{i,n}$ in the next lemma.

Lemma 2. *Suppose $\beta_r < \infty$ for some integer $r \geqq 3$. Then for integers m, s with $m \geqq 0$, $0 \leqq s \leqq r$ and $t \in \mathbb{R}^k$,*

$$||(\mathscr{F}^n \bar{u}_t^{(m)})^{[s]}|| \leqq C(k, m, r, B)|t|^m$$

for $n \geqq B\beta_r^{2/(r-2)}$ where $B > 0$ is any constant.

Proof. Consider $((\mathscr{F}^n \bar{u})^{[a]})_t^{(b)}$ where $0 \leqq a \leqq s$, $0 \leqq b \leqq m$. The operator corresponding to the density $(\mathscr{F}^n \bar{u})^{[a]}$ is $(\mathscr{F}^n \mathscr{W}^{r+1})^{(a)}$ (where $\mathscr{W}$ is the operator associated with W), and from (18)

$$(\mathscr{F}^n \mathscr{W}^{r+1})^{[a]} = \sum_{\alpha \in \Omega_a} C(\alpha) \mathscr{F}^{[a_1]} \cdots \mathscr{F}^{[a_n]} \mathscr{W}^{[a_1']} \cdots \mathscr{W}^{[a_{r+1}']}$$

where $\Omega_a = \{\alpha = (a_1, \cdots, a_n, a_1', \cdots, a_{r+1}'): a_i, a_i' \geqq 0, \sum_{i=1}^{n} a_i + \sum_{i=1}^{r+1} a_i' = a\}$. In each term in the above sum at least one a_i' is zero since $a \leqq r$. Writing $\mathscr{A}_\alpha$ for the product of the $\mathscr{W}^{[a_i']}$ excluding the first operator with $a_i' = 0$ we have the following identity for $(\mathscr{F}^n \bar{u})^{[a]}$:

$$(\mathscr{F}^n \bar{u})^{[a]} = \sum_{\alpha \in \Omega_a} C(\alpha) \mathscr{F}^{[a_1]} \cdots \mathscr{F}^{[a_n]} \mathscr{A}_\alpha \bar{w} .$$

Partition Ω_a into the classes $\Omega_a(l)$, $l = 0, 1, \cdots, a$, where $\alpha \in \Omega_a(l)$ iff exactly l of $a_1, \cdots, a_n$ are nonzero. We have $|\Omega_a(l)| \leqq C_1\binom{n}{l} \leqq C_2 n^l$ so that

$$(20) \qquad ||((\mathscr{F}^n \bar{u})^{[a]})_t^{(b)}|| \leqq C_3 \sum_{l=0}^{a} n^l Q_t(a, b, l)$$

where $Q_t(a, b, l) = \sup_{\alpha \in \Omega_a(l)} ||\mathscr{F}^{[a_1]} \cdots \mathscr{F}^{[a_n]} \mathscr{A}_\alpha \bar{w}_t^{(b)}||$. We shall show that

$$(21) \qquad Q_t(a, b, l) \leqq C_4 |t|^b n^{-l} .$$

If $J \in C^k$, then for all $j \geqq 0$

$$(22) \qquad ||\mathscr{W}^{[j]} J|| \leqq \gamma_j h^{-j} n^{-j/2} ||J|| \leqq C_5 ||J||$$

since $n \geqq 2B\beta_3^2$, and for $0 \leqq j \leqq r$

$$(23) \qquad \begin{aligned} ||\mathscr{F}^{[j]} J|| &\leqq \beta_j n^{-j/2} ||J|| \leqq C_6 n^{-1} ||J|| , & j > 1 \\ &\leqq \int ||J_y^{(1)}|| \, |y| \bar{F}\{dy\} , & j = 1 \end{aligned}$$

(using $n \geqq 2B\beta_j^{2/(j-2)}$ for $3 \leqq j \leqq r$). Suppose $\alpha \in \Omega_a(l)$ and let N be the number of a_i equal to one $(0 \leqq N \leqq l)$. Iterating (22) and (23) and remembering that $(n-l)$ of $a_1, \cdots, a_n$ are zero we find that if $N \geqq 1$

$$||\mathscr{F}^{[a_1]} \cdots \mathscr{F}^{[a_n]}\mathscr{A}_\alpha \bar{w}_t^{(b)}|| \leqq C_7 n^{-(l-N)} \int \cdots \int ||J_Y^{(e)}|| \prod_{j=1}^N |y_j| \bar{F}\{dy_j\}$$

where $Y = (y_1, \cdots, y_N)$, $e = (1, \cdots, 1)$ (N terms), and $J = \mathscr{F}^{n-l}\bar{w}_t^{(b)}$. From Corollary 1 we have $||J_Y^{(e)}|| \leqq C_8|t|^b \prod_{j=1}^N |y_j|$ so that (21) holds for $N \geqq 1$; evidently (21) holds also for $N = 0$. Thus from (20)

$$||((\mathscr{F}^n\bar{u})^{[a]})_t^{(b)}|| \leqq C_9|t|^b .$$

Finally, from (19),

$$||(\mathscr{F}^n\bar{u}_t^{(m)})^{[s]}|| \leqq \sum_{i=0}^{\min(m,s)} C(i)|t|^i||((\mathscr{F}^n\bar{u})^{[s-i]})_t^{(m-i)}|| \leqq C_{10}|t|^m$$

using the previous inequality with $a = s - i$, $b = m - i$.

Put $\tilde{\mathscr{L}}_{i,n} = \tilde{\mathscr{F}}^i\tilde{\mathscr{G}}^{n-i}$. We can deduce the following inequality for the truncated distributions.

COROLLARY 2. *If $\Lambda_n(F) \leqq \rho < \frac{1}{2}$ then for all $n \geqq 1$ and nonnegative integers m, s*

$$||(\bar{\tilde{\mathscr{L}}}_{i,n}\bar{u}_t^{(m)})^{[s]}|| \leqq C(k, m, s, \rho)|t|^m .$$

PROOF. Lemma 2 in fact holds for the mixture $\tilde{\mathscr{L}}_{i,n}$ as long as the relevant moment conditions hold for both $\tilde{F}$ and $\tilde{G}$; that is, if there are positive constants B_1 and B_2 such that $n \geqq B_1\tilde{\beta}_s^{2/(s-2)}$ and $n \geqq B_2\tilde{\alpha}_s^{2/(s-2)}$. But from (16)

$$\tilde{\beta}_s \leqq [2(1-2\rho)^{-\frac{1}{2}}n^{\frac{1}{2}}]^{(s-2)}E|X_1|^2 = (B_1^{-1}n)^{(s-2)/2}$$

where $B_1 = \frac{1}{4}(1-2\rho)$; similarly we may take $B_2 = \frac{1}{4}(1-\rho_1)$.

The next lemma is the main result of this section.

LEMMA 3. *If $\Lambda_n(F) \leqq \rho < \frac{1}{2}$ then for all real $s \geqq 0$*

$$|(\bar{\tilde{\mathscr{F}}}^n - \bar{\tilde{\mathscr{G}}}^n)\bar{u}(x)| \leqq C(k, s, \rho)(1 + |x|^s)^{-1}\tilde{\beta}_3 n^{-\frac{1}{2}} .$$

PROOF. From the identity (1) and expansion (2)

$$\begin{aligned} |(\bar{\tilde{\mathscr{F}}}^n - \bar{\tilde{\mathscr{G}}}^n)\bar{u}(x)| &= |(\bar{\tilde{\mathscr{F}}} - \bar{\tilde{\mathscr{G}}}) \sum_{i=0}^{n-1} \bar{\tilde{\mathscr{L}}}_{i,n-1}\bar{u}(x)| \\ &\leqq \tfrac{1}{6} \sum_{i=0}^{n-1} \int |(\bar{\tilde{\mathscr{L}}}_{i,n-1}\bar{u})_y^{(3)}(x - \theta y)|(\bar{\tilde{F}} + \bar{\tilde{G}})\{dy\} . \end{aligned}$$

Using the bound (15) for $\tilde{X}_1$ and the corresponding bound for $\tilde{Y}_1$ we see that $n^{-\frac{1}{2}}|\tilde{X}_1|$, $n^{-\frac{1}{2}}|\tilde{Y}_1| \leqq \frac{1}{2}|x|$ if $|x| \geqq C_1$ for C_1 sufficiently large. Thus the integration may be taken over $|y| < \frac{1}{2}|x|$. Choose, for each value of x, $c = |x|^{-1}x$; then $|(c, (x - \theta y))| \geqq \frac{1}{2}|x|$ if $|y| < \frac{1}{2}|x|$, so for any $s \geqq 0$

$$|(\bar{\tilde{\mathscr{L}}}_{i,n-1}\bar{u})_y^{(3)}(x - \theta y)| \leqq 2^s|x|^{-s}||(\bar{\tilde{\mathscr{L}}}_{i,n-1}\bar{u}_y^{(3)})^{[s]}|| \leqq C_2|x|^{-s}|y|^3$$

from Corollary 2. For $|x| < C_1$ we may apply a uniform bound instead, and hence the result.

5. Truncation lemma. The purpose of this section is to translate Lemma 3 into a result for the original variables. We consider here general moment and pseudo-moment conditions. Define the class Γ of functions on $[0, \infty)$ by $g \in \Gamma$ iff $g(0) = 0$, $g(1) = 1$ and

(i) $g(x)$ is nonnegative and nondecreasing;
(ii) $x/g(x)$ is defined for all x and is nondecreasing.

It is easily seen that (i) and (ii) are equivalent to

$$0 \leqq h^{-1}[g(x+h) - g(x)] \leqq g(x)/x$$

for all $x \geqq 0$, $h > 0$. Let $g \in \Gamma$, r be an integer at least two and define

$$\eta_n = n^{-(r-2)/2} E|X_1|^r g(n^{-\frac{1}{2}}|X_1|) = n \int |x|^r g(|x|) \bar{F}\{dx\} .$$

Suppose $E|X_1|g(|X_1|) < \infty$; we shall require the bound

$$E|S_n|^r g(|S_n|) \leqq C(k, r)(1 + \eta_n) . \tag{24}$$

In the case $k = 1$, (24) is an immediate corollary of the inequality proved in V. Sazonov (1974). (The inequality (24) was obtained independently by the author.) Define for $r = 0, 1, \cdots$ the function g_r on $[0, \infty)$ by $g_0 = g$, and for $r \geqq 1$

$$g_r(x) = \int_0^x g_{r-1}(y)\, dy = \int_0^x [(x-y)^{r-1}/(r-1)!] g(y)\, dy.$$

For $r \geqq 1$, $g_r(x)$ is nonnegative, nondecreasing and convex. If $a \geqq 1$, $x \geqq 0$ it follows from (i) and (ii) that

$$g_r(ax) \leqq a^{r+1} g_r(x) , \tag{25}$$

$$r!\, g_r(x) \leqq x^r g(x) \leqq (r+1)!\, g_r(x) . \tag{26}$$

If $x = (x_1, \cdots, x_k) \in \mathbb{R}^k$ we have $|x| \leqq \sum_{i=1}^k |x_i| \leqq k^{\frac{1}{2}}|x|$; it is easy to prove the following inequalities for $r \geqq 1$ (using the properties of g_r):

$$\begin{gathered} g_r(|x|) \leqq g_r(\textstyle\sum_{i=1}^k |x_i|) \leqq k^r \sum_{i=1}^k g_r(|x_i|) \\ \textstyle\sum_{i=1}^k g_r(|x_i|) \leqq g_r(\sum_{i=1}^k |x_i|) \leqq k^{(r+1)/2} g_r(|x|) . \end{gathered} \tag{27}$$

Using (26), (27) and (24) for $k = 1$ we can establish (24) for $k > 1$. Note that if $f: \mathbb{R}^k \to \mathbb{R}^1$ is an integrable function such that $f(x) = f_1(|x|)$, where $f_1: \mathbb{R}^1 \to \mathbb{R}^1$, then for any Borel set A in $\mathbb{R}^1$

$$\int_{|x| \in A} f(x)\, dx = 2\pi^{k/2}[\Gamma(\tfrac{1}{2}k)]^{-1} \int_A r^{k-1} f_1(r)\, dr . \tag{28}$$

Suppose now that r is an integer at least three, and $g \in \Gamma$. Put $\varepsilon_n = \beta_3 n^{-\frac{1}{2}}$, $h(t) = 1 + t^r g(t)$, $t \geqq 0$.

LEMMA 4. *Suppose $E|X_1|^r g(|X_1|) < \infty$; then for all $n \geqq 1$*

$$\int h(|x|) |(\mathscr{F}^n - \mathscr{G}^n)\bar{u}(x)|\, dx \leqq C(k, r)(\varepsilon_n + \eta_n) .$$

PROOF. Let $\gamma_n = \varepsilon_n + \eta_n$. If $\gamma_n > \frac{1}{12}$ the result is immediate from (24). If $\gamma_n \leqq \frac{1}{12}$ we may take $\rho = \frac{1}{12}$ in Lemma 3, since $\Lambda_n(F) \leqq \gamma_n$. From Lemma 3 and (28)

$$(29) \qquad \int h(|x|)|(\bar{\bar{\mathscr{F}}}^n - \bar{\bar{\mathscr{G}}}^n)\bar{u}(x)|\, dx \leqq C_1\gamma_n \int h(|x|)[1 + |x|^{k+r+2}]^{-1}\, dx \leqq C_2\gamma_n .$$

Using (24), the existence of an $(r+1)$th order moment for U and $n \int_{|y|>1} h(|y|)\bar{F}\{dy\} \leqq \gamma_n$, one may show from (17) that

$$(30) \qquad \int h(|x|)|(\bar{\mathscr{F}}^n - \bar{\mathscr{F}}_n{}^n)\bar{u}(x)|\, dx \leqq C_3\gamma_n$$

and similarly

$$(31) \qquad \int h(|x|)|(\mathscr{G}^n - \mathscr{G}_n{}^n)\bar{u}(x)|\, dx \leqq C_4\gamma_n .$$

We use the following inequality:

$$(32) \qquad |(\bar{\mathscr{F}}_n{}^n - \mathscr{G}^n)\bar{u}(x)| \leqq |(\bar{\bar{\mathscr{F}}}^n - \bar{\bar{\mathscr{G}}}^n)\bar{u}(a(x))| + |(\mathscr{G}_n{}^n - \mathscr{G}^n)\bar{u}(b(x))| + |\mathscr{U}\mathsf{N}_k(b(x)) - \mathscr{U}\mathsf{N}_k(x)|$$

where $a(x) = T(x - n^{\frac{1}{2}}\mu)$, $b(x) = T_1{}^{-1}a(x)$. Applying the change of variable $y = a(x)$, $y = b(x)$ respectively and using the fact that the determinant of a symmetric matrix is the product of its eigenvalues we find that

$$\int h(|x|)|(\bar{\bar{\mathscr{F}}}^n - \bar{\bar{\mathscr{G}}}^n)\bar{u}(a(x))|\, dx \leqq C_5\gamma_n$$

from (29) and

$$\int h(|x|)|(\mathscr{G}_n{}^n - \mathscr{G}^n)\bar{u}(b(x))|\, dx \leqq C_6\gamma_n$$

from (31). Finally, it may be shown that

$$(33) \qquad |\mathscr{U}\mathsf{N}_k(b(x)) - \mathscr{U}\mathsf{N}_k(x)| \leqq C_7[1 + |x|^{k+r+2}]^{-1}\gamma_n$$

by writing $b(x) = A(x - v)$ where $A = T_1{}^{-1}T$, $v = n^{\frac{1}{2}}\mu$ and $t = (I - A)x + Av$. After some computation one finds that $|t| \leqq C_8(1 + |x|)\gamma_n$. Using

$$|\mathsf{N}_k(b(x) - z) - \mathsf{N}_k(x - z)| = |(\textstyle\sum_{i=1}^k t_i D_i)\mathsf{N}_k(x - z - \theta t)| \leqq C_9|t|\,|x - z - \theta t|e^{-\frac{1}{2}|x-z-\theta t|^2}$$

(since $D_i\mathsf{N}_k(x) = -x_i\mathsf{N}_k(x)$), one can eventually establish (33). Thus from (32) we have

$$\int h(|x|)|(\bar{\mathscr{F}}_n{}^n - \mathscr{G}^n)\bar{u}(x)|\, dx \leqq C_{10}\gamma_n$$

which along with (30) yields the result.

6. Application: convergence of integrals. In this section we obtain speeds of convergence for the expectations of both bounded and certain unbounded functions in B^k. Following the notation of Bhattacharya (1970a) define for any real function ϕ on $\mathbb{R}^k$, $A \subset \mathbb{R}^k$

$$\omega_\phi(A) = \sup\{|\phi(x) - \phi(y)| : x, y \in A\} .$$

In particular, if $A = S(x, \varepsilon)$, the open sphere centered at x, radius ε, put $\omega_\phi{}^\varepsilon(x) = \omega_\phi(A)$. A G-continuous function ϕ is a function in $B_1{}^k$ for which $\mathscr{G}\omega_\phi{}^\varepsilon(0) \to 0$

as $\varepsilon \searrow 0$, and for such a function one has

(34) $$\int \phi(y)\bar{F}^{*n}\{dy\} \to \int \phi(y)G\{dy\}$$

since $\bar{F}^{*n}$ converges weakly to G. It is thus natural to seek the speed of convergence in (34); estimates of this speed were given by Bhattacharya (1970b). A class of P-continuous functions $\mathscr{A}$ is said to be a P-uniformity class if the convergence $\int \phi\, dP_n \to \int \phi\, dP$ is uniform over $\mathscr{A}$ for every sequence $\{P_n\}$ of probability measures converging weakly to P. Bhattacharya (1970b) is able to remove the logarithmic term from his general result for G-continuous functions when the family of translates of ϕ is a G-uniformity class. Using a characterization of P-uniformity due to Billingsley and Topsoe (1967), it is seen that such functions ϕ are precisely those for which $\|\mathscr{G}\omega_\phi^{\ \varepsilon}\| \to 0$ as $\varepsilon \searrow 0$. It may be noted that Bhattacharya's result for these functions follows immediately from Lemma 4 and his smoothing lemma (Lemma 8 in [2]). Recently, Bhattacharya (1975) has obtained rates of convergence for the expectations of certain unbounded functions. We prove here a smoothing lemma which applies to both bounded and unbounded functions, and we are able to remove the logarithmic term from Bhattacharya's results.

Let r be an integer at least three, $g \in \Gamma$, and $h(t) = 1 + t^r g(t)$ $(t \geqq 0)$. If $\phi \in B_k$ put

$$\phi^*(x) = [h(|x|)]^{-1}[\phi(x) - \phi(0)]\,.$$

For any real function ϕ on $\mathbb{R}^k$ define

$$\phi^{s,\varepsilon}(x) = \sup_{|y-x|<\varepsilon} \phi(y)\,, \qquad \phi^{i,\varepsilon}(x) = \inf_{|y-x|<\varepsilon} \phi(y)\,.$$

LEMMA 5. *Let P, Q, K be probability measures in $\mathbb{R}^k$ and suppose $h(|x|)$ is integrable with respect to P, Q and K. Let $\phi \in B^k$ and suppose ϕ^* is bounded. Put $D = P - Q$, $\delta = |\int \phi\, dD|$ and let $a > 1$ satisfy*

$$\alpha = \int_{|x|\leqq a} K\{dx\} > \tfrac{1}{2}\,.$$

Let $0 < \varepsilon < \varepsilon' < a^{-1}$ and $t \leqq (a\varepsilon')^{-1}$ be a nonnegative integer. Put $K_\varepsilon\{dx\} = K\{\varepsilon^{-1}\, dx\}$ and

$$\gamma(\varepsilon) = \|\phi^*\| \int h(|x|)|\mathscr{K}_\varepsilon D|\{dx\}$$
$$\zeta(r) = \|\phi^*\| \int_{|x|\geqq ar} h(|x|)K\{dx\}$$
$$\tau(t) = \sup_{|x|<ta\varepsilon'} \mathscr{Q}\omega_\phi^{\ 2a\varepsilon}(x)\,.$$

Then

$$\delta \leqq (2\alpha - 1)^{-1}[C\gamma(\varepsilon) + A_1\zeta(\varepsilon'\varepsilon^{-1}) + \tau(t)] + A_2\|\phi^*\|[(1-\alpha)/\alpha]^t$$

where A_1, A_2 depend on the integrals of $h(|x|)$ with respect to P and Q.

PROOF. Consider $\delta_j = \sup_{|x|<ja\varepsilon'} |\mathscr{D}\phi(x)|$ for a nonnegative integer $j < t$, where $\mathscr{D}$ is the operator associated with D. Suppose that $\sup_{|x|<ja\varepsilon'} \mathscr{D}\phi(x) = \delta_j$. Assume that $\delta_j > \tau(t)$ and let $\eta \leqq \delta_j - \tau(t)$ be some positive number. Let x_j be such that $\mathscr{D}\phi(x_j) > \delta_j - \eta$ $(|x_j| < ja\varepsilon')$. We have

$$\mathscr{K}_\varepsilon \mathscr{D}\phi^{s,a\varepsilon}(x_j) = \int \mathscr{D}\phi^{s,a\varepsilon}(x_j - y)K_\varepsilon\{dy\}\,.$$

Let I_1, I_2, I_3 be the integrals over the regions $|y| < a\varepsilon$, $a\varepsilon \leqq |y| < a\varepsilon'$, $|y| \geqq a\varepsilon'$ respectively. We have

$$\begin{aligned} I_1 &\geqq \int_{|y|<a\varepsilon} \{\mathscr{D}\phi(x_j) - [\mathscr{Q}\phi^{s,a\varepsilon}(x_j - y) - \mathscr{Q}\phi(x_j - y)]\}K_\varepsilon\{dy\} \\ &\geqq \alpha[\delta_j - \eta - \tau(t)]\,; \\ I_2 &\geqq \int_{a\varepsilon\leqq|y|<a\varepsilon'} \{\mathscr{D}\phi(x_j - y) - [\mathscr{Q}\phi^{s,a\varepsilon}(x_j - y) - \mathscr{Q}\phi(x_j - y)]\}K_\varepsilon\{dy\} \\ &\geqq (1 - \alpha)[-\delta_{j+1} - \tau(t)]\,; \\ |I_3| &\leqq ||\phi^*|| \int_{|y|>a\varepsilon'}\int h(|x_j| + a\varepsilon + |y| + |z|)(P + Q)\{dz\}K_\varepsilon\{dy\} \\ &\leqq A_1||\phi^*|| \int_{|y|>a\varepsilon'} h(|y|)K_\varepsilon\{dy\} \leqq A_1\zeta(\varepsilon'\varepsilon^{-1})\,. \end{aligned}$$

Thus

$$\mathscr{K}_\varepsilon\mathscr{D}\phi^{s,a\varepsilon}(x_j) \geqq \alpha\delta_j - (1 - \alpha)\delta_{j+1} - \tau(t) - A_1\zeta(\varepsilon'\varepsilon^{-1}) - \alpha\eta\,.$$

If $\sup_{|x|<ja\varepsilon'} -\mathscr{D}\phi(x) = \delta_j$ we find in a similar way that this inequality holds with $\mathscr{K}_\varepsilon\mathscr{D}\phi^{s,a\varepsilon}(x_j)$ replaced by $-\mathscr{K}_\varepsilon\mathscr{D}\phi^{i,a\varepsilon}(x_j)$. Also

$$\begin{aligned} \mathscr{K}_\varepsilon\mathscr{D}\phi^{s,a\varepsilon}(x_j)\,, \qquad -\mathscr{K}_\varepsilon\mathscr{D}\phi^{i,a\varepsilon}(x_j) &\leqq ||\phi^*|| \int h(|x_j| + a\varepsilon + |y|)|\mathscr{K}_\varepsilon D|\{dy\} \\ &\leqq C\gamma(\varepsilon) \end{aligned}$$

so that, since η was arbitrary, one has

$$\delta_j \leqq \alpha^{-1}\Lambda + \rho\delta_{j+1}$$

where $\Lambda = C\gamma(\varepsilon) + A_1\zeta(\varepsilon'\varepsilon^{-1}) + \tau(t)$, $\rho = (1 - \alpha)/\alpha < 1$; evidently, the inequality holds trivially when $\delta_j \leqq \tau(t)$. Iterating, one has

$$\begin{aligned} \delta = \delta_0 &\leqq \alpha^{-1}\Lambda(1 + \rho + \rho^2 + \cdots + \rho^{t-1}) + \rho^t\delta_t \\ &\leqq (2\alpha - 1)^{-1}\Lambda + \rho^t A_2||\phi^*|| \end{aligned}$$

and hence the lemma.

In order to use Lemma 5 we need to estimate the G-uniformity of the translates of ϕ within a (finite) sphere. Let $h < 1$, $|x| < 1$; we have

$$\mathscr{G}\omega_\phi{}^h(x) = \int \omega_\phi{}^h(x - y)\mathrm{N}_k(y)\,dy\,.$$

Let J_1, J_2 be the integrals over the regions $-((y - \frac{1}{2}x), x) \leqq 1$, $-((y - \frac{1}{2}x), x) > 1$ respectively. Then

$$\begin{aligned} J_1 &= \int_{-((t+\frac{1}{2}x),x)\leqq 1} \omega_\phi{}^h(-t)\mathrm{N}_k(t + x)\,dt \\ &= \int_{-((t+\frac{1}{2}x),x)\leqq 1} e^{-((t+\frac{1}{2}x),x)}\omega_\phi{}^h(-t)\mathrm{N}_k(t)\,dt \leqq e\mathscr{G}\omega_\phi{}^h(0) \\ J_2 &\leqq C_1||\phi^*|| \int_{-((y-\frac{1}{2}x),x)>1} h(|y|)\mathrm{N}_k(y)\,dy \\ &\leqq C_2||\phi^*|| \int_{|y|>(2|x|)^{-1}} (1 + |y|^{r+1})\mathrm{N}_k(y)\,dy \\ &\leqq C_3||\phi^*||\,|x|^{-(k+r-1)}e^{-\frac{1}{8}|x|^{-2}}\,. \end{aligned}$$

Thus

$$\mathscr{G}\omega_\phi{}^h(x) \leqq e\mathscr{G}\omega_\phi{}^h(0) + C_3||\phi^*||\,|x|^{-(k+r-1)}e^{-\frac{1}{8}|x|^{-2}}\,. \tag{35}$$

We now give the main result of this section. As before let $r \geqq 3$, $g \in \Gamma$, $\varepsilon_n = \beta_3 n^{-\frac{1}{2}}$, $\eta_n = n^{-(r-2)/2}E|X_1|^r g(n^{-\frac{1}{2}}|X_1|)$.

THEOREM 1. *Suppose $E|X_1|^r g(|X_1|) < \infty$. Then for all $\phi \in B^k$ with ϕ^* bounded, and $n \geqq 1$*

$$|\int \phi \, d(\bar{F}^{*n} - G)| \leqq C_1(k, r)[||\phi^*||(\varepsilon_n + \eta_n) + \int \omega_\phi^{C_2\varepsilon_n} dG] .$$

PROOF. If $\varepsilon_n + \eta_n \geqq 1$ the result holds trivially in view of (24). When $\varepsilon_n + \eta_n < 1$ the theorem follows from Lemma 4 and Lemma 5 with $P = \bar{F}^{*n}$, $Q = G$, $K_\varepsilon = \bar{U}$ where $a^2\varepsilon = \varepsilon_n$, $\varepsilon' = \varepsilon^{\frac{1}{2}}$, $t = [(a\varepsilon')^{-\frac{1}{2}}]$, and using (35) to estimate $\tau(t)$.

Setting $r = 3$, $g(x) = \min(x, 1)$ and noting that if ϕ is bounded $||\phi^*|| \leqq \omega_\phi(\mathbb{R}^k)$ we have the following result.

COROLLARY 3. *Suppose $\beta_3 < \infty$. Then for all $\phi \in B_1^k$ and $n \geqq 1$*

$$|\int \phi \, d(\bar{F}^{*n} - G)| \leqq C(k, r)[\omega_\phi(\mathbb{R}^k)\varepsilon_n + \int \omega_\phi^{C_2\varepsilon_n} dG] .$$

If, further, the translates of ϕ form a G-uniformity class, then for all $n \geqq 1$

$$||(\mathscr{F}^{*n} - \mathscr{G})\phi|| \leqq C(k, r)[\omega_\phi(\mathbb{R}^k)\varepsilon_n + ||\mathscr{G}\omega_\phi^{C_2\varepsilon_n}||] .$$

The second inequality in Corollary 3 is due to Bhattacharya (1971).

EXAMPLE. Let $r \geqq 3$, $g \in \Gamma$ and let ψ be a real nondecreasing function on $[0, \infty)$ with

$$\psi^*(x) = [1 + x^r g(x)]^{-1}[\psi(x) - \psi(0)]$$

bounded; put $\phi(x) = \psi(|x|)$. For $0 < h < 1$ it is easily verified that

$$\int \omega_\phi^h \, dG \leqq C(k, r)||\psi^*||h .$$

Thus, if $E|X_1|^r g(|X_1|) < \infty$, we have for all $n \geqq 1$

$$|\int \psi(|x|)(\bar{F}^{*n} - G)\{dx\}| \leqq C(k, r)||\psi^*||(\varepsilon_n + \eta_n) . \tag{36}$$

In particular, (36) contains speeds of convergence for the moments and pseudo-moments $E|S_n|^r g(|S_n|)$.

7. Application: nonuniform rates of convergence. If $\phi \in B_1^k$, put $\phi_R(x) = \phi(R^{-1}x)$ $(R > 0)$. Let Δ_k be the class of functions ϕ in B_1^k for which $\inf\{|x| : \phi(x) \neq 0\} = 1$. We obtain nonuniform speeds of convergence for the classes $\{\phi_R : 0 < R < \infty\}$ where ϕ is any G-continuous function in Δ_k. If $\phi \in \Delta_k$, $\varepsilon < \frac{1}{2}$, $R > 1$ we have

$$\begin{aligned} \int \omega_{\phi_R}^\varepsilon(y) \mathrm{N}_k(y) \, dy &= \int_{|y| > \frac{1}{2}} \omega_\phi^{R^{-1}\varepsilon}(R^{-1}y) \mathrm{N}_k(y) \, dy \\ &\leqq A(R) \int \omega_\phi^\varepsilon(z) \mathrm{N}_k(z) \, dz \end{aligned} \tag{37}$$

where $A(R) = Re^{-(R^2-1)/32}$.

Let $r \geqq 3$, $g \in \Gamma$, $h(t) = 1 + t^r g(t)$ $(t \geqq 0)$ and define ε_n, η_n as in Section 6.

THEOREM 2. *Suppose $E|X_1|^r g(|X_1|) < \infty$, $\phi \in \Delta_k$. Then for all $n \geqq 1$, $R > 0$*

$$|\int \phi_R \, d(\bar{F}^{*n} - G)| \leqq C(k, r)[\omega_\phi(\mathbb{R}^k)h(R)^{-1}(\varepsilon_n + \eta_n) + e^{-C_1R^2} \int \omega_\phi^{C_2\varepsilon_n} dG] .$$

PROOF. Since $\phi \in \Delta_k$, we have

$$\phi_R^*(x) = [h(|x|)]^{-1}[\phi(R^{-1}x) - \phi(0)] \leqq [h(R)]^{-1}\omega_\phi(\mathbb{R}^k) .$$

For $R > 1$, the theorem now follows from Theorem 1 applied to ϕ_R, and (37). If $0 < R \leqq 1$, use the first inequality in Corollary 3.

For any set A in $\mathbb{R}^k$ let $(A)^\varepsilon = \{x: |x - y| < \varepsilon$ for some $y \in A\}$ and let ∂A be the boundary of A. Let Δ_1^k be the class of Borel sets A with $\inf_{x\in A} |x| = 1$.

COROLLARY 4. *Suppose $E|X_1|^r g(|X_1|) < \infty$ and A is a Borel set such that either $A \in \Delta_1^k$ or $A^c \in \Delta_1^k$. Then for all $n \geqq 1$, $R > 0$*

$$|\bar{F}^{*n}\{RA\} - G\{RA\}| \leqq C(k, r)[h(R)^{-1}(\varepsilon_n + \eta_n) + e^{-C_1R^2}G\{(\partial A)^{C_2\varepsilon_n}\}]\,.$$

This result includes the nonuniform estimate for distribution functions obtained by Nagaev (1965) for the case $k = 1$, $\beta_3 < \infty$. By taking $g(x) = \min(x, R)$ for $R > 1$ in Corollary 4, we find that if r is an integer at least three and $E|X_1|^r < \infty$ then

$$\begin{aligned} D_{n,R} \leqq C(k, r)[(1 + R^{r+1})^{-1}(\varepsilon_n + n^{-(r-1)/2} \textstyle\int_{E_{n,R}} |x|^{r+1}F\{dx\}) \\ + (1 + R^r)^{-1}(n^{-(r-2)/2} \textstyle\int_{E_{n,R}^c} |x|^r F\{dx\}) + e^{-C_1R^2}G\{(\partial A\}^{C_2\varepsilon_n}\} \end{aligned}$$

where $D_{n,R} = |\bar{F}^{*n}\{RA\} - G\{RA\}|$, $E_{n,R} = \{x: |x| \leqq n^{\frac{1}{2}}(1 + R)\}$. The above inequality includes the nonuniform estimate of Rotar (1970) (when specialized to identically distributed summands) for convex Borel sets. (It may be shown that $G\{(\partial A)^h\} \leqq Ch$ uniformly over all convex Borel sets in $\mathbb{R}^k$.)

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SURREY
GUILDFORD, SURREY GU2 5XH
ENGLAND