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CONDITIONS FOR SAMPLE-CONTINUITY AND THE CENTRAL LIMIT THEOREM

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Let $\{X(t): t \in [0, 1]\}$ be a stochastic process. For any function f such that $E(X(t) - X(s))^2 \leqq f(|t - s|)$, a condition is found which implies that X is sample-continuous and satisfies the central limit theorem in $C[0, 1]$. Counterexamples are constructed to verify a conjecture of Garsia and Rodemich and to improve a result of Dudley.

1. Introduction. Among the most striking features of the classical central limit theory for real-valued independent identically distributed random variables is the wide class of distributions for which weak convergence occurs. The central limit theorem asserts that there is weak convergence to the normal law if and only if the random variables have finite second moments. Because of the fundamental role of this theorem in probability and statistics, it is only natural to seek an analogue for stochastic processes. Since each stochastic process is a random variable in an appropriate function space, the natural setting for the problem is in terms of function space-valued random variables.

Let M be a complete, separable metric space of real-valued functions on $[0, 1]$. Let $\{X_n, n \geqq 1\}$ be independent M-valued random variables with the same distribution $\mathscr{L}(X)$. Assume that they are defined on the same probability space $(\Omega, \mathscr{F}, \Pr)$. Suppose that for $t \in [0, 1]$, $EX(t) = 0$, $EX^2(t) < \infty$. Let $Z_n = (X_1 + \cdots + X_n)/n^{\frac{1}{2}}$. The sequence $\{X_i\}$ is said to *satisfy the central limit theorem* (CLT) *in* M if there exists a Gaussian process Z with sample paths in M such that $\mathscr{L}(Z_n) \to \mathscr{L}(Z)$ weakly in M; i.e., for every bounded continuous real function g on M

$$Eg(Z_n) \to Eg(Z) .$$

Z is called the limiting Gaussian process. By considering finite-dimensional distributions it is easy to see that if the CLT holds for one such sequence then it holds for all sequences with the same properties. Thus, we can unambiguously say that the CLT holds for X, or $\mathscr{L}(X)$, if a sequence $\{X_i\}$ as above satisfies the CLT.

Two function spaces in which many stochastic processes take their sample paths are $C \equiv C[0, 1]$ and $D \equiv D[0, 1]$. $C(S)$ denotes the space of real-valued continuous functions on a compact metric space S with the supremum norm. D is the space of real-valued functions on $[0, 1]$ which are right continuous with left limits and which is endowed with the Skorohod topology.

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Examples of Dudley (1974) show that the CLT may fail in C even for a process possessing uniformly bounded sample paths and satisfying $E(X(t) - X(s))^2 \leqq |t - s|^\alpha$, $\alpha < \frac{1}{2}$. In Section 4 these examples are extended to any $\alpha < 1$. Such examples emphasize the need for additional assumptions in order to show that a process has central limiting behavior.

The bulk of recent attention has been directed towards Banach spaces and in particular $C[0, 1]$, where new techniques have been developed by Strassen and Dudley (1969), Le Cam (1970), de Acosta (1970), Giné (1974), Dudley (1974), Jain and Marcus (1975), Araujo (1975), Hoffmann-Jørgensen and Pisier (1976), and Pisier (1975). A direct investigation of sufficient conditions for D appears in Hahn (1975). However, an alternative approach is to establish conditions which imply that a D-valued random variable is actually C-valued and then prove a CLT in C which applies to it. This is the approach we take here.

In order to show that a process X satisfies the CLT in C it suffices to show that:

(1) There exists a Gaussian process with sample paths in C and the same covariance as X.

(2) The normalized sums $\{Z_n\}$ are uniformly equicontinuous in probability.

This follows from Billingsley ((1968), pages 54, 55, 126) and the finite-dimensional CLT.

In Section 2 of this paper we find a condition on f so that

$$E|X(t) - X(s)|^r \leqq f(|t - s|) \qquad \text{for some} \quad r \geqq 1 \tag{1.1}$$

implies X is sample-continuous. In the case $r = 2$ such a condition is shown to imply conditions (1) and (2), and hence that the CLT holds in C for X.

Dobrushin (see Loève (1963), page 515) established the following criterion for a separable stochastic process not to have jump discontinuities:

$$\sup P[|X(t + h) - X(t)| \geqq \varepsilon] = o_\varepsilon(h)$$

where the supremum is taken over all intervals $[t, t + h]$ in $[0, 1]$. An application of Chebyshev's inequality now shows that if $E|X(t + h) - X(t)|^r = o(h)$ for some $r > 0$, then X cannot have jump discontinuities. In particular, if X satisfies such a moment condition and also has sample paths in D a.s., then X must be sample-continuous.

Kolmogorov (see Loève (1963), page 519) showed that if $X(t)$ is any process for which there exist $\varepsilon > 0$ and $r > 0$ such that $E|X(t) - X(s)|^r \leqq C|t - s|^{1+\varepsilon}$ for some constant C and $|t - s|$ small, then $X(t)$ is sample-continuous. Loève was apparently the first to weaken Kolmogorov's theorem to

$$E|X(t) - X(s)|^r \leqq C \frac{|t - s|}{|\log |t - s||^{\alpha+\varepsilon}}$$

where $\alpha \geqq 1 + r$ and $\varepsilon > 0$. Delporte (1964) showed that it suffices for $\alpha \geqq 1 \vee r$. However, Dudley (1973) showed that $\alpha = -1 - \varepsilon$ is *not* sufficient to guarantee sample-continuity and he posed the problem of finding the best exponent α. In response, Garsia and Rodemich (1974) conjectured that $\alpha \geqq 1 \vee r$ gives the best possible condition. The counterexamples of Section 3 verify their conjecture when $r = 2$. A modification of these examples shows that $\alpha = 2$ is also the smallest exponent which is sufficient to guarantee that X satisfies the CLT in C.

2. Continuity and the central limit theorem. The sufficient condition for sample-continuity which we derive in Theorem 2.3 can be obtained as a corollary of either a theorem of Garsia ((1976), page 86) or a theorem of Delporte ((1964), pages 179–80). Our derivation here will utilize a slight modification of Delporte's theorem.

Let $A_q(\omega) = \sup_{0 \leqq s < 2^{q-1}} |X((s+1)/2^{q-1}, \omega) - X(s/2^{q-1}, \omega)|$, $s \in \mathbb{Z}$. We use $\|\cdot\|_r$ to denote the usual norm on $L^r(\Omega, \mathrm{Pr})$. Throughout this section $\tilde{X}(t)$ will denote a separable version of $X(t)$.

THEOREM 2.1. *Let $\phi(h)$ be a nonnegative function on* $[0, 1]$ *which is nondecreasing in h for h sufficiently small and such that $\phi(h) \to 0$ as $h \to 0$. If $X(t, \omega)$ is a stochastically continuous process satisfying*

$$\textstyle\sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} \|A_q\|_r < \infty\,, \qquad r \geqq 1\,,$$

then there exists a random variable $A \in L^r(\Omega, \mathrm{Pr})$ such that

$$|\tilde{X}(t, \omega) - \tilde{X}(s, \omega)| \leqq A(\omega)\phi(|t-s|) \quad \text{a.s.}$$

for $|t-s|$ sufficiently small.

This theorem is precisely Delporte's result if the hypothesis of stochastic continuity is replaced by separability. Without the hypothesis of separability, his proof shows that the process X is uniformly continuous on the dyadic rationals. If X is also stochastically continuous, it will have a unique continuous version. Furthermore, $A(\omega) = 3 \sum_{j=1}^{\infty} (\phi(2^{-j-1}))^{-1} A_j(\omega)$.

DEFINITION. A function h from $[0, 1]$ into $[0, \infty]$ is called a modulus (of continuity) if and only if both the following hold:

(i) h is continuous and $h(0) = 0$;
(ii) $h(x) \leqq h(x+y) \leqq h(x) + h(y)$ for all $x, y \geqq 0$ with $x + y \leqq 1$.

DEFINITION. Let $X(t)$, $t \in [0, 1]$ be a stochastic process. h is called a *sample modulus* for X if and only if

(i) h is a modulus;
(ii) for almost all ω there exist finite constants k_ω such that for all $s, t \in [0, 1]$

$$|X(t, \omega) - X(s, \omega)| \leqq k_\omega h(|t-s|)\,.$$

If ϕ, in Delporte's theorem, is a nondecreasing function on $[0, 1]$ then $|\tilde{X}(t) - \tilde{X}(s)| \leqq A\phi(|t-s|)$ a.s. for all s, t. If in addition, ϕ is continuous and $\phi(x+y) \leqq \phi(x) + \phi(y)$ for all $x, y \geqq 0$, then ϕ is a sample modulus for $\tilde{X}(t)$.

COROLLARY 2.2. *Let* $X(t)$ *be a stochastic process with* $E|X(t) - X(s)|^r \leqq f(|t - s|)$ *for some* $r \geqq 1$ *and all* $s, t \in [0, 1]$. *If there exists a nonnegative, nondecreasing function* ϕ *on* $[0, 1]$ *such that* $\phi(h) \to 0$ *as* $h \to 0$ *and such that*

$$\sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} 2^{q/r} (f(2^{-q+1}))^{1/r} < \infty , \tag{2.1}$$

then $X(t)$ *is sample-continuous and there exists a random variable* $A \in L^r(\Omega, \Pr)$ *with*

$$|\tilde{X}(t, \omega) - \tilde{X}(s, \omega)| \leqq A(\omega)\phi(|t - s|) \quad \text{a.s.}$$

Furthermore, $||A||_r$ *is bounded by a constant times the series in* (2.1).

PROOF.

$$\begin{aligned} ||A_q||_r &= (E(\sup_{0 \leqq s < 2^{q-1}} |X((s+1)2^{-q+1}) - X(s2^{-q+1})|^r))^{1/r} \\ &\leqq (\textstyle\sum_{s=0}^{2^{q-1}-1} E|X((s+1)2^{-q+1}) - X(s2^{-q+1})|^r)^{1/r} \\ &\leqq (\textstyle\sum_{s=0}^{2^{q-1}-1} f(2^{-q+1}))^{1/r} = 2^{(q-1)/r}(f(2^{-q+1}))^{1/r} . \end{aligned}$$

Since $\sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} 2^{(q-1)/r} (f(2^{-q+1}))^{1/r} < \infty$, Theorem 2.1 implies X is sample-continuous and furthermore

$$|\tilde{X}(t) - \tilde{X}(s)| \leqq A\phi(|t - s|)$$

where $A(\omega) = 3 \sum_{q=1}^{\infty} [\phi(2^{-q-1})]^{-1} A_q(\omega) \in L^r$. □

Here is the main sample-continuity result.

THEOREM 2.3.[1] *Let* f *be a nonnegative function on* $[0, 1]$ *which is nondecreasing in a neighborhood of* 0. *Let* $X(t)$ *be a stochastic process such that for some* $r \geqq 1$, $E|X(t) - X(s)|^r \leqq f(|t - s|)$. *If*

$$\int_0 y^{-(r+1)/r} f^{1/r}(y)\, dy < \infty$$

then there exist a nondecreasing continuous function ϕ *on* $[0, 1]$ *with* $\phi(0) = 0$, *which depends* only *on* f, *and a random variable* $A \in L^r(\Omega, \Pr)$ *such that*

$$|\tilde{X}(s) - \tilde{X}(t)| \leqq A\phi(|t - s|) .$$

Moreover, $||A||_r$ *is bounded above by a constant depending only on* f *and* ϕ.

PROOF. Since f is nondecreasing near 0, the integral condition is equivalent to the condition

$$\sum_{q=1}^{\infty} 2^{q/r} (f(2^{-q+1}))^{1/r} < \infty .$$

This in turn implies that there exists a sequence $c_q \nearrow \infty$ as $q \nearrow \infty$ such that

$$\sum_{q=1}^{\infty} c_q 2^{q/r} (f(2^{-q+1}))^{1/r} < \infty .$$

Define ϕ by setting $\phi(2^{-q-1}) = c_q^{-1}$, and linearly in between. An application of Corollary 2.2 finishes the proof. □

[1] This theorem is used in Hahn and Klass (1977) to obtain a best possible sufficient condition for sample-continuity when the only known information about a process $X(t)$ is of the form $E(X(t) - X(s))^2 \leqq f(|t - s|)$.

Define $\log_k |x|$ inductively by $\log_0 |x| = |x|$, $\log_1 |x| = \log |x|$, the usual natural logarithm, and $\log_k |x| = \log |\log_{k-1} |x||$. Define $e_k(u)$ inductively by $e_1(u) = e^u$ and $e_k(u) = \exp(e_{k-1}(u))$. Let $E_k = \exp(-e_{k-1}(u))$. We will suppress the 1 if $u = 1$.

For $k = 1, 2, \cdots$ and $\varepsilon > 0$ let

$$f_{k,\varepsilon}(x) = |x|/(|\log |x||^2 \cdots |\log_{k-1} |x||^2 |\log_k |x||^{2+\varepsilon}) .$$

Note that $f_{k,\varepsilon}(x)$ decreases to 0 as $x \to 0$ for $x < E_k$. We now consider the special case of Theorem 2.3 when $r = 2$ and $f(x) = O(f_{k,\varepsilon}(x))$ as $x \to 0$.

COROLLARY 2.4. *Given $k \in \mathbb{N}$ and $\varepsilon > 0$, let $X(t)$ be a stochastic process with $E(X(t) - X(s))^2 = O(f_{k,\varepsilon}(|t - s|))$ as $|t - s| \to 0$. Then $X(t)$ is sample-continuous. Furthermore if ϕ is a nonnegative, nondecreasing function on* $[0, 1]$ *with $\phi(u) = |\log_{k+1} |u||^{-\varepsilon'/2}$ for $|u| < E_k$ and $\varepsilon' < \varepsilon$, then there exist $\delta > 0$ with $\delta < E_k$ and a random variable $A \in L^2(\Omega, \mathscr{F}, \Pr)$ such that for $|t - s| < \delta$,*

$$|\tilde{X}(t) - \tilde{X}(s)| \leqq A\phi(|t - s|) \quad \text{a.s.}$$

PROOF. Let $\gamma < E_k$ be such that $|t - s| < \gamma$ implies $E(X(t) - X(s))^2 \leqq Cf_{k,\varepsilon}(|t - s|)$. Sample-continuity follows from Theorem 2.3 by setting $f(|x|) = Cf_{k,\varepsilon}(|x|)$ for $|x| < \gamma$. A routine computation shows that f and ϕ satisfy the hypotheses of Corollary 2.2; hence, the Lipschitz condition holds. □

This result is best possible, in the sense that for each $k \in \mathbb{N}$ there is a process satisfying

$$E(X(t) - X(s))^2 = O(f_{k,0}(|t - s|)) \qquad \text{as} \quad |t - s| \to 0$$

which has no continuous version. One class of examples is constructed in Section 3 and another class is constructed in Hahn and Klass (1977). For both classes of examples the processes fail to have versions with finite right and left limits at all points. This is unavoidable due to the theorem of Dobrushin which was mentioned in the introduction.

THEOREM 2.5. *Let f be a nonnegative function on* $[0, 1]$ *which is nondecreasing near* 0. *Let $X(t)$ be a stochastic process with mean* 0, *finite second moments, and sample paths in D, satisfying*

$$E(X(t) - X(s))^2 \leqq f(|t - s|) \qquad \textit{for} \quad |t - s| \quad \textit{small}$$

and

$$\int_0 y^{-\frac{3}{2}} f^{\frac{1}{2}}(y)\, dy < \infty .$$

Then X is sample-continuous and satisfies the CLT *in C.*

PROOF. According to Theorem 2.3, X is sample-continuous. Let $\{X_i\}$ be a sequence of independent, identically distributed C-valued random variables with law $\mathscr{L}(X)$. Let $Z_n = (X_1 + \cdots + X_n)/n^{\frac{1}{2}}$. $E(Z_n(t) - Z_n(s))^2 = E(X(t) - X(s))^2$ for all n; so Theorem 2.3 implies that there exist a nondecreasing continuous function ϕ on [0, 1] with $\phi(0) = 0$, which depends only on f, and random variables $A^{(n)} \in L^2(\Omega, \Pr)$ such that

$$|Z_n(t) - Z_n(s)| \leqq A^{(n)}\phi(|t - s|) .$$

Since $||A^{(n)}||_2 \leqq M < \infty$ with M depending only on f and ϕ, a simple application of Chebyshev's inequality will show that the $\{Z_n\}$ are uniformly equicontinuous in probability.

Let Z be a Gaussian process with the same covariance as X. Since $E(Z(t) - Z(s))^2 = E(X(t) - X(s))^2$, Z satisfies the hypotheses of Theorem 2.3; hence, Z is sample-continuous. Therefore, X satisfies the CLT in C. □

3. Counterexamples. We now provide examples to show that the exponents of the logarithms appearing in Corollary 2.4 are best possible with respect to sample-continuity, thus confirming the conjecture of Garsia and Rodemich (1974).

PROPOSITION 3.1. *For any* $k \in \mathbb{N}$ *there is a process* $X(t, \omega)$, $0 \leqq t \leqq 1$, *which satisfies*

$$E(X(t) - X(s))^2 \leqq Cf_{k,0}(|t - s|)$$

but which is not sample-continuous.

PROOF. Let the definitions of $\log_k |x|$ and $E_k(x)$ be as in Section 2. Define

$$\begin{aligned} X(t, \omega) = X_k(t, \omega) &= \log_{k+1} |(1 + 2t)/4 - \omega| \\ &\qquad \text{if } \ |(1 + 2t)/4 - \omega| \leqq E_k , \quad \omega \neq (1 + 2t)/4 \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

where $t \in [0, 1]$ and $\omega \in ([0, 1]$, Lebesque). For each ω, $\frac{1}{4} \leqq \omega \leqq \frac{1}{2}$, $X(t, \omega)$ is discontinuous at $t = 2\omega - \frac{1}{2}$ and is unbounded in a neighborhood of that point. Estimation of the mean-square differences involves only calculus type facts, hence we omit the computations. □

A slight modification of the above processes, $X_k(t, \omega)$, yields sample-continuous, mean zero processes which satisfy the same moment conditions; hence, showing that the exponents of the logarithms appearing in Corollary 2.4 are also best possible with respect to the CLT in C.

PROPOSITION 3.2. *For any* $k \in \mathbb{N}$, *there exists a continuous stochastic process* $Y(t, \omega)$ *which does not satisfy the* CLT *in* $C[0, 1]$ *but such that*

$$E(Y(t) - Y(s))^2 \leqq Cf_{k,0}(|t - s|) .$$

PROOF. For $X(t, \omega) \equiv X_k(t, \omega)$ as in the proof of Proposition 3.1, let

$$\begin{aligned} \tilde{X}(t, \omega) &= X(t, \omega) && \text{if } \ |(1 + 2t)/4 - \omega| \geqq \delta_\omega , \quad \omega \neq \tfrac{3}{4} \\ &= \log_{k+1} \delta_\omega^{-1} && \text{if } \ |(1 + 2t)/4 - \omega| \leqq \delta_\omega , \quad \omega \neq \tfrac{3}{4} \\ &= 0 && \text{if } \ \omega = \tfrac{3}{4} \end{aligned}$$

where $\delta_\omega = E_{k+1}(8/(3 - 4\omega))$.

Let

$$Y(t, \underline{\omega}) = Y(t, \omega \times j) = \tilde{X}(t, \omega) \quad \text{if } j = 0$$
$$= -\tilde{X}(t, \omega) \quad \text{if } j = 1$$

where $\underline{\omega} \in ([0, 1] \times \{0, 1\}$, Lebesgue $\times (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1))$. Let $Y^{(i)}(t)$, $i = 1, 2, \cdots$ denote i.i.d. copies of $Y(t)$ and $Z_n(t) = (1/n^{\frac{1}{2}}) \sum_{i=1}^{n} Y^{(i)}(t)$. In order to show that $Y(t)$ does not satisfy the CLT it suffices to show that $\{Z_n\}$ is not uniformly bounded in probability; i.e., there exists $\varepsilon > 0$ such that for $a > 0$ there is an $n(a)$ for which $P\{\sup_t |Z_{n(a)}(t)| \geqq a\} > \varepsilon$. This can be verified by first showing that for any $b \in \mathbb{R}^+$, there exists N_b such that $n \geqq N_b$ implies that

$$P\{\max_{1 \leqq i \leqq n} \sup_t (1/n^{\frac{1}{2}})|Y^{(i)}(t)| \geqq 2b\} > \tfrac{1}{4},$$

and then applying the Lévy inequality for processes (see Dudley (1967), Lemma 4.4, page 300; or Kahane (1968), Lemma 1, page 12). □.

The above examples strongly suggest that Lipschitz conditions on second moments of the increments alone imply that the CLT holds in C only if they also imply sample-continuity. This is indeed the case as verified independently by Hahn (1976) and Pisier (1976). Neither approach, however, gives examples of the Strassen–Dudley type with uniformly bounded sample paths.

4. Counterexamples with uniformly bounded sample paths. The examples in the previous section show that certain Lipschitz conditions on the second moments of increments do not imply that X satisfies the CLT in C. Thus, it is necessary to place additional assumptions on X.

We say X is uniformly sample-bounded if there exists $C \geqq 0$ such that $\sup_{t,\omega} |X(t, \omega)| \leqq C$.

Below we show that even under the strong assumption of X being uniformly sample-bounded, the condition $E(X(t) - X(s))^2 \leqq |t - s|^\alpha$ for $\alpha < 1$ does not imply that X satisfies the CLT in C. If $\alpha > 1$, X satisfies the hypotheses of Theorem 2.5 and hence the CLT in C. The case of $\alpha = 1$, which includes those processes for which Brownian motion is the limiting Gaussian process, remains unsolved. In fact, we do not know of a single process with the covariance of Brownian motion for which the CLT fails.

The following examples are based on the same scheme as those in Strassen and Dudley (1969) and Dudley (1974).

PROPOSITION 4.1. *For any* $\alpha < 1$ *there is a process* $X(t, \omega)$, $0 \leqq t \leqq 1$, *with continuous sample paths*, $|X(t, \omega)| \leqq 1$ *for all* t *and* ω, *and* $E(X(s) - X(t))^2 \leqq |s - t|^\alpha$ *for all* $s, t \in [0, 1]$ *such that the* CLT *does not hold for* X.

PROOF. The following definition of a process X depending on a choice of constants $\{k_n, n = 1, 2, \cdots\}$ is taken verbatim from Dudley ((1974), pages 56–57). For each $n = 1, 2, \cdots$, (we) shall decompose $[0, 1]$ into a set I_n of N_n equal subintervals, where $N_n = \prod_{s=1}^{n} 6k_s$, k_s integers. Thus each interval in I_{n-1} is decomposed into $6k_n$ equal subintervals to form I_n, where $I_0 = \{[0, 1]\}$.

For each n and each $j = 0, \cdots, k_n - 1$, (we) define a piecewise linear continuous function g_{nj} as follows. Let

$$\begin{aligned} g_{nj}(x) &= 0 && \text{if } N_n x/3 \text{ is an integer} \\ &= 1 && \text{if } 6i + 1 \leqq N_n x \leqq 6i + 2 \\ &= -1 && \text{if } 6i + 4 \leqq N_n x \leqq 6i + 5 \end{aligned}$$

where

$$i = j + rk_n , \quad r = 0, 1, \cdots, N_{n-1} - 1 ,$$

and let g_{nj} be continuous and linear on those closed intervals for which it was previously defined only at the endpoints, namely $6i + u \leqq N_n x \leqq 6i + u + 1$, $u = 0, 2, 3, 5$.

Note that for each j, inside every interval in I_{n-1} is an interval in I_n on which $g_{nj} = 1$ and another on which $g_{nj} = -1$.

Let $p_n = cn^{-\beta}$ where $1 < \beta < 2$ and $c = 1/\sum_{n=1}^{\infty} n^{-\beta}$. To be definite, (we) take $\beta = \frac{3}{2}$.

Now (we) define a probability measure μ on $C[0, 1]$ by setting $\mu(\{g_{nj}\}) = \mu(\{-g_{nj}\}) = p_n/2k_n$ for $n = 1, 2, \cdots$ and each $j = 0, \cdots, k_n - 1$. Let X be a random variable with distribution μ. Then clearly $|X(t)| \leqq 1$. Also for each t, $EX(t) = 0$ since X is symmetric and bounded.

Dudley proves that the CLT never holds for X with the given p_n, for any $k_n \geqq 2$. What remains is for us to choose an appropriate sequence $\{k_n, n = 1, 2, \cdots\}$ and then estimate the mean-square differences. Define k_n inductively by:

$$\begin{aligned} k_1 &= 2 \qquad N_1 = 6k_1 = 12 \\ k_n &= N_{n-1}^{\alpha/(1-\alpha)} \quad \text{for} \quad n \geqq 2 \quad \text{and} \quad \alpha \quad \text{fixed} \quad < 1 . \end{aligned}$$

Note that

$$N_n = 6k_n N_{n-1} = 6N_{n-1}^{1/(1-\alpha)} , \qquad n \geqq 2 . \tag{4.1}$$

Now we estimate the mean-square differences. Given $s, t \in [0, 1]$, take n such that

$$1/N_{n+1} < |s - t| \leqq 1/N_n , \qquad \text{where} \quad N_0 = 1 .$$

As Dudley has shown (page 58),

$$\begin{aligned} E(X(s) - X(t))^2 \leqq \textstyle\sum_{m<n} 2p_m k_m^{-1}(6N_m|t - s|)^2 + 2p_n k_n^{-1} N_n^2 |s - t|^2 \\ + 8(\textstyle\sum_{m>n}' p_m k_m^{-1}) . \end{aligned}$$

For $n = 0$, we have by (4.1)

$$E(X(s) - X(t))^2 \leqq 4 = 48N_1^{-1} \leqq 48|s - t| \leqq 48|s - t|^\alpha .$$

For $n = 1$,

(1) The first term is vacuous,

(2) $2p_1 k_1^{-1} N_1^2 |s - t|^2 \leqq N_1^2 |s - t|^2 \leqq 144|s - t|^2 \leqq 144|s - t|^\alpha$,

(3) $8p_{n+1}k_{n+1}^{-1} \leqq 8N_n^{-\alpha/(1-\alpha)} \leqq 48N_{n+1}^{-\alpha} \leqq 48|s-t|^\alpha$,

(4) $8\sum_{m=n+2}^{\infty} p_m k_m^{-1} = 8\sum_{m=n+2}^{\infty} p_m N_{m-1}^{-\alpha/(1-\alpha)} \leqq 8N_{n+1}^{-\alpha/(1-\alpha)} \sum_{m=n+2}^{\infty} p_m$

$$< 8N_{n+1}^{-\alpha/(1-\alpha)} \leqq 8|s-t|^{\alpha/(1-\alpha)} \leqq 8|s-t|^\alpha .$$

Therefore $E(X(s)-X(t))^2 \leqq 200|s-t|^\alpha$.

For $n \geqq 2$,

(1′) $2p_1k_1^{-1}(6N_1|s-t|)^2 \leqq (36 \cdot 144)|t-s|^2 \leqq 5184|t-s|^\alpha$,

(1″) $\sum_{1<m<n} 2p_m k_m^{-1} 36 N_m^2 |s-t|^2 = \sum_{1<m<n} 72 p_m N_{m-1}^{-\alpha/(1-\alpha)} N_m^2 |s-t|^2$

$$\leqq |s-t|^\alpha \sum_{1<m<n} 432 p_m N_m^{2-\alpha} |s-t|^{2-\alpha}$$

$$\leqq |s-t|^\alpha 432 \sum_{1<m<n} p_m \leqq 432|s-t|^\alpha ,$$

(2′) $2p_n k_n^{-1} N_n^2 |s-t|^2 \leqq N_{n-1}^{-\alpha/(1-\alpha)} N_n^2 |s-t|^2 \leqq 6N_n^{-\alpha} N_n^2 |s-t|^2$

$$= 6N_n^{2-\alpha}|s-t|^2 \leqq 6|s-t|^\alpha ,$$

(3) and (4) are as for the case $n = 1$ above.

Therefore $E(X(s)-X(t))^2 \leqq 5678|s-t|^\alpha$.

Replace X by $X/76$ to get rid of the constant. □

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