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ENTRANCE LAWS FOR MARKOV CHAINS

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Let S be a countable set and let Q be a stochastic matrix on $S \times S$. An entrance law for Q is a collection $\boldsymbol{\mu} = \{\mu_n\}_{n \in \mathbb{Z}}$ of probability measures on S such that $\mu_n Q = \mu_{n+1}$ for all $n \in \mathbb{Z}$. There is a natural correspondence between entrance laws and Markov chains ξ_n with stationary transition probabilities Q and time parameter set $\mathbb{Z}$. The set $\mathscr{L}(Q)$ of entrance laws is examined in the discrete and continuous time setting. Criteria are given which insure the existence of nontrivial entrance laws.

0. Introduction. Let S be a countable set, and let Q be a stochastic matrix on $S \times S$. $\mathbb{Z}$ will denote the set of integers, with $\mathbb{Z}^+(\mathbb{Z}^-)$ the nonnegative (nonpositive) integers. For any two functions $u, v : S \to R$ we will write

$$uQ(y) = \sum_{x \in S} u(x)Q(x, y) , \qquad Qv(x) = \sum_{y \in S} Q(x, y)v(y)$$

whenever the sums exist. We follow Dynkin [6] in making

DEFINITION 0.1. An *entrance law* for Q is a family $\boldsymbol{\mu} = \{\mu_n\}_{n \in \mathbb{Z}}$ of probability measures on S such that

$$\mu_n Q = \mu_{n+1} , \quad n \in \mathbb{Z} . \tag{0.1}$$

(We write $\mu_n(x)$ for $\mu_n(\{x\})$.)

The motivation for studying entrance laws is simple: there is a natural correspondence between entrance laws and (time homogeneous) Markov chains ξ_n which have time parameter set $\mathbb{Z}$ instead of $\mathbb{Z}^+$. To be precise, let $\Omega = S^{\mathbb{Z}}$, and let $\mathscr{F}$ be the σ-algebra on Ω generated by finite cylinder sets. If $\boldsymbol{\mu}$ is a Q entrance law, define $\mathscr{P}$ on $(\Omega, \mathscr{F})$ by

$$\begin{aligned}\mathscr{P}(\{\omega : \omega_n = x_0, \omega_{n+1} = x_1, \cdots, \omega_{n+k} = x_k\}) \\ = \mu_n(x_0)Q(x_0, x_1) \cdots Q(x_{k-1}, x_k)\end{aligned} \tag{0.2}$$

for all $n \in \mathbb{Z}$, $k \in \mathbb{Z}^+$, and $x_i \in S$. It is a simple matter to check using (0.1) and the Kolmogorov extension theorem that $\mathscr{P}$ is a well defined probability measure on $(\Omega, \mathscr{F})$. Letting ξ_n be the coordinate process $\xi_n(\omega) = \omega_n$, it is easy to see that the Markov property holds,

$$\mathscr{P}[\xi_{n+1} = y \mid \xi_k, k \leqq n] = Q(\xi_n, y) \quad \mathscr{P}\text{-a.e.} \tag{0.3}$$

Conversely, if $\mathscr{P}$ and ξ_n defined on some probability space satisfy (0.3), then $\boldsymbol{\mu}$ defined by

$$\mu_n(x) = \mathscr{P}[\xi_n = x] \tag{0.4}$$

is an entrance law for Q. Hence the correspondence is established.

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If Q has an invariant probability measure π (π will always be used to denote an invariant probability measure of Q, if one exists) then it is a simple matter to construct an entrance law $\boldsymbol{\mu}$, just set $\mu_n = \pi$ for all n. An entrance law of this form will be called *trivial*; all others will be called *nontrivial*. A trivial entrance law leads to the construction of a stationary Markov chain. The fact that nontrivial entrance laws for strictly positive matrices exist was established by Spitzer in [15].

Our purpose here is to investigate the set $\mathscr{L} = \mathscr{L}(Q)$ of entrance laws for Q. Both Föllmer [10] and Spitzer [15] refer to entrance laws in their papers, but use them primarily as a tool in their investigation of countable one-dimensional Markov random fields.

In [6], [7] and [8] Dynkin introduces and studies Markov processes with random birth and death times. The entrance laws presented here correspond to his processes with birth time $-\infty$, death time $+\infty$. We refer the reader to his papers for results in the general case.

In Section 1 the (convex) set $\mathscr{L}$ is shown to be completely determined by its extreme points. In addition a useful method for computing the extreme points is obtained. Section 2 attempts to characterize the existence of entrance laws in terms of the convergence of certain hitting times. We show that nontrivial entrance laws "come down from infinity." In Section 3 Q is replaced by a semigroup $\{p(t)\}_{t \geqq 0}$ and the previous results are shown to remain true in this setting. Birth and death processes are examined in Section 4. Our results become simpler and more concrete for these semigroups. Examples are given in Section 5 to illustrate our results.

1. The representation theorem. Let $s_1, s_2, \cdots$ be any fixed enumeration of S, and write $|x| = k$ if $x = s_k$. We give $\mathscr{L}$ the topology of pointwise convergence, and the corresponding σ-algebra of Borel subsets. That is, if $\boldsymbol{\mu}^k$, $\boldsymbol{\mu} \in \mathscr{L}$, then $\boldsymbol{\mu}^k \to \boldsymbol{\mu}$ as $k \to \infty$ means

$$\lim_{k\to\infty} \mu_n{}^k(x) = \mu_n(x)\ , \qquad n \in \mathbb{Z}\ , \quad x \in S\ . \tag{1.1}$$

Since $\mathbb{Z} \times S$ is countable this actually determines a metric topology on $\mathscr{L}$. This topology is related to the usual weak convergence ($\Rightarrow$) of probability measures in the following way: $\boldsymbol{\mu}^k \to \boldsymbol{\mu}$ in $\mathscr{L}$ if and only if $\mu_n{}^k \Rightarrow \mu_n$ for all $n \in \mathbb{Z}$.

Defining addition in the natural way makes $\mathscr{L}(Q)$ a convex set. If $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{L}(Q)$, $0 \leqq \alpha \leqq 1$, then $\boldsymbol{\varphi} \in \mathscr{L}(Q)$, where

$$\varphi_n(x) = \alpha\mu_n(x) + (1 - \alpha)\nu_n(x)\ . \tag{1.2}$$

The set of extreme points of $\mathscr{L}(Q)$ will be denoted by $\mathscr{E}(Q)$.

$\mathscr{L}(Q)$ is also closed under "translations." That is, if $\boldsymbol{\mu} \in \mathscr{L}(Q)$, and $\nu_n = \mu_{n+1}$, then $\boldsymbol{\nu} \in \mathscr{L}(Q)$. Thus if $\mathscr{L}(Q)$ contains one nontrivial entrance law it must contain infinitely many nontrivial ones. We can now state

THEOREM 1.1. *Assume Q is irreducible. If $\boldsymbol{\mu} \in \mathscr{L}(Q)$ then there is a unique* *probability measure λ on $\mathscr{E}(Q)$ such that*

$$\mu_n(x) = \int_{\mathscr{E}(Q)} \varphi_n(x)\, d\lambda(\varphi)\,, \qquad n \in \mathbb{Z}\,, \quad x \in S\,. \tag{1.3}$$

Any probability measure λ on $\mathscr{E}(Q)$ determines an element $\boldsymbol{\mu}$ of $\mathscr{L}(Q)$ via (1.3). *Furthermore, if $\boldsymbol{\mu} \in \mathscr{E}(Q)$, then there is a sequence $x_1, x_2, \cdots$ of element of S such that*

$$\mu_n(x) = \lim_{k\to\infty} Q^{n+k}(x_k, x)\,, \qquad n \in \mathbb{Z}\,, \quad x \in S\,. \tag{1.4}$$

REMARKS. (1) If in addition Q is aperiodic and $\boldsymbol{\mu} \in \mathscr{E}(Q)$ is nontrivial, then the sequence x_k in (1.4) has the property that $|x_k| \to \infty$. For if not, then there must be an element $y \in S$ and a subsequence $x_{k'}$ of x_k with $x_{k'} \equiv y$. In this case (1.4) becomes

$$\begin{aligned} \mu_n(x) = \lim_{k'\to\infty} Q^{n+k'}(y, x) &= \pi(x)\,, \qquad Q \text{ positive recurrent} \\ &= 0 \qquad\qquad\;\; \text{otherwise} \end{aligned}$$

by Theorem 69 in [11]. This contradicts $\boldsymbol{\mu} \in \mathscr{E}(Q)$ being nontrivial.

(2) Theorem 1.1 is actually a special case of Theorem 3.1 in [7]. The method of proof presented here (which consists of identifying the solutions of $\mu_n Q = \mu_{n+1}$ with the harmonic functions of a related operator p) is used by Föllmer in [10]. The technique was first outlined by Dynkin in [6]. Preston uses different methods to derive an integral representation theorem for random fields in [14].

LEMMA 1.2. *If Q is irreducible, and $\mathscr{L}(Q) \neq \emptyset$, then there exists $\boldsymbol{\nu} \in \mathscr{L}(Q)$ which is strictly positive*, i.e., $\nu_n(x) > 0$ *for all $n \in \mathbb{Z}$, $x \in S$.*

PROOF. Fix $m \in \mathbb{Z}$, $y \in S$. Since $\mathscr{L}(Q)$ is convex, it suffices to produce $\boldsymbol{\varphi} \in \mathscr{L}(Q)$ such that $\varphi_m(y) > 0$. Assume $\boldsymbol{\mu} \in \mathscr{L}(Q)$, choose $z \in S$ such that $\mu_0(z) > 0$. Since Q is irreducible there is a positive integer N with $Q^N(z, y) > 0$. Hence,

$$\mu_N(y) \geqq \mu_0(z) Q^N(z, y) > 0\,.$$

Since $\mathscr{L}(Q)$ is also closed under translations, define $\boldsymbol{\varphi}$ by

$$\varphi_n(x) = \mu_{n+N-m}(x)\,.$$

Then $\boldsymbol{\varphi} \in \mathscr{L}(Q)$ and $\varphi_m(y) > 0$. □

Let $E = \{[n, x] : n \in \mathbb{Z}^-, x \in S\}$, with $\boldsymbol{\nu} \in \mathscr{L}(Q)$ strictly positive. Define a function p on $E \times E$ by

$$\begin{aligned} p([n, x], [m, y]) &= 0 \qquad\qquad\qquad\quad m \neq n - 1 \\ &= \frac{\nu_{n-1}(y) Q(y, x)}{\nu_n(x)} \quad\; m = n - 1\,. \end{aligned} \tag{1.5}$$

p is a well-defined transient stochastic matrix on $E \times E$, which (intuitively) governs a Markov chain which is a kind of reverse space-time chain for the original Q Markov chain. The operator p certainly depends upon the choice of $\boldsymbol{\nu} \in \mathscr{L}$; nevertheless our results will not.

Define a *standard measure* γ on E by

$$\begin{aligned} \gamma(\eta) &= 0\,, \qquad\;\; \eta \neq [0, x] \\ &= \nu_0(x)\,, \quad \eta = [0, x]\,. \end{aligned} \tag{1.6}$$

A function f on E will be called *normalized* if

$$\sum_{\eta\in E} f(\eta)\gamma(\eta) = 1 . \tag{1.7}$$

A *nonnegative* function f defined on E such that

$$pf(\eta) = f(\eta) \tag{1.8}$$

will be called p-harmonic. Let $\mathscr{H} = \mathscr{H}(Q)$ denote the set of normalized p-harmonic functions, with the topology of pointwise convergence. $\mathscr{H}$ is convex and $\mathscr{E}(\mathscr{H})$ will denote its extreme points.

LEMMA 1.3. *There is a one-to-one onto homeomorphism ψ between $\mathscr{L}$ and $\mathscr{H}$. If $\boldsymbol{\mu} \in \mathscr{L}$, then $\psi(\boldsymbol{\mu}) = f$, where*

$$f([n, x]) = \frac{\mu_n(x)}{\nu_n(x)} . \tag{1.9}$$

PROOF. Assume $\boldsymbol{\mu} \in \mathscr{L}$ and define f by (1.9). Then f is clearly nonnegative and

$$\begin{aligned} pf([n, x]) &= \sum_{y\in S} p([n, x], [n-1, y])f([n-1, y] \\ &= \sum_{y\in S} \frac{\nu_{n-1}(y)Q(y, x)}{\nu_n(x)} \cdot \frac{\mu_{n-1}(y)}{\nu_{n-1}(y)} \\ &= \frac{\mu_{n-1}Q(x)}{\nu_n(x)} \\ &= f([n, x]) \end{aligned}$$

since $\mu_{n-1}Q = \mu_n$. The normalization (1.7) follows immediately from the fact that μ_0 is a probability measure on S. Hence $f \in \mathscr{H}$.

Conversely, assume $f \in \mathscr{H}$, and for $n \in \mathbb{Z}^-$ define μ_n by (1.9). For $n \in \mathbb{Z}^+$, set $\mu_n = \mu_0 Q^n$ so that $\mu_n Q = \mu_{n+1}$ certainly holds for $n \in \mathbb{Z}^+$. For $n \in \mathbb{Z}^-$ we simply reverse the steps in the previous paragraph to obtain $\mu_n Q = \mu_{n+1}$. It remains to show that each μ_n is actually a probability measure on S. For $n = 0$ this is equivalent to (1.7). For $n \in \mathbb{Z}^+$,

$$\begin{aligned} \sum_{y\in S} \mu_n(y) &= \sum_{y\in S} \mu_0 Q^n(y) \\ &= \sum_{x\in S} \sum_{y\in S} \mu_0(x)Q^n(x, y) \\ &= 1 . \end{aligned}$$

For $n \in \mathbb{Z}^-$,

$$\begin{aligned} \sum_{x\in S} \mu_n(x) &= \sum_{x\in S} \mu_n(x) \sum_{y\in S} Q^{-n}(x, y) \\ &= \sum_{y\in S} \mu_n Q^{-n}(y) \\ &= 1 \end{aligned}$$

since $\mu_n Q^{-n} = \mu_0$. Hence $\boldsymbol{\mu} \in \mathscr{L}$, and ψ is clearly one-to-one and onto.

To show that ψ is a homeomorphism, it suffices to show that if $\boldsymbol{\mu}^k$, $\boldsymbol{\mu} \in \mathscr{L}$, and $\psi(\boldsymbol{\mu}^k) = f_k$ and $\psi(\boldsymbol{\mu}) = f$, then $\boldsymbol{\mu}^k \to \boldsymbol{\mu}$ if and only if $f_k \to f$. Now if $\boldsymbol{\mu}^k \to \boldsymbol{\mu}$, then

$$\begin{aligned} f_k([n, x]) &= \frac{\mu_n^k(x)}{\nu_n(x)} \\ &\to \frac{\mu_n(x)}{\nu_n(x)} \\ &= f([n, x]) . \end{aligned}$$

Conversely, this argument shows that if $f_k \to f$, and $n \in \mathbb{Z}^-$, then $\mu_n^k(x) \to \mu_n(x)$. For $n \in \mathbb{Z}^+$,

$$\begin{aligned} \mu_n^k(x) &= \mu_0^k Q^n(x) \\ &= \sum_{y \in S} \mu_0^k(y) Q^n(y, x) . \end{aligned}$$

Now we already know that $\mu_0^k(y) \to \mu_0(y)$ for all $y \in S$, and μ_0 is a probability measure on S. Hence $\mu_0^k \Rightarrow \mu_0$, and since $Q^n(\cdot, x)$ is a bounded function on S,

$$\begin{aligned} \sum_{y \in S} \mu_0^k(y) Q^n(y, x) &\to \sum_{y \in S} \mu_0(y) Q^n(y, x) \\ &= \mu_n(x) . \end{aligned}$$

Therefore $\boldsymbol{\mu}^k \to \boldsymbol{\mu}$. □

The standard Martin boundary theorems about harmonic functions can now be translated into theorems about entrance laws. Before doing this, we sketch a minimal amount of the Martin theory, following Dynkin [5].

Let G be the usual Green's function,

$$G(\eta, \zeta) = \sum_{n=0}^{\infty} p^n(\eta, \zeta) , \quad \eta, \zeta \in E .$$

The Martin kernel K relative to γ is

$$K(\eta, \zeta) = \frac{G(\eta, \zeta)}{\gamma G(\zeta)} ,$$

where γ is the standard measure in (1.6). A computation using (1.5) and (1.6) gives

$$\begin{aligned} K([n, x], [m, y]) &= 0 , & m > n \\ &= \frac{Q^{n-m}(y, x)}{\nu_n(x)} , & m \leqq n . \end{aligned} \tag{1.10}$$

Briefly, the Martin theory says that $\mathscr{H}$ is determined by its extreme points, $\mathscr{E}(\mathscr{H})$, and that these extreme points are given by certain limits of the Martin kernel. The details of the complete theory are presented in [5]. We quote one of the main results, stated in terms of harmonic functions instead of the usual "boundary" terminology.

LEMMA 1.4. *$\mathscr{H}$ consists of exactly those functions of the form*

$$f = \int_{\mathscr{E}(\mathscr{H})} h \, d\lambda(h) \tag{1.11}$$

where λ is a probability measure on the Borel subsets of $\mathscr{E}(\mathscr{H})$. This representation is unique. If $f \in \mathscr{E}(\mathscr{H})$, then there exists a sequence $x_1, x_2, \cdots$ of elements of S

such that

(1.12) $$f([n, x]) = \lim_{k\to\infty} K([n, x], [-k, x_k])\,.$$

PROOF. See Theorem 6 and the remark preceding it in [5]. □

We can now use the correspondence between $\mathscr{L}$ and $\mathscr{H}$ to convert Lemma 1.4 into a statement about entrance laws. Note that the map ψ restricted to $\mathscr{E}(Q)$ is a one-to-one onto homeomorphism between $\mathscr{E}(Q)$ and $\mathscr{E}(\mathscr{H})$.

PROOF OF THEOREM 1.1. We may assume $\mathscr{L}(Q) \neq \emptyset$ and p is constructed from some strictly positive $\nu \in \mathscr{L}$. For $n \in \mathbb{Z}^-$ (1.3) follows directly from (1.9), (1.11) and the fact that ψ is a homeomorphism between $\mathscr{E}(Q)$ and $\mathscr{E}(\mathscr{H})$. For $n \in \mathbb{Z}^+$

$$\begin{aligned}\mu_n(x) &= \textstyle\sum_{y\in S}\mu_0(y)Q^n(y, x)\\ &= \textstyle\int_{\mathscr{E}(Q)}[\sum_{y\in S}\varphi_0(y)Q^n(y, x)]\,d\lambda(\varphi)\\ &= \textstyle\int_{\mathscr{E}(Q)}\varphi_n(x)\,d\lambda(\varphi)\end{aligned}$$

by Fubini and the fact that (1.3) holds for μ_0.

In a similar manner (1.4) follows from (1.9), (1.10) and (1.12), at least for $n \in \mathbb{Z}^-$. To show that (1.4) holds for $n \in \mathbb{Z}^+$ we use the fact that it holds for μ_0, which implies that $Q^k(x_k, \cdot) \Rightarrow \mu_0(\cdot)$. Hence,

$$\begin{aligned}Q^{n+k}(x_k, x) &= \textstyle\sum_{y\in S}Q^k(x_k, y)Q^n(y, x)\\ &\to \textstyle\sum_{y\in S}\mu_0(y)Q^n(y, x)\\ &= \mu_n(x)\end{aligned}$$

because $Q^n(\cdot, x)$ is bounded. □

A trivial consequence of Theorem 1.1 is the fact that $\mathscr{L}(Q) = \{$the trivial entrance law$\}$ if S is finite and Q is irreducible and aperiodic. This follows because of the remark after Theorem 1.1 ($|x_k| \to \infty$ is impossible when S is finite). If the entries of Q are strictly positive then this is an immediate corollary of a theorem of Dobrushin [3] about Markov random fields.

Note that $\mathscr{L}(Q) = \emptyset$ exactly when all the limits μ_n in (1.4) are *not* probability measures. This is in general difficult to check (but see [2]), and the following result due to Kesten (about Markov random fields) is simpler.

THEOREM 1.5. *Assume Q has strictly positive entries but is not positive recurrent. If there exists $\delta > 0$ and $m \geqq 1$ such that*

$$\textstyle\sum_{n=1}^{m} Q^n(x, x) \geqq \delta\,, \quad x \in S\,,$$

then $\mathscr{L}(Q) = \emptyset$.

PROOF. See Theorem 2 in [13]. □

2. Limit theorems. In this section we present several results which try to capture the idea that a nontrivial entrance law must "come down from infinity" in a predictable way. Throughout this section P^x and E^x will denote the probability law and expectation operator of a Markov chain $\{X_n\}_{n=0,1,\cdots}$ with transition probabilities Q and initial state x. τ_y will denote the first hitting time of y.

DEFINITION 2.1. A sequence $z_1, z_2, \cdots$ of elements of S will be called an *entry from infinity* for Q if there is a family of probability measures $\{\Phi_y\}_{y \in S}$ on $\mathbb{Z}$ and a sequence of integers $n_k \to \infty$ such that for all $j \in \mathbb{Z}$, $y \in S$,

$$\lim_{k\to\infty} P^{z_k}[\tau_y - n_k = j] = \Phi_y(j) . \tag{2.1}$$

Intuitively, the states z_k tend to some "boundary point" of S in such a way that Φ_y captures the (infinite) time it takes the Markov chain to reach y from this "point." It is a simple matter to check that if z_k is an entry from infinity, then $|z_k| \to \infty$ as $k \to \infty$.

We shall see that the above definition can be relaxed if Q is recurrent. We now state the main results of this section.

THEOREM 2.2. *Let Q be irreducible and aperiodic, $\boldsymbol{\mu} \in \mathscr{L}(Q)$, $x \in S$. If Q is positive recurrent, and $\boldsymbol{\mu}$ is nontrivial and extremal, then*

$$\lim_{n\to-\infty} \mu_n(x) = 0 , \qquad \lim_{n\to+\infty} \mu_n(x) = \pi(x) . \tag{2.2}$$

If Q is not positive recurrent, then

$$\lim_{n\to-\infty} \mu_n(x) = 0 , \qquad \lim_{n\to+\infty} \mu_n(x) = 0 . \tag{2.3}$$

THEOREM 2.3. *Let Q be irreducible. If Q has an entry from infinity, and there is a state $z \in S$ such that for all $j \in \mathbb{Z}^+$,*

$$\lim_{|x|\to\infty} Q^j(x, z) = 0 , \tag{2.4}$$

then Q has a nontrivial entrance law.

THEOREM 2.4. *Let Q be irreducible, aperiodic, and recurrent. If Q has a nontrivial entrance law, then for all $j \in \mathbb{Z}^+$, $y \in S$,*

$$\liminf_{|x|\to\infty} Q^j(x, y) = 0 , \tag{2.5}$$

and Q has an entry from infinity.

The preceding two theorems will be combined into one when we deal with birth and death semigroups. However, in the present setting, the gap between the two theorems is real, as is shown by Example 5.1. Before proving these results we state and prove a fact concerning Definition 2.1.

LEMMA 2.5. *Let Q be irreducible and recurrent, $z \in S$, and Φ a probability measure on $\mathbb{Z}$. If there is a sequence z_k of elements of S and integers $n_k \to \infty$ such that*

$$\lim_{k\to\infty} P^{z_k}[\tau_z - n_k = j] = \Phi(j) , \tag{2.6}$$

then some subsequence *of z_k is an entry from infinity for Q.*

PROOF. Fix $\varepsilon > 0$, $y \in S$. By assumption there are integers M and N such that for all k,

$$P^{z_k}[|\tau_z - n_k| \leqq M] \geqq 1 - \varepsilon$$
$$P^{z}[\tau_y \leqq N] \geqq 1 - \varepsilon .$$

Using these estimates we obtain

$$\begin{aligned} P^{z_k}[|\tau_y - n_k| \leqq M + N] &\geqq P^{z_k}[|\tau_y - n_k| \leqq M + N, \tau_z < \tau_y] \\ &\geqq \textstyle\sum_{i=-(M+N)}^{M+N} \sum_{j=-M}^{i} P^{z_k}[\tau_z - n_k = j]P^z[\tau_y = i - j] \\ &\geqq \textstyle\sum_{j=-M}^{M} P^{z_k}[\tau_z - n_j = j] \sum_{i=j}^{M+N} P^z[\tau_y = i - j] \\ &\geqq P^{z_k}[|\tau_z - n_k| \leqq M]P^z[\tau_y \leqq N] \\ &\geqq 1 - 2\varepsilon \, . \end{aligned}$$

Hence the sequence of probability measures σ_k on $\mathbb{Z}$, $\sigma_k(j) = P^{z_k}[\tau_y - n_k = j]$, is *tight*. The standard weak convergence theorem shows that there must exist a subsequence $\sigma_{k'}$ of σ_k and a probability measure Φ_y on $\mathbb{Z}$ such that

$$\sigma_{k'} \Rightarrow \Phi_y \, .$$

We apply the above construction to $y = s_1$, and then repeat the process using s_1 for z and s_2 for y. Since S is countable, the diagonal argument can be applied to produce a single subsequence $z_{u(k)}$ of z_k and a family of probability measures $\{\Phi_y\}_{y \in S}$ on $\mathbb{Z}$ such that for all $y \in S$,

$$\lim_{k\to\infty} P^{z_{u(k)}}[\tau_y - n_{u(k)} = j] = \Phi_y(j) \, ,$$

as desired. □

REMARK. Lemma 2.5 can fail if Q is transient.

PROOF OF THEOREM 2.2. The second limits in (2.2) and (2.3) follow trivially from the fact that $Q^n(x, y) \to \pi(y)$ or 0 as $n \to \infty$ depending on whether or not Q is positive recurrent.

Now if the first part of (2.2) fails, then for some $x \in S$, $\varepsilon > 0$, positive integers $n_k \to \infty$, we have $\mu_{-n_k}(x) \to \varepsilon$. Fix $n \in \mathbb{Z}$, let k be large enough so that $n + n_k > 0$. Then for $y \in S$,

$$\begin{aligned} \mu_n(y) &= \mu_{-n_k} Q^{n+n_k}(y) \\ &\geqq \mu_{-n_k}(x) Q^{n+n_k}(x, y) \, . \end{aligned}$$

Let k tend to infinity to obtain

$$\mu_n(y) \geqq \varepsilon\pi(y) \, .$$

If $\varepsilon = 1$, this inequality coupled with the fact that both μ_n and π are probability measures implies $\mu_n = \pi$, a contradiction. So we may assume $0 < \varepsilon < 1$, and define

$$\varphi_n(y) = \frac{\mu_n(y) - \varepsilon\pi(y)}{1 - \varepsilon} \, .$$

It is easy to check that $\boldsymbol{\varphi}$ is an entrance law for Q, and that

$$\mu_n = (1 - \varepsilon)\varphi_n + \varepsilon\pi \, .$$

But this contradicts the fact that $\boldsymbol{\mu}$ is extremal. Hence (2.2) must hold.

Similarly, assume the first part of (2.3) fails, let $\mu_{-n_k}(x) \to \varepsilon$, and choose any finite set $F \subset S$. Since $Q^{n_k}(x, F) \to 0$, $(Q(x, F) = \sum_{y \in F} Q(x, y))$,

$$\mu_0(F) = \mu_{-n_k} Q^{n_k}(F)$$

$$\leqq \mu_{-n_k}(x)Q^{n_k}(x, F) + (1 - \mu_{-n_k}(x))$$
$$\to 1 - \varepsilon .$$

This implies $\mu_0(S) \leqq 1 - \varepsilon$, which is impossible. □

PROOF OF THEOREM 2.3. Let z_k, n_k, $\{\Phi_y\}_{y\in S}$ be as in Definition 2.1. For $y \in S$, $n \in \mathbb{Z}$, $n + n_k > 0$,

$$Q^{n+n_k}(z_k, y) = \sum_{j=-\infty}^{n} P^{z_k}[\tau_y - n_k = j]Q^{n-j}(y, y)$$
$$\to \sum_{j=-\infty}^{n} \Phi_y(j)Q^{n-j}(y, y)$$

as $k \to \infty$. Define $\boldsymbol{\mu}$ by

$$\mu_n(y) = \sum_{j=-\infty}^{n} \Phi_y(j)Q^{n-j}(y, y) = \lim_{k\to\infty} Q^{n+n_k}(z_k, y) . \tag{2.7}$$

We will show $\boldsymbol{\mu} \in \mathscr{L}(Q)$ by first showing each μ_n is a probability measure on S, and $\mu_n Q = \mu_{n+1}$. $\boldsymbol{\mu}$ is clearly nontrivial since the first equality in (2.7) shows that $\mu_n(y) \to 0$ as $n \to -\infty$.

Fix $n \in \mathbb{Z}$, $\varepsilon > 0$, let b be a positive integer large enough so that for all k

$$P^{z_k}[|\tau_z - (n_k + n)| > b] < \varepsilon . \tag{2.8}$$

This can be done because of the existence of the entry from infinity. Let $F \subset S$ be a finite set large enough so that

$$\sup_{x\in S\backslash F} \sum_{j=0}^{b} Q^j(x, z) < \varepsilon \tag{2.9}$$

and

$$\sum_{j=0}^{b} Q^j(z, S\backslash F) < \varepsilon , \tag{2.10}$$

where $S\backslash F = \{x \in S\colon x \notin F\}$. These estimates follow from (2.4) and the fact that each $Q^j(z, \cdot)$ is a probability measure on S. For k large enough,

$$\begin{aligned}
&\sum_{x\in S\backslash F} Q^{n_k+n}(z_k, x)\\
&\quad= \sum_{x\in S\backslash F} P^{z_k}[X_{n_k+n} = x]\\
&\quad= \sum_{x\in S\backslash F} P^{z_k}[|\tau_0 - (n_k + n)| > b, X_{n_k+n} = x]\\
&\qquad\quad+ \sum_{x\in S\backslash F} P^{z_k}[|\tau_0 - (n_k + n)| \leqq b, X_{n_k+n} = x]\\
&\quad\leqq P^{z_k}[|\tau_0 - (n_k + n)| > b]\\
&\qquad\quad+ \sum_{x\in S\backslash F} P^{z_k}[X_{n_k+n} = x, -b \leqq \tau_0 - (n_k + n) \leqq b]\\
&\quad< \varepsilon + \sum_{x\in S\backslash F} \sum_{j=-b}^{0} P^{z_k}[\tau_0 - (n_k + n) = j]Q^{-j}(0, x)\\
&\qquad\quad+ \sum_{x\in S\backslash F} \sum_{j=0}^{b} P^{z_k}[X_{n_k+n} = x]P^{x}[\tau_0 = j]\\
&\quad\leqq \varepsilon + \sum_{j=0}^{b} Q^j(0, S\backslash F) + \sum_{x\in S\backslash F} \Big(\sum_{j=0}^{b} Q^j(x, 0)\Big)P^{z_k}[X_{n_k+n} = x]\\
&\quad< \varepsilon + \varepsilon + \sup_{x\in S\backslash F} \sum_{j=0}^{b} Q^j(x, 0)\\
&\quad< 3\varepsilon ,
\end{aligned}$$

using (2.8), (2.9), and (2.10). Hence, for k large enough,

$$Q^{n+n_k}(z_k, F) \geqq 1 - 3\varepsilon .$$

Since F is finite, let k tend to infinity, and recall (2.7) to obtain

$$\mu_n(F) \geqq 1 - 3\varepsilon ,$$

which shows μ_n is a probability measure on S.

Finally,

$$\begin{aligned}\mu_{n+1}(y) &= \lim_{k\to\infty} Q^{n_k+n+1}(z_k, y) \\ &= \lim_{k\to\infty} \textstyle\sum_{x\in S} Q^{n_k+n}(z_k, x)Q(x, y) \\ &= \textstyle\sum_{x\in S} \mu_n(x)Q(x, y) \\ &= \mu_n Q(y) . \qquad \square\end{aligned}$$

PROOF OF THEOREM 2.4. In view of Theorem 1.1 we may assume that there exists $\boldsymbol{\mu} \in \mathscr{E}(Q)$ which is nontrivial and given by (1.4). If we assume that (2.5) fails, then there must exist $z \in S$, $\varepsilon > 0$, and $j \in \mathbb{Z}^+$ such that

$$\liminf_{|x|\to\infty} Q^j(x, z) = \varepsilon . \tag{2.11}$$

If Q is positive recurrent, then

$$\begin{aligned}\mu_n(x) &= \lim_{k\to\infty} Q^{j+n-j+k}(x_k, x) \\ &\geqq \liminf_{k\to\infty} Q^j(x_k, z)Q^{n-j+k}(z, x) \\ &\geqq \varepsilon\pi(x) .\end{aligned}$$

Here we have used the fact that $|x_k| \to \infty$. The above inequality contradicts Theorem 2.2 (let $n \to -\infty$).

If Q is null recurrent, and $F \subset S$ is finite, then

$$\begin{aligned}\mu_0(F) &= \lim_{k\to\infty} Q^k(x_k, F) \\ &= \lim_{k\to\infty} [Q^j(x_k, z)Q^{k-j}(z, F) + \textstyle\sum_{x\in S\setminus\{z\}} Q^j(x_k, x)Q^{k-j}(x, F)] \\ &\leqq \limsup_{k\to\infty} [Q^{k-j}(z, F) + Q^j(x_k, S\setminus\{z\})] \\ &\leqq 1 - \varepsilon\end{aligned}$$

by (2.11). This implies $\mu_0(S) \leqq 1 - \varepsilon$, which is impossible. Therefore (2.5) must hold.

We will now show that some *subsequence* of x_k is an entry from infinity for Q. This will be done by showing that the sequence of probability measures σ_k,

$$\sigma_k(j) = P^{x_k}[\tau_y - n_k = j] ,$$

is tight, and appealing to Lemma 2.5.

Fix $\varepsilon > 0$, $y \in S$. In view of (1.4) there must exist a finite set $F \subset S$ such that for all k,

$$Q^k(x_k, F) > 1 - \varepsilon .$$

Letting τ_F denote the first hitting time of F this implies

$$P^{x_k}[\tau_F \leqq k] > 1 - \varepsilon .$$

Since F is finite and Q is recurrent there must exist an integer b such that

$$P^x[\tau_y \leqq b] > 1 - \varepsilon , \quad x \in F .$$

Using these estimates we obtain

$$\begin{aligned}P^{x_k}[\tau_y - k \leqq b] &\geqq P^{x_k}[\tau_F \leqq k, \tau_y \leqq k + b] \\ &\geqq \textstyle\sum_{x\in F} \sum_{j=0}^{k} P^{x_k}[\tau_F = j, X_{\tau_F} = x]P^x[\tau_y \leqq b] \\ &> (1 - \varepsilon)P^{x_k}[\tau_F \leqq k] \\ &> 1 - 2\varepsilon .\end{aligned}$$

Therefore,

$$P^{x_k}[\tau_y - k \leqq b] \geqq 1 - 2\varepsilon . \tag{2.12}$$

We now seek a similar bound below, but the method will depend on Q.

If Q is *positive* recurrent, then there must exist an integer j_0 such that $Q^j(y, y) \geqq \pi(y)/2$ for $j \geqq j_0$. Using this estimate we obtain

$$\begin{aligned}\mu_n(y) &= \lim_{k\to\infty} Q^{k+n}(x_k, y)\\ &= \lim_{k\to\infty} \sum_{j=-\infty}^{n} P^{x_k}[\tau_y - k = j]Q^{n-j}(y, y)\\ &= \lim_{k\to\infty} \sum_{j=0}^{\infty} P^{x_k}[\tau_y - k = n - j]Q^j(y, y)\\ &\geqq \limsup_{k\to\infty} \sum_{j=j_0}^{\infty} P^{x_k}[\tau_y - k = n - j]\pi(y)/2 .\end{aligned}$$

Thus

$$\limsup_{k\to\infty} P^{x_k}[\tau_y - k \leqq n - j_0] \leqq \frac{2\mu_n(y)}{\pi(y)} .$$

By Theorem 2.2 $\mu_n(y) \to 0$ as $n \to -\infty$. This means there is a positive integer b_1 such that

$$\liminf_{k\to\infty} P^{x_k}[\tau_y - k \geqq -b_1] \geqq 1 - \varepsilon . \tag{2.13}$$

If Q is *null* recurrent, choose a finite subset $F \subset S$ which contains y such that $\mu_0(F) \geqq 1 - \varepsilon$. Then

$$\begin{aligned}1 - \varepsilon &\leqq \lim_{k\to\infty} Q^k(x_k, F)\\ &= \lim_{k\to\infty} \sum_{j=0}^{\infty} \sum_{x\in F} P^{x_k}[\tau_F - k = -j, X_{\tau_F} = x]Q^j(x, F) .\end{aligned}$$

Choose j_1 such that $Q^j(x, F) < \varepsilon$ for $j \geqq j_1$, $x \in F$. This implies

$$1 - \varepsilon \leqq \liminf_{k\to\infty} \sum_{j=0}^{j_1} \sum_{x\in F} P^{x_k}[\tau_F - k = -j, X_{\tau_F} = x]Q^j(x, F) + \varepsilon$$

or

$$1 - 2\varepsilon \leqq \liminf_{k\to\infty} P^{x_k}[\tau_F - k \geqq -j_1] .$$

Since $\tau_F \leqq \tau_y$ this in turn implies

$$\liminf_{k\to\infty} P^{x_k}[\tau_y - k \geqq -j_1] \geqq 1 - 2\varepsilon . \tag{2.14}$$

Inequalities (2.12), (2.13) and (2.14) prove the desired tightness. Hence there must exist a subsequence $x_{k'}$ of x_k and a probability measure Φ on $\mathbb{Z}$ such that

$$\lim_{k'\to\infty} P^{x_{k'}}[\tau_y - k' = j] = \Phi(j)$$

Lemma 2.5 now asserts that a subsequence of $x_{k'}$ is an entry from infinity for Q. □

3. Continuous time entrance laws. We now replace the matrix Q with a stochastic semigroup $\{p(t)\}_{t\geqq 0}$. The state space is still S, and (using the terminology of [11]) $p(t)$ is *standard* and *irreducible*. We make the additional assumption that all states are *stable*. In this case there is a strong Markov process $\{X(t)\}_{t\geqq 0}$ which is governed by $p(t)$ and which has right continuous sample paths (Theorem 7.41 in [11]). As usual, P^x and E^x will denote the probability law and expectation operator of the process starting at x, and τ_y is the first hitting time of y.

An entrance law for $p(t)$ is a family $\boldsymbol{\mu} = \{\mu_s\}_{s\in\mathbb{R}}$ of probability measure on S such that for all $s \in \mathbb{R}$, $t \geqq 0$,

$$\mu_s p(t) = \mu_{s+t} .$$

$\mathscr{L} = \mathscr{L}(p)$ will denote the set of all entrance laws for $p(t)$. $\mathscr{L}$ is convex and $\mathscr{E}(p)$ will be the set of its extreme points. We give $\mathscr{L}$ the usual pointwise convergence topology.

Let Q be the matrix obtained from the semigroup at time $t = 1$,

$$Q(x, y) = p(1)(x, y) .$$

(Note that Q has nothing to do with the so-called infinitesimal generator of $p(t)$.)

The following result, first noticed by Dynkin in [6], shows that all of the results of Sections 1 and 2 (with the obvious modifications) carry over to the present setting.

LEMMA 3.1. *There is a one-to-one onto homeomorphism ϕ^* between $\mathscr{L}(Q)$ and $\mathscr{L}(p)$. In particular, $\phi^*(\boldsymbol{\mu}) = \boldsymbol{\nu}$, where*

$$\nu_s = \mu_n p(s-n) , \qquad s \in \mathbb{R} , \quad n \in \mathbb{Z} , \quad n \leqq s . \tag{3.1}$$

PROOF. Assume $\boldsymbol{\mu} = \{\mu_n\}_{n\in\mathbb{Z}} \in \mathscr{L}(Q)$ and define $\boldsymbol{\nu} = \{\nu_s\}_{s\in\mathbb{R}}$ by (3.1). Then $\boldsymbol{\nu}$ is well defined, since

$$\begin{aligned} \mu_n p(s-n) &= \mu_n p(m-n)p(s-m) \\ &= \mu_n Q^{m-n} p(s-m) \\ &= \mu_m p(s-m) \end{aligned}$$

whenever n, m are integers with $n \leqq m$. Each ν_s is clearly a probability measure on S, and

$$\begin{aligned} \nu_s p(t) &= \mu_n p(s-n)p(t) \\ &= \mu_n p(s+t-n) \\ &= \nu_{s+t} , \end{aligned}$$

which shows $\boldsymbol{\nu} \in \mathscr{L}(p)$.

On the other hand, if $\boldsymbol{\nu} \in \mathscr{L}(p)$, define $\boldsymbol{\mu}$ by $\mu_n = \nu_n$; simply restrict the time parameter from $\mathbb{R}$ to $\mathbb{Z}$. It is trivial to check that $\boldsymbol{\mu} \in \mathscr{L}(Q)$ and $\phi^*(\boldsymbol{\mu}) = \boldsymbol{\nu}$. Hence ϕ^* is one-to-one and onto. We must now show ϕ^* preserves topologies.

Assume $\boldsymbol{\mu}^k, \boldsymbol{\mu} \in \mathscr{L}(Q)$ and $\boldsymbol{\mu}^k \to \boldsymbol{\mu}$ in $\mathscr{L}(Q)$. Then, if $\phi^*(\boldsymbol{\mu}^k) = \boldsymbol{\nu}^k$, $\phi^*(\boldsymbol{\mu}) = \boldsymbol{\nu}$,

$$\begin{aligned} \nu_s^k(y) &= \mu_n^k p(s-n)(y) \\ &= \textstyle\sum_{x\in S} \mu_n^k(x) p(s-n)(x, y) \\ &\to \textstyle\sum_{x\in S} \mu_n(x) p(s-n)(x, y) \\ &= \nu_s(y) . \end{aligned}$$

Hence $\boldsymbol{\nu}^k \to \boldsymbol{\nu}$ in $\mathscr{L}(p)$.

It is even easier to show that $\boldsymbol{\nu}^k \to \boldsymbol{\nu}$ in $\mathscr{L}(p)$ implies $\boldsymbol{\mu}^k \to \boldsymbol{\mu}$ in $\mathscr{L}(Q)$. This completes the proof. □

Note that an entry from infinity is now a sequence z_k for which there is a sequence of reals $a_k \to \infty$ and a family of probability measures $\{\Phi_y\}_{y \in S}$ on $\mathbb{R}$ such that

$$P^{z_k}[\tau_y - a_k \in du] \Rightarrow \Phi_y(du) .$$

Using Lemma 3.1 it is a simple matter to check that the obvious analogues of Theorems 1.1, 2.2, 2.3 and 2.4 are true. Some additional work is required but it is all straightforward.

4. Birth and death processes. A very nice class of semigroups to work with is the class of birth and death processes on $S = \mathbb{Z}^+$, as described in [9]. For these semigroups we will show that Theorems 2.3 and 2.4 combine into one simple statement, and then give a sufficient condition in terms of the rates for the existence of nontrivial entrance laws.

We assume strictly positive birth rates β_k and death rates δ_k (except $\delta_0 = 0$), and define

$$\pi_0 = 1 , \qquad \pi_k = \frac{\beta_0 \beta_1 \cdots \beta_{k-1}}{\delta_1 \delta_2 \cdots \delta_k} , \qquad \tilde{\pi}(n) = \pi_n / \textstyle\sum_{k=0}^\infty \pi_k . \tag{4.1}$$

It is well known (see [12]) that if the rates satisfy

$$\textstyle\sum_{k=0}^\infty \pi_k < \infty , \qquad \sum_{k=0}^\infty \dfrac{1}{\beta_k \pi_k} = \infty , \tag{4.2}$$

then $p(t)$ is well defined and ergodic with limiting measure $\tilde{\pi}$. Our main results are:

THEOREM 4.1. *If $p(t)$ has rates which satisfy* (4.2), *then the following are equivalent*:

(i) *$p(t)$ has a nontrivial entrance law.*

(ii) *$p(t)$ has an entry from infinity.*

(iii) *There is a sequence of reals $a_k \to \infty$ and a probability measure Φ on $\mathbb{R}$ such that $P^k[\tau_0 - a_k \in du] \Rightarrow \Phi(du)$ as $k \to \infty$.*

THEOREM 4.2. *If $p(t)$ has rates which satisfy* (4.2) *and in addition*

$$\begin{gathered} \textstyle\sum_{n=0}^\infty \dfrac{1}{\beta_n \pi_n} \sum_{k=n+1}^\infty \pi_k = \infty \\ \textstyle\sum_{n=0}^\infty \dfrac{1}{\beta_n \pi_n} \sum_{k=n}^\infty \dfrac{1}{\beta_k \pi_k} (\sum_{j=k+1}^\infty \pi_j)^2 < \infty , \end{gathered} \tag{4.3}$$

then $p(t)$ has a nontrivial entrance law.

PROOF OF THEOREM 4.1. (i) $\Rightarrow$ (ii): Immediate from the analogue of Theorem 2.4.

(ii) $\Rightarrow$ (iii): Assume z_k is an entry from infinity for $p(t)$. Since $z_k \to \infty$, we may (taking a subsequence if necessary) assume $z_1 < z_2 < \cdots$. Hence there are reals $b_k \to \infty$ and a probability measure Φ_0 on $\mathbb{R}$ such that

$$P^{z_k}[\tau_0 - b_k \in du] \Rightarrow \Phi_0(du) . \tag{4.4}$$

Our problem is to "interpolate" between the z_k.

Let $\sigma_1, \sigma_2, \cdots$ be independent random variables defined on a probability space $(\Omega^*, \mathscr{F}^*, P^*)$ such that

$$P^*[\sigma_n \leqq t] = P^n[\tau_{n-1} \leqq t] . \tag{4.5}$$

That is, σ_n is a copy of the time it takes the process $X(t)$ to go from n to $n-1$. By the strong Markov property, if $m \leqq n$,

$$P^n[\tau_m \leqq t] = P^*[\sigma_{m+1} + \cdots + \sigma_n \leqq t] , \tag{4.6}$$

which shows we are dealing with sums of independent random variables.

In view of (4.6) we may rewrite (4.4) as

$$P^*[\textstyle\sum_{n=1}^{z_k} \sigma_n' \in du] \Rightarrow \Phi_0(du) , \qquad \sigma_n' = \sigma_n - (b_n - b_{n-1}) , \quad b_0 = 0 . \tag{4.7}$$

This is a much stronger statement than the hypothesis of Theorem 3.2.9 in [4] which implies the existence of centering constants which make $\sum \sigma_n'$ converge P^* a.e. This means there are constants a_k (it is easy to check $a_k \to \infty$) such that

$$\lim_{k\to\infty} \textstyle\sum_{n=1}^{k} \sigma_n - a_k$$

exists P^* a.e., and hence the measures

$$P^*[\textstyle\sum_{n=1}^{k} \sigma_n - a_k \in du]$$

must converge weakly to a probability measure on $\mathbb{R}$. This is because convergence in distribution and converge a.e. are equivalent for sums of independent random variables. In view of (4.6), (iii) must hold.

(iii) $\Rightarrow$ (ii): If (iii) holds then (4.6) now implies that

$$\lim_{k\to\infty} \textstyle\sum_{n=1}^{k} \sigma_n - (a_n - a_{n-1})$$

exists P^* a.e. ($a_0 = 0$). Hence for each $y \in S$,

$$\lim_{k\to\infty} \textstyle\sum_{n=y}^{k} \sigma_n - (a_n - a_{n-1}) - a_y$$

must exist P^* a.e., and Φ_y will be its (P^*) distribution. This implies (via 4.6) that

$$P^k[\tau_y - a_k \in du] \Rightarrow \Phi_y(du) ,$$

which shows that $z_k = k$ is an entry from infinity.

(ii) $\Rightarrow$ (i): Since $p(t)$ is irreducible, it suffices to show that $\lim_{k\to\infty} p(t)(k, 0) = 0$ and apply the analogue of Theorem 2.3. Using the fact that (ii) $\Rightarrow$ (iii) we can write

$$\begin{aligned} \limsup_{k\to\infty} p(t)(k, 0) &\leqq \limsup_{k\to\infty} P^k[\tau_0 \leqq t] \\ &= \limsup_{k\to\infty} P^k[\tau_0 - a_k \leqq t - a_k] \\ &= 0 . \end{aligned}$$

This completes the proof. □

It seems impossible to give a reasonable condition on the rates β_k and δ_k which is equivalent to (ii) or (iii). Nevertheless it is possible to give a condition which implies (iii), and hence the existence of nontrivial entrance laws. We first state

LEMMA 4.3. *If* (4.2) *holds and in addition*

$$\sum_{n=0}^{\infty} \frac{1}{\beta_n \pi_n} (\sum_{k=n+1}^{\infty} \pi_k)^2 < \infty , \tag{4.8}$$

then $E^N[\tau_0^2] < \infty$ *for* $N = 1, 2, \cdots,$ *and*

$$E^N[\tau_0] = \sum_{n=0}^{N-1} \frac{1}{\beta_n \pi_n} \sum_{k=n+1}^{\infty} \pi_k$$

$$E^N[\tau_0^2] = 2 \sum_{n=0}^{N-1} \frac{1}{\beta_n \pi_n} \sum_{k=n+1}^{\infty} \pi_k E^k[\tau_0]$$

$$\begin{aligned}\mathrm{Var}^N [\tau_0] = &\sum_{n=0}^{N-1} \frac{1}{\beta_n \pi_n} \sum_{k=n+1}^{\infty} \frac{1}{\beta_k \pi_k} (\sum_{j=k+1}^{\infty} \pi_j)^2 \\ &+ \sum_{n=0}^{N-1} \frac{1}{\beta_n \pi_n} \sum_{k=n}^{\infty} \frac{1}{\beta_k \pi_k} (\sum_{j=k+1}^{\infty} \pi_j)^2 .\end{aligned}$$

PROOF. The formulas for the first and second moments can be determined by standard difference equation techniques, or they can be derived from the integral formulas in [12]. The formula for the variance is obtained from the first two equations. □

PROOF OF THEOREM 4.2. Condition (4.3) simply says that $\lim_{k\to\infty} E^k[\tau_0] = \infty$ and $\lim_{k\to\infty} \mathrm{Var}^k [\tau_0] < \infty$. In view of (4.6) this implies that $\sum_{k=1}^{\infty} E^*[\sigma_k] = \infty$, $\sum_{k=1}^{\infty} \mathrm{Var}^* [\sigma_k] < \infty$. By Theorem 3.2.3 in [4], $\sum_{k=1}^{\infty} (\sigma_k - E^*[\sigma_k])$ converges P^* a.e., and so (4.6) now implies that condition (iii) in Theorem 4.1 holds with $a_k = E^k[\tau_0]$. □

5. Examples. Examples of stochastic matrices or semigroups which possess nontrivial entrance laws are not difficult to construct. Theorem 1.1 can be used to explicitly determine $\mathscr{L}(Q)$ in some nontrivial cases, including certain subcritical branching processes with different types of immigration (see [2]). In [15] Spitzer gave several examples in the continuous time setting, and Theorem 4.2 can be used to construct birth and death processes (even with unbounded birth rates) which possess nontrivial entrance laws. See [1] for an explicit example in the Markov random field setting.

In this section we will content ourselves with two examples; the first illustrates the "gap" between Theorems 2.3 and 2.4, and the second shows that even a transient process can have nontrivial entrance laws.

EXAMPLE 5.1. Let $S = \mathbb{Z}^+$, let Q be defined by

$$\begin{aligned} Q(i, j) &= p_i , & j &= i + 1 \\ &= 1 - p_i , & j &= 0 \\ &= 0 , & &\text{otherwise} \end{aligned}$$

where $p_0 = 1$ and $0 < p_i < 1$, $i \geqq 1$. Let $x_1 = 2$, $x_{k+1} = x_k + k + 1$, $a_k = \prod_{i=x_k}^{x_k+k-1} p_i$. Now let $p_{x_k+k} = 1/k$, $k \geqq 2$, and choose the remaining p_i to insure that $a_k \to 1$ as $k \to \infty$.

It is a simple matter to check that Q is irreducible, aperiodic and recurrent, that x_k is an entry from infinity for Q, and that

$$\liminf_{k\to\infty} Q(k, 0) = 0 , \qquad \limsup_{k\to\infty} Q(k, 0) \neq 0 .$$

This means we can not apply Theorem 2.3. To show that there are no nontrivial entrance laws it suffices to show that there are none of the form

$$\mu_n(x) = \lim_{k\to\infty} Q^{n+k}(z_k, x) ,$$

$z_k \to \infty$. We rewrite the above as

$$\mu_n(x) = \lim_{k\to\infty} \sum_{j=0}^{\infty} P^{z_k}[\tau_0 - k = n - j]Q^j(0, x)$$

and so for any finite set $F \subset S$,

$$\mu_n(F) = \lim_{k\to\infty} \sum_{j=0}^{\infty} P^{z_k}[\tau_0 - k = n - j]Q^j(0, F) .$$

This implies that

$$\mu_n(F) \leqq \liminf_{k\to\infty} P^{z_k}[\tau_0 - k \leqq n] \equiv u(n) .$$

This cannot hold unless $u(n) \equiv 1$, in which case

$$\begin{aligned}\mu_n(x) &= \lim_{k\to\infty} \sum_{j=j_0}^{\infty} P^{z_k}[\tau_0 - k = n - j]Q^j(0, x) \\ &\leqq \sup_{j \geqq j_0} Q^j(0, x)\end{aligned}$$

for every positive integer j_0. Finally, we let $j_0 \to \infty$ to obtain

$$\begin{aligned}\mu_n(x) &\leqq \pi(x) , && Q \text{ positive recurrent} \\ &\leqq 0 , && Q \text{ null recurrent.}\end{aligned}$$

In either case μ is *not* a nontrivial entrance law.

EXAMPLE 5.2. Let $S = \mathbb{Z}$, and let $p(t)$ be the birth and death semigroup on S generated by rates

$$\begin{aligned}\beta_k &= 1 & k &\geqq 0 & \delta_k &= k + 1 & k &\geqq 0 \\ &= \alpha^{-k} & k &< 0 & &= 1 - \alpha^{-k} & k &< 0\end{aligned}$$

where $0 < \alpha < 1$. $p(t)$ is standard and irreducible with all states stable. The associated Markov chain X_t is transient because

$$\begin{aligned}P^0[X_t \text{ never returns to } 0] &= \prod_{k=0}^{\infty} \frac{\delta_{-k}}{\delta_{-k} + \beta_{-k}} \\ &= \tfrac{1}{2} \prod_{k=1}^{\infty} (1 - \alpha^k) \\ &> 0 .\end{aligned}$$

Furthermore, $z_k = k$, $k = 1, 2, \cdots$ is an entry from infinity. This is because the formulas in Lemma 4.3 are valid here, and therefore the measures $P^k[\tau_0 - E^k[\tau_0] \in du]$ converge weakly to a probability measure on $\mathbb{R}$ as $k \to \infty$. Imitation of the proof of Theorem 4.1 will prove that z_k is an entry from infinity. Finally, for each $t \geqq 0$,

$$\lim_{|n|\to\infty} p(t)(n, 0) = 0 .$$

The analogue of Theorem 2.3 can now be applied to show that $p(t)$ has nontrivial entrance laws.

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