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RAW TIME CHANGES OF MARKOV PROCESSES[1]

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Let A_t be a nonadapted continuous additive functional of a right continuous strong Markov process X_t, and let τ_t denote the right continuous inverse of A_t. We give general sufficient conditions for the time-changed process X_{τ_t} to again be a strong Markov process with a new transition semigroup. We give several examples and show that birthing a process at a last exit time and killing a process at a cooptional time may be realized as raw time changes.

0. Introduction. Additive functionals have long been used as intrinsic measures of "time" for a Markov process. That is, instead of running the Markov process $t \to X_t$, we run the Markov process $t \to X_{\tau_t}$, where τ_t is the right continuous inverse of a continuous additive functional. This process is again a strong Markov process [2]. Certain natural transformations of Markov processes which incorporate some knowledge of the future have been exhibited recently [4, 8, 11], and it seems entirely natural to investigate the effect of time-changing a Markov process by a nonadapted or *raw* continuous additive functional. We present a general approach to this matter in Section 1. Our approach has been influenced by the techniques used in the recent papers of Getoor and Sharpe [6, 7]. Theorem (1.5) presents general sufficient conditions for the time change of a strong Markov process to again be a strong Markov process. In general, the transformed process has a different transition semigroup and may fail to have the strong Markov property at time $t = 0$. This is unavoidable and an entirely natural occurence in many cases (cf. Section 2).

In Sections 2, 3, and 4, we verify the hypotheses of Theorem (1.5) for certain raw additive functionals. In particular, in Sections 2 and 3, we show that the procedures given by Meyer, Smythe and Walsh [11] of birthing the process at a coterminal time and killing the process at a cooptional time may be reformulated in terms of a raw time change. In Section 4, we show how to delete pieces of paths which do not reenter a set C in a certain manner, and we show how to shorten paths in the complement of C. These examples serve as prototypes for a large class of transformations. We have not tried to achieve maximum generality in constructing examples. Rather, we choose to present those which we feel offer some insight into the range of available raw time changes and which require a bit of thought in order to verify the hypotheses of Theorem (1.5).

The remainder of this section contains the notations, definitions, and hypotheses of the paper. First, we shall henceforth abbreviate additive functional and continuous additive functional as AF and CAF, respectively. We have chosen an "algebraic" framework in which to present the results; i.e., we have assumed that equalities hold identically in many places. We indicate in some remarks scattered throughout how these conditions may be relaxed.

Let $(E, \mathscr{E})$ be a Lusin topological space together with its Borel field to which we adjoin an isolated point Δ to act as cemetery. A function $f \in b\mathscr{E}^+$ is a bounded positive Borel function on E (similar conventions hold for other σ-algebras). We denote by $\mathscr{E}^*$ the universal completion of $\mathscr{E}$, and we let $\mathscr{M}$ denote the collection of finite measures on $(E_\Delta, \mathscr{E}_\Delta)$. All functions on E are assumed to vanish at Δ.

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Let Ω denote the collection of right continuous paths from R^+ to $E \cup \{\Delta\}$ such that if $\omega \in \Omega$ and $\omega(s) = \Delta$, then $\omega(s+t) = \Delta$ for all $t \ge 0$: we allow the lifetime ζ to be infinite. Let $(X_t)_{t\ge0}$ denote the usual family of coordinate mappings, $X_t(\omega) = \omega(t)$, and let $\mathscr{F}_t^0 = \sigma(X_s\colon s \le t)$, $\mathscr{F}^0 = \sigma(X_s\colon s \ge 0)$. Let $\mathscr{F}^*$ denote the universal completion of $\mathscr{F}^0$. We make the conventions that $X_\infty = \Delta$, and that $F \circ \theta_\infty = 0$ for all $F \in \mathscr{F}^*$. Let $(P_t)_{t\ge0}$ be a semigroup mapping Borel functions on E to Borel functions on E, and let $(P^x)_{x\in E}$ be a family of probability measures on $(\Omega, \mathscr{F}^0)$ under which X_t is a strong Markov process with semigroup P_t and such that $P^x(X_0 = x) = 1$.

If $\mu \in \mathscr{M}$, let $\mathscr{F}^\mu$ denote the P^μ-completion of $\mathscr{F}^0$, and let $\mathbf{N}^\mu$ denote the P^μ-null sets in $\mathscr{F}^\mu$. We define:

$$\mathscr{F}_t^\mu = \mathscr{F}_t^0 \vee \mathbf{N}^\mu$$

$$\mathscr{F}_t = \cap_{\mu\in\mathscr{M}} \mathscr{F}_t^\mu$$

$$\mathscr{F} = \cap_{\mu\in\mathscr{M}} \mathscr{F}^\mu$$

$$\mathbf{N} = \cap_{\mu\in\mathscr{M}} \mathbf{N}^\mu.$$

The class of $(\mathscr{F}_t^\mu)$-optional processes (resp. $(\mathscr{F}_t^\mu)$-predictable processes; $(\mathscr{F}_t^0)$-optional processes; $(\mathscr{F}_t^0)$-predictable processes) will be denoted by $\mathscr{O}(\mathscr{F}_t^\mu)$ (resp. $\mathscr{P}(\mathscr{F}_t^\mu)$; $\mathscr{O}(\mathscr{F}_t^0)$; $\mathscr{P}(\mathscr{F}_t^0)$). Following [7], we say that Z is an $\mathscr{F}_t$-optional process (and write $Z \in \mathscr{O}(\mathscr{F}_t)$) if for each measure μ on E, Z is P^μ-indistinguishable from a process $Z \in \mathscr{O}(\mathscr{F}_t^\mu)$.

We define a raw CAF $(A_t)_{t\ge0}$ to be an increasing process in $\mathscr{B}(R^+) \otimes \mathscr{F}^*$ which satisfies $A_0 = 0$; $A_{t+s} = A_t + A_s \circ \theta_t$ identically; $A_t = A_\zeta$ for all $t > \zeta$; $t \to A_t$ is continuous. Usually one requires (A_t) to be only $\mathscr{B}(R^+) \otimes \mathscr{F}$-measurable, but *in this paper we require* $A_t \in \mathscr{B}(R^+) \otimes \mathscr{F}^*$ (see Remark (6) and the Appendix). We shall say that a raw CAF $(A_t)_{t\ge0}$ is σ-integrable if there exists a right continuous process $Z \in \mathscr{O}(\mathscr{F}_t)$ with $Z > 0$ on $[0, \zeta)$ such that $E^x \int_0^\infty Z_s\, dA_s < \infty$ for all x. Each raw CAF given in Sections 2 through 4 is σ-integrable (take $Z_t = e^{-t}$).

We shall need the canonical killing operators $k_t\colon \Omega \to \Omega$ defined by

$$\begin{aligned} X_s(k_t\omega) &= X_s(\omega) && \text{if} \quad s < t \\ &= \Delta && \text{if} \quad s \ge t. \end{aligned}$$

If $F \in \mathscr{F}^0$, then $F \in \mathscr{F}_{t-}^0$ if and only if $F = F \circ k_t$ [1].

Finally, we need to introduce the random times and associated σ-algebras we shall discuss. If $R \in \mathscr{F}^+$ is a random time, we define a σ-algebra $\mathscr{F}_R$ and a σ-algebra $\mathscr{F}_{R-}$ as follows. A random variable $Z \in \mathscr{F}$ is $\mathscr{F}_R$-measurable (resp. $\mathscr{F}_{R-}$-measurable) if and only if there is a process $Y \in \mathscr{O}(\mathscr{F}_t)$ (resp. $Y \in \mathscr{P}(\mathscr{F}_t)$) with $Y_R = Z$ on $\{R < \infty]$. Let $\Gamma \subset R^+ \times \Omega$ be in $\mathscr{B}(R^+) \otimes \mathscr{F}$. If $1_\Gamma(s, \theta_t\omega) = 1_\Gamma(s+t, \omega)$ for all $s > 0$, $t \ge 0$, then Γ is said to be homogeneous. A cooptional time L is defined to be the end of a homogeneous set: $L = \sup\{t\colon (t, \omega) \in \Gamma\}$ ($\sup \varnothing = 0$). Then L satisfies $L \circ \theta_t = (L - t)^+ = \max(L - t, 0)$. If Γ is also an $(\mathscr{F}_t)$-optional set, then L is said to be a coterminal time.

1. Raw time changes. Let A_t be a σ-integrable raw CAF, and let $\tau_t(\omega) = \inf\{u\colon A_u(\omega) > t\}$. Then τ_t is the right continuous inverse of A, $A_{\tau_t} = t$ on $\{\tau_t < \infty\}$, and $t \to X_{\tau_t}$ is right continuous. The following is a standard fact [2].

(1.1) LEMMA. $\tau_{t+s} = \tau_t + \tau_s \circ \theta_{\tau_t}$,

PROOF. If $\tau_t = \infty$, then $\tau_{t+s} = \infty$. Therefore, we assume $\tau_t < \infty$. Since $A_{\tau_t+s} = A_{\tau_t} + A_s \circ \theta_{\tau_t} = t + A_s \circ \theta_{\tau_t}$, we have $A_{\tau_t+s} > t + v$ if and only if $A_s \circ \theta_{\tau_t} > v$. But $\tau_s \circ \theta_{\tau_t} = \inf\{q\colon A_q \circ \theta_{\tau_t} > s\} = \inf\{q\colon A_{q+\tau_t} > t + s\} = \tau_{t+s} - \tau_t$.

Let $Y \in b\,\mathscr{O}(\mathscr{F}_t)^+$, and let $F \in b\mathscr{F}^*$. Our first objective is to show that

$$E^x \int Y_{\tau_t} F \circ \theta_{\tau_t}\, dt = E^x \int Y_{\tau_t} K(X_{\tau_t}, F)\, dt \tag{1.2}$$

for some appropriate kernel $K(x, d\omega)$ from Ω to E. The left-hand side of (1.2) may be rewritten as

$$E^x \int Y_t F \circ \theta_t \, dA_t,$$

by the time-change theorem [3]. Define a new raw CAF by setting

$$B_t = \int_0^t F \circ \theta_u \, dA_u \in \mathscr{B}(R^+) \otimes \mathscr{F}^*.$$

Then the left-hand side of (1.2) becomes

$$E^x \int Y_t \, dB_t = E^x \int Y_t \, dB_t^1,$$

where B^1 denotes the dual optional projection of the additive functional B. It is by now a standard fact that B^1 is an AF, as is A^1 (here we are using the σ-integrability of the AF's) [12]. Moreover, B^1 and A^1 are continuous, since if T is an $(\mathscr{F}_t)$-optional time,

$$E^x \int 1_{[T]} \, dB_t^1 = E^x \int 1_{[T]} \, dB_t = 0.$$

If $h \in b\mathscr{E}^+$, then $E^x \int h(X_u) \, dB_u^1 = E^x \int h(X_u) F \circ \theta_u \, dA_u$, and this integral is zero whenever $E^x \int h(X_u) \, dA_u = E^x \int h(X_u) \, dA_u^1$ is zero. Thus B^1 is absolutely continuous with respect to A^1, and by Motoo's theorem, there exists a function $g \in \mathscr{E}^e$ (the σ-algebra on E generated by the p-excessive functions, $p \geq 0$) such that B^1 is indistinguishable from

$$\int_0^t g(X_s) \, dA_s^1.$$

Note that g depends only on the function F (and, of course, the AF A_t). It is a standard fact [5] that there exists a bounded kernel $K(x, d\omega)$ from $(\Omega, \mathscr{F})$ to $(E, \mathscr{E}^*)$ such that B_t^1 is indistinguishable from

$$\int_0^t K(X_u, F) \, dA_u^1.$$

Thus the left-hand side of (1.2) now becomes

$$E^x \int Y_t K(X_t, F) \, dA_t = E^x \int Y_{\tau_t} K(X_{\tau_t}, F) \, dt.$$

(Since $K(x, F)$ is only $\mathscr{E}^*$-measurable, the measurability in the line above requires some justification. Define a measure Q on $(E, \mathscr{E})$ by setting

$$Q(g) = E^x \int e^{-t} Y_t g(X_t) \, dA_t = E^x \int e^{-t} Y_{\tau_t} g(X_{\tau_t}) \, dt$$

for $g \in \mathscr{E}^+$. For any $f \in b\mathscr{E}^*$, there are functions $g_1, g_2 \in \mathscr{E}$ such that $g_1 \leq f \leq g_2$ and $Q(g_2 - g_1) = 0$. Therefore, the line above is well defined.) We define a family of kernels $K_s(x, dy)$ from $(E, \mathscr{E})$ to $(E, \mathscr{E}^*)$ by setting $K_s(x, f) = K(x, f(X_{\tau_s}))$. We shall often write $K_s(x, f)$ as $K_s f(x)$. Notice that $s \to K_s(x, f)$ is right continuous for all bounded continuous functions f, and hence the map $(s, x) \to K_s f(x)$ is jointly measurable.

(1.3) Proposition. *Assume that for each $t > 0$, $A_u \circ k_{\tau_t} = A_u$ for all $u \leq \tau_t$. Then the time-changed family of σ-algebras $(\mathscr{F}_{\tau_t})_{t>0}$ is increasing.*

Proof. Fix μ a finite measure on $(E, \mathscr{E})$, and define a measure Q on $(\Omega, \mathscr{F}^0)$ as follows.

If $F \in b\mathscr{F}^0$, $Q(F) = E^\mu[F + F \circ k_{\tau_{t+s}}]$. It is a simple matter to construct a continuous process $(B_u)_{u \geq 0} \in \mathscr{B}(R^+) \otimes \mathscr{F}^0$ which is Q-indistinguishable from $(A_u)_{u \geq 0}$. Thus if we define $\sigma_s(\mu) = \inf\{u \geq 0: B_u > s\}$, then $\sigma_s(\mu) = \tau_s$ a.s. (P^μ). Moreover, $B_u \circ k_{\tau_{t+s}} = A_u \circ k_{\tau_{t+s}} = A_u = B_u$ on $[0, \tau_{t+s}]$ a.s. (P^μ).

We define two σ-algebras $\mathscr{G}_s^\mu$ and $\mathscr{G}_{s-}^\mu$ on Ω as follows. A random variable $G \in \mathscr{F}^\mu$ is $\mathscr{G}_s^\mu$-measurable (resp. $\mathscr{G}_{s-}^\mu$-measurable) if and only if there is a $Z \in \mathscr{O}(\mathscr{F}_t^\mu)$ (resp. $Z \in \mathscr{P}(\mathscr{F}_t^\mu)$ such that $G = Z_{\tau_s}$ on $\{\tau_s < \infty\}$. It follows that $\mathscr{G}_s^\mu = \mathscr{F}_{\tau_s} \vee \mathbf{N}^\mu$ (resp. $\mathscr{G}_{s-}^\mu = \mathscr{F}_{\tau_s-} \vee \mathbf{N}^\mu$).

It is evident that $\mathscr{G}_{s-}^\mu \subset \mathscr{G}_s^\mu$. We now show that $\mathscr{G}_s^\mu \subset \mathscr{G}_{(t+s)-}^\mu$ for $t > 0$. For each u, define a process

$$T_t^u = u \qquad \text{on} \quad \{(t, \omega): B_u(k_t\omega)\, 1_{\{u \leq t\}} > s\}$$
$$= \infty \qquad \text{elsewhere.}$$

Then T_t^u is predictable since the process $t \to B_u \circ k_t$ is predictable [1]. For fixed t, $u \to T_t^u$ is left continuous, so the process T_t defined by $T_t = \inf\{T_t^u: u > 0\}$ is P^μ-indistinguishable from the $(\mathscr{F}_t^\mu)$-predictable process $R_t = \inf\{T_t^u: u \text{ a positive rational}\}$. Therefore, $T_t \in \mathscr{P}(\mathscr{F}_t^\mu)$. Since we may rewrite T_t again as $\inf\{u \leq t: B_u \circ k_t > s\}$ on $\{T_t < \infty\}$, $T_{\sigma_{t+s}(\mu)} = T_{\tau_{t+s}} = \inf\{u \leq \tau_{t+s}: B_u \circ k_{\tau_{t+s}} > s\} = \inf\{u \leq \tau_{t+s} \quad B_u > s\} = \tau_s = \sigma_s(\mu)$ a.s. (P^μ) on $\{\tau_s < \infty\}$. Now if $Y \in \mathscr{O}(\mathscr{F}_t^\mu)$, then $Y(T_t) \in \mathscr{P}(\mathscr{F}_t^\mu)$. To see this, it suffices to consider the case where $Y = 1_{[S,\infty)}$, with S an $(\mathscr{F}_t^\mu)$-optional time. Then $\{(t, \omega): Y(T_t) = 1\} = \{(t, \omega): T_t \geq S\} = \{(t, \omega): T_t \geq S \wedge t\}$. Since $t \to S \wedge t$ is a predictable process, this set is predictable, and the claim follows. Therefore, to complete the proof of the claim that $\mathscr{G}_s^\mu \subset \mathscr{G}_{(t+s)-}^\mu$, let $Z \in \mathscr{G}_s^\mu$. There is a process $W \in \mathscr{O}(\mathscr{F}_t^\mu)$ such that $Z = W_{\tau_s}$ on $\{\tau_s < \infty\}$. By the discussion above, $W(T_t) \in \mathscr{P}(\mathscr{F}_t^\mu)$, and

$$W(T_{\tau_{t+s}}) = W_{\tau_s} = Z \text{ a.s. } (P^\mu).$$

The "a.s. (P^μ)" is necessary in the expression above since $T_{\tau_{t+s}} = \tau_s$ only a.s. (P^μ). However, $\mathscr{G}_{(t+s)-}^\mu$ contains all P^μ-null sets, and therefore $Z \in \mathscr{G}_{(t+s)-}^\mu$.

We have proved that for each μ, $(\mathscr{G}_t^\mu)_{t>0}$ and $(\mathscr{G}_{t-}^\mu)_{t>0}$ are filtrations. Therefore,

$$\mathscr{H}_t = \cap_{\mu \in \mathscr{M}} \mathscr{G}_t^\mu$$

and

$$\mathscr{H}_{t-} = \cap_{\mu \in \mathscr{M}} \mathscr{G}_{t-}^\mu$$

are filtrations. It is clear that $\mathscr{F}_{\tau_s} \subset \mathscr{H}_s$, and we now know $\mathscr{F}_{\tau_s} \subset \mathscr{H}_s \subset \mathscr{H}_{(t+s)-}$ and $\mathscr{F}_{\tau_{t+s}-} \subset \mathscr{F}_{\tau_{t+s}}$. By adapting a standard argument in Markov process theory, we shall now show that $\mathscr{H}_{(t+s)-} = \mathscr{F}_{\tau_{t+s}-}$, which will complete the proof.

Clearly, $\mathscr{F}_{\tau_s-} \subset \mathscr{H}_{s-}$ and $\mathbf{N} \subset \mathscr{F}_{\tau_s-}$. To show $\mathscr{H}_{s-} \subset \mathscr{F}_{\tau_s-}$, let $H \in b\mathscr{H}_{s-}$. Define yet another σ-algebra $\mathscr{F}_{\tau_s-}^0$ as follows. A random variable $R \in \mathscr{F}^0$ belongs to $\mathscr{F}_{\tau_s-}^0$ if and only if there is a process $Z \in \mathscr{P}(\mathscr{F}_t^0)$ such that $R = Z_{\tau_s}$ on $\{\tau_s < \infty\}$. Note that $\mathscr{F}_{\tau_s-}^0$ is separable since $\mathscr{P}(\mathscr{F}_t^0)$ is separable. Define a family of measures Q^x on $\mathscr{F}_{\tau_s-}^0$ by setting $Q^x(G) = E^x[GH]$ for $G \in b\mathscr{F}_{\tau_s-}^0$. Since $Q^x \ll P^x$, $\mathscr{F}_{\tau_s-}^0$ is separable, and $x \to Q^x(G)$ and $x \to E^x[G]$ are $\mathscr{E}^*$-measurable, Doob's lemma guarantees the existence of a density $p(x, \omega) \in \mathscr{E}^* \otimes \mathscr{F}_{\tau_s-}^0$ such that $E^x[GH] = E^x[p(x, \cdot)G]$ for all $G \in b\mathscr{F}_{\tau_s-}^0$. If we set $H^*(\omega) = p(X_0(\omega), \omega)$, then $H^* \in \mathscr{F}_{\tau_s-}^0 \vee \mathscr{F}_0 \subset \mathscr{F}_{\tau_s-}$. Since $E^\mu[GH] = E^\mu[GH^*]$ for all $G \in b\mathscr{F}_{\tau_s-}^0$, it follows that $E^\mu[GH] = E^\mu[GH^*]$ for all $G \in b\mathscr{F}_{\tau_s-}^0 \vee \mathbf{N}^\mu = \mathscr{G}_{s-}^\mu$. In particular, take $G = H - H^* \in \mathscr{G}_{s-}^\mu$. Then $E^\mu[(H - H^*)^2] = 0$ implies $H - H^* \in \mathbf{N}^\mu$. Since this holds for every finite measure μ, $H - H^* \in \mathbf{N}$. Because $\mathbf{N} \subset \mathscr{F}_{\tau_s-}$, we conclude $H \in \mathscr{F}_{\tau_s-}$. Therefore $\mathscr{F}_{\tau_s-} = \mathscr{H}_{s-}$. □

(1.4) LEMMA. *Assume that for each $t > 0$, $A_u \circ k_{\tau_t} = A_u$ for all $u \leq \tau_t$. Then for each $x \in E$, there is an optional process V_t^x such that $V_{\tau_t}^x 1_{\{\tau_t < \infty\}} = t\, 1_{\{\tau_t < \infty\}}$* a.s. (P^x).

PROOF. Fix $\mu = \epsilon_x$ on E, and let $(B_u) \in \mathscr{B}(R^+) \otimes \mathscr{F}^0$ be the process obtained in the proof of (1.3). Set $R_t^u = B_u \circ k_t 1_{\{u \le t\}}$. Since R^u is predictable, $V_t^x = \sup\{R_t^u: u$ a positive rational$\}$ is predictable. Since $u \to R_t^u$ is left continuous a.s. (P^x), V_t^x is P^x-indistinguishable from $\sup\{R_t^u: u \downarrow 0\} = B_t \circ k_t$. But $V_{\tau_t}^x = B_{\tau_t} \circ k_{\tau_t} = B_{\tau_t} = t$ on $\{\tau_t < \infty\}$ a.s. (P^x).

We now state and prove the main theorem of this section, after which we shall make a series of remarks about the hypotheses of the theorem and sundry generalizations. Recall that $\mathscr{F}_{\tau_{t+}} = \cap_{s>t} \mathscr{F}_{\tau_s}$.

(1.5) THEOREM. *Let A_t be a σ-integrable raw* CAF. *Assume*

(a) *For each $t > 0$, $A_u \circ k_{\tau_t} = A_u$ for all $u \le \tau_t$;*

(b) *For each s, and for each bounded, positive continuous function f on E, $t \to K_s f(X_{\tau_t})$ is almost surely right continuous on $(0, \infty)$.*

Then $(X_{\tau_t})_{t>0}$ is a strong Markov process over the filtration $(\mathscr{F}_{\tau_{t+}})$.

PROOF. Hypothesis (a) and Proposition (1.3) guarantee that $(\mathscr{F}_{\tau_{t+}})$ is indeed a filtration. Since $f(X_{\tau_{t+s}}) = f(X_{\tau_s}) \circ \theta_{\tau_t}$, we have shown that

$$E^x \int f(X_{\tau_{t+s}})\, Y_{\tau_t}\, dt = E^x \int K_s f(X_{\tau_t}) Y_{\tau_t}\, dt$$

for all $Y \in b\,\mathcal{O}(\mathscr{F}_t)^+$ such that $E^x \int Y_t\, dA_t < \infty$ for all x, and for all functions f which are bounded, positive and continuous on E. Fix a right continuous strictly positive optional process Z such that $E^x \int Z_t\, dA_t < \infty$ for all x, and let $Y \in b\,\mathcal{O}(\mathscr{F}_t)^+$. For $\alpha > 0$, define $Y_t^\alpha = \exp(-\alpha V_t^x)$, where V_t^x is the process defined in Lemma (1.4). Replacing Y with $Z \cdot Y^\alpha \cdot Y$ in the equation above, and applying Fubini's theorem, we find that

$$\int e^{-\alpha t} E^x[f(X_{\tau_{t+s}}) Z_{\tau_t} Y_{\tau_t}]\, dt$$

$$= \int e^{-\alpha t} E^x[K_s f(X_{\tau_t}) Z_{\tau_t} Y_{\tau_t}]\, dt,$$

for all $\alpha > 0$. Since both integrands are right continuous on $(0, \infty)$, we conclude by uniqueness of the Laplace transform that

$$E^x[f(X_{\tau_{t+s}}) Z_{\tau_t} Y_{\tau_t}] = E^x[K_s f(X_{\tau_t}) Z_{\tau_t} Y_{\tau_t}]$$

for each $t > 0$. Since $\mathscr{F}_{\tau_t} = \sigma\{W_{\tau_t}: W_s$ is a right continuous process adapted to $(\mathscr{F}_s)\}$, it follows from the right continuity and strict positivity of Z that

$$(1.6) \qquad E^x[f(X_{\tau_{t+s}}) \mid \mathscr{F}_{\tau_t}] = K_s f(X_{\tau_t}) \text{ a.s. } P^x \qquad \text{for all } x.$$

The standard argument [2] shows that $(X_{\tau_t})_{t>0}$ is a strong Markov process; we give a brief sketch of the argument here. Let $T > 0$ be an optional time for the filtration $(\mathscr{F}_{\tau_{t+}})$. Define a sequence of $(\mathscr{F}_{\tau_t})$-optional times (T_n) decreasing to T by setting

$$\begin{aligned} T_n &= (k+1)2^{-n} && \text{on } \{k2^{-n} \le T < (k+1)2^{-n}\} \\ &= \infty && \text{on } \{T = \infty\}. \end{aligned}$$

Let f be a bounded continuous function on E, and compute

$$\begin{aligned} &E^x \int e^{-\alpha t} f(X_{\tau_{T+t}})\, dt \\ &= \lim_{n\to\infty} E^x \int e^{-\alpha t} f(X_{\tau_{T_n+t}})\, dt \\ &= \lim_{n\to\infty} E^x \textstyle\sum_{k=1}^{\infty} \left[\displaystyle\int e^{-\alpha t} f(X_{\tau_{k2^{-n}+t}})\, dt;\ T_n = k2^{-n} \right]. \end{aligned}$$

Using the simple Markov property (1.6), this becomes

$$\lim_{n\to\infty} E^x \sum_{k=1}^{\infty} \left[\int e^{-\alpha t} K_t f(X_{\tau_{k2^{-n}}})\, dt;\ T_n = k2^{-n} \right]$$

$$= \lim_{n\to\infty} E^x \int e^{-\alpha t} K_t f(X_{\tau_{T_n}})\, dt$$

$$= \int e^{-\alpha t} E^x[K_t f(X_{\tau_T})]\, dt,$$

the last equality holding by the right continuity of $s \to K_t f(X_{\tau_s})$ for each t, and Fubini's theorem. Since $t \to E^x[f(X_{\tau_{T+t}})]$ and $t \to E^x\{K_t f(X_{\tau_T})]$ are right continuous on $(0, \infty)$,

$$E^x[f(X_{\tau_{T+t}})] = E^x[K_t f(X_{\tau_T})] \tag{1.7}$$

for each $t > 0$. If G is open in E, we let (f_n) be a sequence of continuous functions increasing to 1_G, whence (1.7) and the monotone convergence theorem imply that

$$P^x(X_{\tau_{T+t}} \in G) = E^x[K_t 1_G(X_{\tau_T})].$$

REMARKS. (1). $(X_{\tau_t})_{t\geq 0}$ is a strong Markov process if, for each s, $t \to K_s f(X_{\tau_t})$ is right continuous on $[0, \infty)$. This right continuity at zero is simply not true in several natural situations (cf. Section 2).

(2). We may weaken (1.5a) as follows. If we replace (1.5a) by

(1.8) For each $t > 0$, $P^x(A_u \circ k_{\tau_t} = A_u \text{ for all } u \leq \tau_t) = 1$ for all x,

then the proof of (1.3) shows that if $Z \in \mathscr{F}_{\tau_t}$, there exists $Z' \in \mathscr{F}_{\tau_{t+}}$ such that $Z = Z'$ almost surely. Thus if $\mathscr{F}^x_{\tau_t}$ denotes the σ-algebra $\mathscr{F}_{\tau_t}$ augmented with the P^x-null sets in the P^x-completion of $\mathscr{F}$, then $(\mathscr{F}^x_{\tau_t})$ is a filtration. Moreover, the process V^x_t defined in Lemma (1.4) has the property that $V^x_{\tau_t} = \mathrm{t}$ a.s. (P^x) on $\{\tau_t < \infty\}$. The proof of Theorem (1.5) then shows that $(\mathrm{X}_{\tau_t})_{t>0}$ is a strong Markov process over the filtration $(\mathscr{F}^x_{\tau_{t+}})$ for each x. In the examples we present in the succeeding sections, it is easy to see that (1.5a) is satisfied, except in Section 3, where we need to introduce a technical device in order to satisfy the hypothesis.

(3). We may weaken the hypothesis (1.5b), also. For example, if $t \to K_x f(X_{\tau_t})$ is almost surely right continuous except at a finite collection $(T_1, \ldots, T_n)$ of times which are optional for the filtration $(\mathscr{F}_{\tau_{t+}})$, then the strong Markov property holds at an optional time T if and only if $P^x(\cup_k\{T = T_k\}) = 0$ for all x. It may be of some use in this situation to appeal to the "usual" compactification arguments to explore the behavior of the process at these times.

(4). The fact that A is an AF is used in two places in the proof. It is used to show that

(i) its right continuous inverse τ_t satisfies

$$\tau_{t+s} = \tau_t + \tau_s \circ \theta_{\tau_t};$$

(ii) A^1 is an AF.

Thus we might rephrase the statement of Theorem (1.5) in terms of a continuous increasing process A satisfying (i) and (ii) above.

(5). If A is an *adapted* CAF, then the hypotheses of Theorem (1.5) are always satisfied (take $K_s f(x) = E^x[f(X_{\tau_s})]$). This recovers the usual time-change theorem for CAF's. [2].

(6). As mentioned in the introduction, the requirement that $A_u \in \mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}^*$ is a stronger measurability condition than the usual requirement that $A_u \in \mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}$. Meyer [9] has examined in certain cases when one can choose a version of A_u satisfying the stronger measurability condition. In the appendix at the end of the paper we show that any raw AF satisfying a mild integrability condition has a version which is $\mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}^*$-

measurable. As a corollary, every cooptional time has a version which is $\mathscr{F}^e$ measurable. We also show that if u is a Borel natural potential of X, then the natural additive functional A_t generating u may be chosen so that $A_t \in \mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}^0$.

The stronger measurability condition is used only to verify that the $\mathscr{F}_{\tau_t}$ increase and to produce an optional process V_t^x. These facts can sometimes be verified by hand. In this case, if hypothesis (1.5b) holds, then the conclusion of Theorem (1.5) remains true.

2. Birthing a Markov process. We show that "birthing" a Markov process at a coterminal time may be realized as a nonadapted time-change. Let L be a coterminal time (the canonical example of which is the last exit time from a Borel set). Set

$$A_t = \int_0^t 1_{\{L \le u\}}\, du.$$

The reader may easily verify that A is a raw CAF, and a version of A may be chosen in $\mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}^*$ by the corollary in the appendix.

In this section, however, the reader can also verify directly that the $\mathscr{F}_{\tau_t}$ increase and that there is an $(\mathscr{F}_t)$-optional process V_t^x with $V_{\tau_t}^x = t$ a.s. (P^x) on $\{\tau_t < \infty\}$. We now compute the kernel K_s.

$$\begin{aligned} E^x \int Y_u f(X_{\tau_s}) \circ \theta_u \, dA_u &= E^x \int Y_u f(X_{\tau_s}) \circ \theta_u 1_{\{L \le u\}}\, du \\ &= E^x \int Y_u E^{X_u}[f(X_{\tau_s}); L = 0]\, du \\ &= E^x \int Y_u E^{X_u}[f(X_s); L = 0]\, du. \end{aligned}$$

On the other hand,

$$E^x \int Y_u K_s f(X_u) 1_{\{L \le u\}}\, du = E^x \int Y_u K_s f(X_u) P^{X_u}(L = 0)\, du.$$

Thus we set

$$\begin{aligned} K_s f(x) &= \frac{1}{P^x(L=0)} E^x[f(X_s); L = 0] &\quad \text{if} \quad P^x(L = 0) > 0; \\ &= f(\Delta) &\quad \text{if} \quad P^x(L = 0) = 0. \end{aligned}$$

We now rewrite $K_s f$ in the form obtained in [11]. Define $T_L = \inf\{t > 0: L(k_t\omega) > 0\}$. Then T_L is a perfect, exact terminal time, and $T_L = \infty$ on $\{L = 0\}$ [11]. Thus, $E^x[f(X_s); L = 0] = E^x[f(X_s); T_L = \infty; s < T_L] = E^x[f(X_s) P^{X_s}(L = 0); s < T_L]$. Therefore, if V_t is the semigroup P_t killed at T_L, $V_t f(x) = E^x[f(X_t); t < T_L]$, then K_s is the semigroup V_t conditioned by the function $g(x) = P^x(L = 0) = P^x(T_L = \infty)$.

Since $\tau_t = L + t$, $L \circ k_{\tau_t} = L$ for all $t > 0$, and it follows that (1.5a) is satisfied. To check (1.5b), we adapt an argument of Getoor [4]. Let f be bounded positive continuous function on E, and define $R^\alpha fg(x) = \int_0^\infty e^{-\alpha u} V_{u+s} fg(x)\, du$. Then $R^\alpha fg$ is α-excessive for the semigroup V_t. Therefore, $t \to R^\alpha fg(X_t) 1_{\{t < T_L\}}$ is right continuous a.s., and hence is an optional process. Since $s \to fg(X_s)$ is right continuous (g is excessive!), $s \to V_s fg(x)$ is right continuous. Therefore, $\lim_{n\to\infty} nR^n fg(X_t) 1_{\{t<T_L\}} = V_s(fg)(X_t) 1_{\{t < T_L\}}$ is an optional process. Let (T_n) be a sequence of optional times decreasing to T. Then $E^x\{E^{X(T_n)}[f(X_s); L = 0]; T_n < T_L\} = E^x[f(X_{s+T_n}); L \le T_n, T_n < T_L]$, and this converges to $E^x[f(X_{s+T}); L \le T, T < T_L] = E^x\{E^{X(T)}[f(X_s); L = 0]; T < T_L\}$ as n increases. Therefore, $t \to E^{X_t}[f(X_s); L = 0] 1_{\{t < T_L\}}$ is right continuous almost surely [3]. Set $\Lambda = \{t \to E^{X_t}[f(X_s); L = 0]$ is not right continuous$\}$. Then $0 = P^x(P^{X(r)}(\Lambda; T_L = \infty)) = P^x(\Lambda \circ \theta_r; T_L \circ \theta_r = \infty) \ge P^x(\Lambda \circ \theta_r; L < r)$. Therefore, $t \to E^{X_t}[f(X_s); L = 0]$ is a.s. right continuous on (L, ∞). Since $g(x)$ is excessive, $g(X_t)$ is optional, and a similar argument shows that $t \to g(X_t)$ is right continuous on $[0, \infty)$. Therefore, $t \to K_s f(X_{\tau_t})$ is right continuous on $(0, \infty)$ if $X_{L+t} = X_{\tau_t} \notin \{x: g(x) = 0\}$ for all $t > 0$. Getoor [4] gives the following argument to show this. Let $T = \inf\{t: g(X_t) = 0\}$. For each μ, T is an $(\mathscr{F}_t^\mu)$-optional time. We need only show that for each r, $\Lambda = \{L < r, r + T\circ\theta_r < \infty\}$ is a P^μ-null set for all μ. Since $T_L\circ\theta_r = \infty$ on $\{L < r\}$, and $g(X_T)\circ\theta_r = 0$ on $\{r + T\circ\theta_r < \infty\}$ by right continuity of $g(X_t)$, $\Lambda \subset \{T_L\circ\theta_r = \infty, r + T\circ\theta_r < \infty, g(X_t)\circ\theta_r = 0\} \subset \{T_L\circ\theta_T\circ\theta_r = \infty, T\circ\theta_r < \infty, g(X_T)\circ\theta_r = 0\}$, the last containment holding because $r + T\circ\theta_r \geq r$. Thus $P^\mu(\Lambda) \leq E^\mu P^{X_r}(T_L\circ\theta_T = \infty, T < \infty, g(X_T) = 0) = E^\mu E^{X_r}[P^{X_T}(T_L = \infty); T < \infty; g(X_T) = 0] = E^\mu E^{X_r}[g(X_T); T < \infty; g(X_T) = 0] = 0$.

Interesting variants of the birthing procedure are suggested by the following example. Let B now be an open set, and assume that the boundary of B is a polar set for the process. (We are deliberately avoiding (interesting) difficulties at the boundary in this way; see the example at the end of this section.) Let L be the last time the process leaves B, and set

$$A_t = \int_0^t 1_B(X_u)\,du + \int_0^t 1_{\{L\leq u\}}\,du.$$

If τ_t denotes the right continuous inverse of A, then forming X_{τ_t} amounts to deleting all of the excursions from B except for the last one. We leave it to the reader to verify (1.5a) and to check that

$$K_s f(x) = \frac{E^x[f(X_s); L = 0] + E^x[f(X_{\tau_s})]1_B(x)}{1_B(x) + P^x(L = 0)} \qquad \text{if} \quad x \notin \partial B,$$

and to check that $t \to K_s f(X_{\tau_t})$ is right continuous on $[0, \infty)$ P^x a.s. for all $x \notin \partial B$.

If $x \in \partial B$, $t \to K_s f(X_{\tau_t})$ is right continuous on $(0, \infty)$ P^x a.s. The reader may consider various special cases when $t \to K_s f(X_{\tau_t})$ is right continuous on $[0, \infty)$, but this is not true in general.

A moment's thought will convince the reader that some hypotheses on the set B are necessary above in order to obtain the strong Markov property. Let X denote Brownian motion on R^1 killed at an independent exponential time, and let $B = \{0\}$. Then the CAF A_t above should be replaced with

$$M_t + \int_0^t 1_{\{L\leq u\}}\,du,$$

where M_t denotes the local time of the Brownian motion at 0. If we time-change, the process X_{τ_t} sits at zero for some positive time, and then moves away *continuously*. Such a process cannot be a strong Markov process.

3. Killing at a cooptional time. Let L be a cooptional time, and let A_t be the raw CAF given by

$$A_t = L \wedge t = \int_0^t 1_{\{u<L\}}\,du.$$

Again we compute $K_s f$:

$$E^x \int Y_u f(X_{\tau_s})\circ\theta_u 1_{\{u<L\}}\,du = E^x \int Y_u E^{X_u}[f(X_{\tau_s})1_{\{L\geq 0\}}\,du,$$

and

$$E^x \int Y_u K_s f(X_u) 1_{\{u<L\}}\,du = E^x \int Y_u K_s f(X_u) P^{X_u}(L > 0)\,du.$$

But $E^x[f(X_{\tau_s}); L > 0] = E^x[f(X_s); L > s] + E^x[f(X_\infty); L \leq s] = E^x[f(X_s)P^{X_s}(L > 0)]$.

Thus we set

$$K_s f(x) = \frac{1}{P^x(L > 0)} E^x[f(X_s) P^{X_s}(L > 0)] \qquad \text{if} \quad P^x(L > 0) > 0,$$
$$= f(\Delta) \qquad \text{if} \quad P^x(L > 0) = 0.$$

Right continuity of $K_s f(X_t)$ is easily verified for continuous functions f, and (1.5b) is therefore satisfied.

The reader will easily check that $A_u \circ k_{\tau_t} \neq A_u$ on $[0, \tau_t)$ in general (even when L is a last exit time). However, we may show directly that the $\mathscr{F}_{\tau_t}$ increase. If $Z \in \mathscr{F}_{\tau_t}$, then $Z = W_t$ on $\{\tau_t < \infty\}$ for some optional process W since $\tau_t = t$ on $\{\tau_t < \infty\}$. Set $\tilde{W}_t = W_{t-s} 1_{\{t \geq s\}}$. Then $\tilde{W}_{\tau_{t+s}} = \tilde{W}_{t+s} = W_t$ on $\{\tau_{t+s} < \infty\} \subset \{\tau_t < \infty\}$. Thus $Z \in \mathscr{F}_{\tau_{t+s}}$. Moreover, if we set $V_t^x = t$, then $V_{\tau_t}^x = t$ on $\{\tau_t < \infty\}$.

With but a few technical machinations, we can adjust the example so that hypothesis (1.5a) is satisfied. The reader may cringe at what we do now since we have just verified enough by hand to guarantee that the conclusion of Theorem (1.5) is true in this case, but the technique may be of use in more complicated situations where the outcome is not so obvious. Adjoin an isolated point $\tilde{\Delta}$ to $E \cup \{\Delta\}$ and extend the semigroup P_t so that $P_t(\tilde{\Delta}, \{\tilde{\Delta}\}) = 1$. Let $\tilde{\Omega}$ denote the collection of right continuous paths from R^+ to $E \cup \{\Delta, \tilde{\Delta}\}$ such that for $\tilde{\omega} \in \tilde{\Omega}$,

(i) if $\tilde{\omega}(s) = \Delta$, then $\tilde{\omega}(s + t) \in \{\Delta, \tilde{\Delta}\}$ for all $t > 0$;

(ii) if $\tilde{\omega}(s) = \tilde{\Delta}$, then $\tilde{\omega}(s + t) = \tilde{\Delta}$ for all $t > 0$;

(iii) $\tilde{\omega}(\infty)$ is defined to be $\tilde{\Delta}$.

Notice that each path $\tilde{\omega} \in \tilde{\Omega}$ can be identified with a path $i(\tilde{\omega}) \in \Omega$ as follows. Let $\tilde{\zeta}(\tilde{\omega}) = \inf\{t: \tilde{\omega}(t) \in \{\Delta, \tilde{\Delta}\}\}$, and set $i(\tilde{\omega}) = k_{\tilde{\zeta}(\tilde{\omega})} \tilde{\omega}$, where k_t is the usual killing operator on $\tilde{\Omega}$ using Δ as cemetery.

Define new killing operators $\tilde{k}_t : \tilde{\Omega} \to \tilde{\Omega}$ by setting

$$(\tilde{k}_t \tilde{\omega})(s) = \tilde{\omega}(s) \qquad \text{if} \quad s < t$$
$$= \tilde{\Delta} \qquad \text{if} \quad s \geq t.$$

We extend the coordinate mappings, the σ-algebras generated by them, and the shift operators in the obvious manner. Each measure P^x, $x \in E$, charges only $\Omega \subset \tilde{\Omega}$ so it seems as though we have appended a useless piece to Ω. However, if L is the end of a homogeneous set $\Gamma \subset R^+ \times \Omega$, define a homogeneous set $\tilde{\Gamma} \subset R^+ \times \tilde{\Omega}$ by setting $\tilde{\Gamma}(\tilde{\omega}) = \Gamma(i(\tilde{\omega})) \cup \{t: X_t(\tilde{\omega}) = \tilde{\Delta}\}$. Let $\tilde{L} = \sup \tilde{\Gamma}$ and set $\tilde{A}_t = \tilde{L} \wedge t$. Then $\tilde{L} \circ \tilde{k}_t = \infty$ for each $t \geq 0$. Thus $\tilde{A}_u \circ \tilde{k}_{\tau_t} = \tilde{L} \circ \tilde{k}_{\tau_t} \wedge u = u$ and $\tilde{A}_u = u$ on $\{u \leq \tau_t < \infty\}$: (1.5a) is satisfied on $\tilde{\Omega}$.

Here is an interesting special case of killing. Let Λ be an invariant event with $h(x) = P^x(\Lambda) > 0$ for all x. That is, $\Lambda \circ \theta_t = \Lambda$. Then the set $\Gamma = [0, \infty) \times \Lambda$ is homogeneous, and the end of Γ,

$$L = L_\Gamma = 0 \quad \text{on } \Lambda^c$$
$$= \infty \quad \text{on } \Lambda.$$

Therefore,

$$X_{\tau_t}(\omega) = X_t(\omega) \qquad \text{if} \quad \omega \in \Lambda$$
$$= \Delta \qquad \text{if} \quad \omega \in \Lambda^c,$$

and X_{τ_t} has semigroup $K_s f(x) = (1/h(x)) E^x[f(X_s) h(X_s)]$. Thus X_{τ_t} is a realization of the h-transform of the process X_t by the invariant function $h(x)$.

4. Resection of paths. Let $D \subset C$ be two open sets in E, each having boundaries (denoted by ∂C, ∂D) which are polar sets for the process X_t (i.e., the process cannot enter or leave C or D continuously). Set

$$Z_t = 1 \qquad \text{if} \quad X_{T_C} \circ \theta_t \in D \qquad \text{or if} \quad T_C \circ \theta_t = \infty,$$
$$= 0 \qquad \text{if} \quad X_{T_C} \circ \theta_t \in C - D,$$

and define a raw CAF by setting $A_t = \int_0^t Z_s 1_{\{s<\zeta\}}\, ds \in \mathscr{B}(R^+) \otimes \mathscr{F}^0$. If we time-change by τ_t, the right continuous inverse of A, then, roughly speaking, we are excising all of the excursions from C which return to $C - D$ and pieces of path in $C - D$. Let $W = 1_{\{X(T_C) \in D\} \cup \{T_C = \infty\}}$. Then

$$E^x \int Y_u f(X_{\tau_s}) \circ \theta_u W \circ \theta_u \, du = E^x \int Y_u E^{X_u}[f(X_{\tau_s}) W] \, du,$$

while

$$E^x \int Y_u K_s f(X_u) W \circ \theta_u \, du = E^x \int Y_u K_s f(X_u) E^{X_u}[W] \, du.$$

Therefore, we set

$$\begin{aligned} K_s f(x) &= \frac{1}{P^x(W)} E^x[f(X_{\tau_s}) W] && \text{if} \quad P^x(W) > 0, \\ &= f(\Delta) && \text{if} \quad P^x(W) = 0. \end{aligned}$$

The right continuity of $K_s f(X_{\tau_t})$ requires careful attention in this case. Let (T_n) be a sequence of optional times decreasing to T. Then

$$E^x[P^{X(T_n)}(W)] = P^x(X_{T_n + T_C \circ \theta_{T_n}} \in D) + P^x(T_C \circ \theta_{T_n} = \infty).$$

But the first term in the sum converges to $P^x(X_{T+T_C \circ \theta_T} \in D)$ since the boundary of D is a polar set (similarly for the second term in the sum), and this implies that $P^{X_t}(W)$ is right continuous. Now we examine

$$E^x E^{X(T_n)}[f(X_{\tau_s}) W] = E^x[f(X_{T_n + \tau_s \circ \theta_{T_n}}) W \circ \theta_{T_n}]$$

for continuous functions f on E. It follows exactly as in the argument above that $W \circ \theta_{T_n}$ converges to $W \circ \theta_T$ almost surely as n increases. Since f is continuous, we need only show that $\pi_n = \tau_s \circ \theta_{T_n}$ converges to $\pi = \tau_s \circ \theta_T$ as n increases. Recall that π_n is the largest number such that

$$\int_{T_n}^{T_n + \pi_n} Z_t \, dt = s.$$

Since $T \le T_n$, it follows that $T + \pi \le T_n + \pi_n$, whence $T + \pi \le T + \liminf_{n\to\infty} \pi_n \le T + \limsup_{n\to\infty} \pi_n$. Let (π_{n_k}) be a subsequence of (π_n) converging to $\limsup_{n\to\infty} \pi_n$. Then

$$s = \lim_{k\to\infty} \int 1_{[T_{n_k}, T_{n_k} + \pi_{n_k}]} \quad Z_t \, dt = \int 1_{[T, T + \limsup_{n\to\infty} \pi_n]} Z_t \, dt.$$

This implies that $\limsup_{n\to\infty} \pi_n \le \pi$, and we conclude that $\pi = \lim_{n\to\infty} \pi_n$. The right continuity of $E^{X_t}[f(X_{\tau_s}) W]$ follows. Note than we have shown that $t \to K_s f(X_{\tau_t})$ is right continuous on $[0, \infty)$ P^x a.s. for all x not in $\partial C \cup \partial D$ since $X_{\tau_t} \notin \{x: P^x(W) = 0\}$ (by virtue of the fact that $\partial C \cup \partial D$ is a polar set).

In order to verify (1.5a), we let $S = \sup\{u: X_u \circ k_{\tau_t} \in C\}$. Then $A_u = A_u \circ k_{\tau_t}$ for all $u \in [0, S]$. Since $X_u \circ k_{\tau_t} \notin C$ for all $u \in (S, \infty)$, $Z_u \circ k_{\tau_t} = 1$ on (S, ∞). Since Z_u is constant on excursions of X outside C, and $Z_{\tau_t} = 1$, $Z_u = 1$ on (S, τ_t). Therefore, $A_u = A_u \circ k_{\tau_t}$ for all $u \in [0, \tau_t]$.

REMARKS. (1). Let $\mathscr{H}$ denote the collection of raw CAF's satisfying the hypotheses of Theorem (1.5). It is simple to see that $\mathscr{H}$ is not a positive cone. For if we set

$$\begin{aligned} \tilde{Z}_t &= 2 \quad \text{if} \quad X_{T_C} \circ \theta_t \in C - D \quad \text{or if} \quad T_C \circ \theta_t = \infty, \\ &= 0 \quad \text{if} \quad X_{T_C} \circ \theta_t \in D, \end{aligned}$$

and $\tilde{A}_t = \int_0^t \tilde{Z}_s \, ds$, then $\tilde{A}_t$ also satisfies the conditions of Theorem (1.5). If $B_t = A_t + \tilde{A}_t$

satisfies the conditions of (1.5), then there exists an optional process Y_t such that $Y_{\tau_t} = t$, where τ_t denotes the right continuous inverse of the CAF B. Since B_t is strictly increasing, $Y(\tau_{B_t}) = Y_t$, and therefore, $Y_t = B_t$. But B_t is definitely not adapted.

(2). For a related transformation, the reader should consult the work of Knight and Pittenger [8]. We briefly describe their procedure here for the purpose of discussion. Let A and B be Borel sets with $\bar{A} \cap \bar{B} = \emptyset$. Define $S_0 = L_0 = T_0 = 0$,

$$S_1 = D_B = \inf\{t \geq 0 \colon X_t \in B\}$$

$$L_1 = \sup\{t < S_1 \colon X_t \in A\}$$

$$T_1 = \inf\{t > S_1 \colon X_t \in A\},$$

and by recurrence we define

$$S_{n+1} = T_n + S_1 \circ \theta_{T_n}$$

$$L_{n+1} = T_n + L_1 \circ \theta_{T_n}$$

$$T_{n+1} = T_n + T_1 \circ \theta_{T_n}.$$

Then intervals of the form $[L_n, T_n)$ are the time intervals of excursions from A which hit B. Delete these intervals by letting $Z_t(\omega) = 1$ if and only if $(t, \omega) \in (\cup_n [L_n, T_n))^c$, and setting $A_t = \int_0^t Z_s\, ds$. Then A is *not* an AF. However, Knight and Pittenger show that X_{τ_t} is a strong Markov process, where τ_t denotes the right continuous inverse of A. One might hope initially that Remark (4) following Theorem (1.5) is relevant here since $\tau_{t+s} = \tau_t + \tau_s \circ \theta_{\tau_t}$. However, A^1 is not an AF. For $E^x[A_\infty] = E^x[A^1_\infty]$, and therefore if A^1 is an AF, $E^x[A_\infty]$ must be an excessive function. We give a trivial example to show that this is not the case. Let X_t denote uniform motion to the right with speed one on (0, 5) with death occurring when the process strikes {5}. Let $B = (0, 1) \cup (4, 5)$, and let $A = (2, 3)$. Then $E^{1/2}[A_\infty] = 1$, but $E^1[A_\infty] = 2$. Thus $E^x[A_\infty]$ is not decreasing and cannot be excessive.

We briefly indicate a transformation which "shortens" the paths a bit. Let C be a finely open set in E with the fine boundary of E being a polar set, and set

$$Z_t = 1 \quad \text{if} \quad T_C \circ \theta_t \geq a,$$
$$\quad = 0 \quad \text{otherwise},$$

$$A_t = \int_0^t Z_s\, ds.$$

We leave it to the reader to choose the kernel K_s and to check that the hypotheses of Theorem (1.5) are verified. Note that if one replaces the condition $T_C \circ \theta_t \geq a$ with $T_C \circ \theta_t \leq a$, hypothesis (1.5a) is no longer satisfied.

APPENDIX

Let A_t be a raw AF which is only $\mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}$-measurable, and set $m_t = \exp(-A_t)$.

THEOREM. *Assume that for some* $\alpha > 0$,

$$E^x \int e^{-\alpha t} m_t\, dt < \infty \text{ for all } x.$$

Then there is a process $B_t \in \mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}^e \subset \mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}^*$ *which is indistinguishable from* A_t.

PROOF. Let $f_1, f_2, f_3, \cdots, f_n$ be bounded positive continuous functions on E, and let $t_1, t_2, \cdots, t_n$ be positive real numbers. Set $H = f_1(X_{t_1}) f_2(X_{t_2}) \cdots f_n(X_{t_n})$. Then

$$G_b(x) = E^x \int_0^\infty e^{-bt} H \circ \theta_t m_{t+u}\, dt$$

is a b-excessive function. Since $t \mapsto e^{-\alpha t} H \circ \theta_t m_{t+u}$ is bounded and right continuous, $\lim_{b \to \infty} bG_{b+\alpha}(x) = E^x[H \cdot m_u]$ and $E^x[H \cdot m_u] \in \mathscr{E}^e$, being a limit of b-excessive functions. A monotone class argument shows that $E^x[F \cdot m_u] \in \mathscr{E}^e$ for all bounded $\mathscr{F}^0$-measurable functions F. Similarly, $P^x(F) \in \mathscr{E}^e$. By Doob's lemma, for each rational u, there is a density $p_u(x, w) \in \mathscr{E}^e \otimes \mathscr{F}^0$ such that $E^x[F \cdot m_u] = E^x[F \cdot p_u(x, w)]$. Then $c_u(w) = p_u(X_0(w), w) \in \mathscr{F}^e$ and $E^x[F \cdot m_u] = E^x[F \cdot c_u]$. Set

$$d_u(w) = c_u(w) \quad \text{if } c_s(w) \text{ is nondecreasing as } s \text{ runs through the rationals;}$$
$$= 0 \quad \text{otherwise.}$$

Define $B_u(w) = \lim_{s \downarrow \downarrow u; s \in Q} d_s(w)$. It is simple to check that $B_u \in \mathscr{B}(\mathbb{R}^+) \otimes \mathscr{F}^e$ and B_t is indistinguishable from A_t. □

COROLLARY. *If L is a cooptional time, then there is a random variable $M \in \mathscr{F}^e$ such that $L = M$* a.s.

PROOF. Simply take the additive functional $A_t = 1_{\{0 < L \le t\}}$, and apply the preceding theorem.

The following result of Benveniste and Jacod may be found in [9], page 182. Let $\mathscr{F}_t^e = \sigma\{f(X_s);\, s \le t;\, f \text{ is } \alpha\text{-excessive}, \alpha \ge 0\}$.

PROPOSITION. *Let (M_t) be an exact multiplicative functional. Then there is a multiplicative functional N_t adapted to $\mathscr{F}_{t+}^e$ which is indistinguishable from M_t.*

We shall indicate the proof of the following result, which has hypotheses and conclusion slightly different from those of Benveniste and Jacod. No hypothesis of exactness is required.

PROPOSITION. *Let X be either a Hunt process or a Ray process with Borel transition semigroup on E. Let A_t be a natural additive functional with finite potential $u(x) = E^x[A_\infty] \in \mathscr{E}$. Then there exists a right continuous additive functional B_t adapted to $\mathscr{F}_{t+}^0$ which is indistinguishable from A_t.*

We briefly sketch part of the "general theory" construction of the additive functional B_t having potential $u(x)$, paying attention to certain points of measurability.

PROOF. For each $x \in E$, define a measure m^x on $(R^+ \times \Omega, \mathscr{P}(\mathscr{F}_t^0))$ by specifying the measure of predictable stochastic intervals. If S and T are two optional times in $\mathscr{F}^0$ with $S \le T$, define $m^x((S, T]) = E^x[u(X_S) - u(X_T)]$. In fact, m^x does extend to be a measure on $(R^+ \times \Omega, \mathscr{P}(\mathscr{F}_t^0))$ [3]. Since $m^x((S, T]) \in \mathscr{E}$, it follows by a monotone class argument that $m^x(Z) \in \mathscr{E}$ for every $Z \in b\mathscr{P}(\mathscr{F}_t^0)$. We extend m^x to a measure $\bar{m}^x$ on $(R^+ \times \Omega, \mathscr{B}(R^+) \otimes \mathscr{F}^0)$ by setting $\bar{m}^x(Z) = m^x({}^3Z)$, where ${}^3Z$ denotes the predictable projection of Z. Now, in fact, it follows from the hypotheses and Dawson's formula [10], page 533, that ${}^3Z$ may be chosen to be in $\mathscr{P}(\mathscr{F}_t^0)$. Therefore, $\bar{m}^x(Z) \in \mathscr{E}$ for all $Z \in b\mathscr{B}(R^+) \otimes \mathscr{F}^0$. For each rational t, define a measure Q_t^x on $(\Omega, \mathscr{F}_t^0)$ by setting $Q_t^x(H) = \bar{m}^x([0, t] \times H)$ whenever $H \in \mathscr{F}_t^0$. Since $Q_t^x \ll P^x$ and $\mathscr{F}_t^0$ is separable, Doob's lemma guarantees the existence of a density $p_t(x, w) \in \mathscr{E} \otimes \mathscr{F}_t^0$ so that $Q_t^x(H) = P^x[p_t(x, \cdot)H]$ for $H \in b\mathscr{F}_t^0$. Set $C_t(w) = p_t(X_0(w), w) \in \mathscr{F}_t^0$, and observe that $Q_t^x(H) = E^x[C_t H]$ for $H \in b\mathscr{F}_t^0$. It is simple to show that $C_t \le C_{t+s}$ a.s., and for each real t, we define

$$B_t = \liminf_{s \downarrow \downarrow t;\, s \text{ rational}} C_s.$$

Then $B_t \in \mathscr{F}_{t+}^0$, and the rest of the proof follows as usual to verify that B_t is indeed the additive functional with potential $u(x)$.

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