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GENERAL ONE-SIDED LAWS OF THE ITERATED LOGARITHM[1]

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Let $\{X_i\}$ be a sequence of independent, identically distributed nondegenerate random variables and $S_n = \sum_{i=1}^n X_i$. We consider the question for various centering sequences $\{\alpha_n\}$: when is it possible to find a positive, monotone sequence $\{\beta_n\}$ such that lim sup $\beta_n^{-1}(S_n - \alpha_n) = c$ a.s., c a finite nonzero constant? If $\alpha_n =$ med S_n, we obtain a necessary and sufficient condition for this. An important corollary is a one-sided version of the Hartman-Wintner law of the iterated logarithm: if $E(X^+)^2 < \infty$, then it is always possible to find such a norming sequence. Explicit norming sequences are given which are easy to obtain. Necessary and sufficient conditions are also given for being able to find a norming sequence $\{\beta_n\}$ for the two-sided problem (lim sup $\beta_n^{-1}|S_n - \alpha_n| = c$ a.s.) when $\alpha_n = ES_n$ and $\alpha_n = 0$. The two-sided problem with $\alpha_n =$ med S_n was solved by Kesten. The one-sided problem remains open for $\alpha_n = ES_n$ and $\alpha_n = 0$. Examples are given which illustrate the advantage of considering different centering sequences. A one-sided version of Strassen's converse to the law of the iterated logarithm is also given: if lim sup $S_n/\sqrt{2n \log\log n} = 1$ a.s., then $EX = 0$, $EX^2 = 1$.

1. Introduction. Let $\{X_i\}$ be a sequence of independent, identically distributed nondegenerate random variables and $S_n = \sum_{i=1}^n X_i$. We will let F denote the distribution function of X_1 and X will be a random variable with this distribution. The object of this paper is to study

$$\limsup_{n\to\infty} \frac{S_n - \alpha_n}{\beta_n} \tag{1.1}$$

for various centering sequences $\{\alpha_n\}$ and norming sequences $\{\beta_n\}$. The only assumption that will be made about $\{\beta_n\}$ is that it is positive and monotone. The general approach will be to consider certain specific centering sequences such as $\alpha_n = 0$, $\alpha_n = ES_n$, or $\alpha_n =$ median S_n and then ask, for a given distribution F, whether it is possible to find a norming sequence $\{\beta_n\}$ such that the lim sup in (1.1) is a finite, nonzero constant. When such a norming sequence exists we will construct one that works and investigate some of its properties. When no such norming sequence exists we will generally be able to give a simple criterion which will decide the question of whether the lim sup in (1.1) is zero or infinity for a given sequence $\{\beta_n\}$.

The classical results in this area are concerned with the specific norming sequence

$$\beta_n = (2n \log\log n)^{1/2}. \tag{1.2}$$

The early work was done by Khintchine [12] in the Bernoulli case and Kolmogorov [17].

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This culminated in the Hartman-Wintner law of the iterated logarithm [7]: if $EX = 0$, $EX^2 = 1$, then with β_n as in (1.2)

$$\text{(1.3)} \qquad \limsup_{n\to\infty} \beta_n^{-1} S_n = 1, \qquad \liminf_{n\to\infty} \beta_n^{-1} S_n = -1 \text{ a.s.}$$

There is also a converse which is due to Strassen [22]: if (1.3) holds with β_n as in (1.2) then $EX = 0$, $EX^2 = 1$. For a history of the development of the classical results, see [2].

The present study was motivated by a result of Kesten [11]: if α_n satisfies

$$\text{(1.4)} \qquad P\{S_n \geq \alpha_n\} \geq \epsilon, \qquad P\{S_n \leq \alpha_n\} \geq \epsilon$$

for some $\epsilon > 0$, then it is possible to find a norming sequence $\{\beta_n\}$ such that

$$\text{(1.5)} \qquad 0 < \limsup_{n\to\infty} \frac{|S_n - \alpha_n|}{\beta_n} < \infty \text{ a.s.}$$

if and only if F is in the domain of partial attraction of the normal distribution. To describe this condition in terms of F, let

$$\text{(1.6)} \qquad G(x) = P\{|X| > x\}, \qquad K(x) = x^{-2} \int_{|y| \leq x} y^2 dF(y).$$

Then F is in the domain of partial attraction of the normal distribution if and only if

$$\text{(1.7)} \qquad \liminf_{x\to\infty} \frac{G(x)}{K(x)} = 0.$$

One of the main results of this study is a one-sided analogue of Kesten's result: if $\{\alpha_n\}$ satisfies (1.4) then it is possible to find a norming sequence $\{\beta_n\}$ such that (1.1) is positive and finite if and only if

$$\text{(1.8)} \qquad \liminf_{x\to\infty} \frac{P\{X > x\}}{K(x) + G(x)} = 0.$$

An important special case of this is the following one-sided version of the Hartman-Wintner result:

THEOREM. *If $E(X^+)^2 < \infty$, then there is a norming sequence $\{\beta_n\}$ such that*

$$\text{(1.9)} \qquad \limsup_{n\to\infty} \frac{S_n - \operatorname{med} S_n}{\beta_n} = 1 \text{ a.s.}$$

In the Hartman-Wintner theorem, the centering could be at either the expectation or the median of S_n but in the present case the mean may not exist. The norming sequence $\{\beta_n\}$ in (1.9) must depend on the negative tail of F; as this tail of F increases, so will β_n.

The one-sided problem is quite different from the two-sided one since the latter reduces to considering symmetric distributions. Also the conditions (1.7) and (1.8) appear to be somewhat similar but are fundamentally different. Thus (1.7) does not occur too easily because of the relation between G and K. But (1.8) will be satisfied whenever the negative tail dominates the positive tail infinitely often. As an example, consider a distribution where (1.7) fails so that it is impossible to get a norming sequence for (1.5). Without changing the distribution of $|X|$ (and thus not changing the failure of (1.7)), one can construct a distribution F so that (1.8) is true and also its analogue for $-X$ by putting the mass on the positive reals for a while, then on the negative reals, etc. This means that it is possible to find a norming sequence for (1.1) with $\alpha_n = \operatorname{med} S_n$ and another (necessarily distinct) norming sequence for the lim inf even though there is no norming sequence for the two-sided problem.

Another general result relating to the classical theory is a one-sided version of Strassen's converse: if the lim sup in (1.3) is one a.s. with $\{\beta_n\}$ as in (1.2), then $EX = 0$, $EX^2 = 1$. This means that for the classical norming sequence, the first statement in (1.3) implies the second. This is in contrast to the situation with most norming sequences where it is common to have the lim sup one and the lim inf minus infinity.

The norming sequences that must be used in general in Kesten's theorem are not very nice. Feller [5] (also see Theorem 7 in [11]) considered the question of when one could find well-behaved norming sequences in the symmetric case. This is equivalent to considering the two-sided problem for general distributions if the centering is at the median of S_n. He found that it was unusual to be able to find a nice norming sequence that would work for the two-sided problem once the distribution had infinite variance. In fact, it is not possible even for many distributions in the domain of attraction of the normal distribution (i.e., with lim inf replaced by lim in (1.7)) which come very close to having finite variance. In the one-sided problem it is still the case that under the necessary and sufficient condition (1.8) the norming sequences need not be very nice. But there are nice norming sequences which do work quite generally for the one-sided problem. In particular, nice sequences can always be used in the theorem stated above for $E(X^+)^2 < \infty$ or any time the positive tail is smaller that the negative tail divided by a factor slightly larger than log x. Furthermore these nice sequences are easy to obtain. Results concerning these nice norming sequences for the one-sided problem with centering at the expectation when it exists have been obtained by Klass [13, 14].

The reason for studying (1.1) is to obtain information about the large values in the sequence $\{S_n\}$. Note that if a norming sequence $\{\beta_n\}$ can be found such that (1.1) is positive and finite for a particular centering sequence $\{\alpha_n\}$, then for any other centering sequence $\{\alpha'_n\}$ with $\alpha'_n \le \alpha_n$ the sequence $\{\beta'_n\}$ defined by

$$\beta'_n = \max\{\beta_n, \alpha_n - \alpha'_n\}$$

will serve as an appropriate norming sequence although it may not be monotone. We obtain the best bound from (1.1) when $\{\alpha_n\}$ is chosen so that $\{\beta_n\}$ is as small as possible. This is the reason that we consider different centering sequences. Comparisons of the various norming sequences are made in Section 9. Also Example 9.3 illustrates how more information is obtained when $\{\beta_n\}$ is smaller.

It should be pointed out that typically there will be no best place to center in the one-sided problem. Thus even if the $\{X_i\}$ are normal with mean zero, variance one, then (1.3) is true but zero is not the best centering sequence. The integral test of Kolmogorov (for a statement see [2]) implies that

$$\limsup_{n\to\infty} \frac{S_n - \sqrt{2n \log\log n}}{\sqrt{n}\,\log\log\log n/\sqrt{\log\log n}} = \frac{3}{2\sqrt{2}} \text{ a.s.}$$

This idea can be repeated so as to obtain better and better centering sequences. Of course the ultimate answer to the problem is to have an integral test that will determine for a given distribution F and increasing sequence $\{\beta_n\}$ whether $S_n > \beta_n$ i.o. But this is far too much to hope for at present. Indeed, it is not yet possible to even obtain the constant value of the lim sup when using a nice norming sequence. (Of course an integral test is usually rather easy to come by when lim sup $\beta_n^{-1}S_n$ is zero or infinity for all norming sequences.) An integral test was obtained by Lipschutz-Yevick [18] for a subset of the distributions in the domain of attraction of a stable law.

Many of the results that are in the literature are included here as this can be done with very little extra effort. The organization of the paper will now be described briefly. Section 2 consists of the basic facts we need about the functions G and K defined in (1.6) and some related functions. Also the norming sequences to be used are defined here and some of their properties are obtained. Section 3 contains the required probability estimates. In Section 4 we prove the fundamental convergence lemmas which can be used in a large variety of problems. These are basically lim sup results for normed sums of truncated variables when centered at truncated means. The unusual feature is that the truncation is one-sided. However, they have been formulated so they can be used in proving both one- and two-sided results. The lemma which deals with the case when the lim sup is necessarily zero or infinity is also in this section. The two-sided results are given in Section 5. These include Kesten's theorem as well as necessary and sufficient conditions for (1.5) when $\alpha_n = 0$ and $\alpha_n = ES_n$. This solves a problem listed by Kesten in [11]. The proof that (1.8) is necessary and sufficient in the one-sided case for a centering sequence satisfying (1.4) is given in Section 6. The theorems giving the nice norming sequences are developed in Section 7 for centering at the median of S_n and at the expectation when it exists. Centering at zero is somewhat different and this is discussed in Section 8. Of course, if $E|X| < \infty$ then centering at zero corresponds to the strong law if $EX \neq 0$ or to centering at ES_n if $EX = 0$. Thus we only consider this case when $E|X| = \infty$. This problem was solved by Fristedt and the author [6] under a very weak tail condition for the case of negative random variables. Erickson has since obtained an elegant criterion [1] for determining when S_n is positive infinitely often. We give a slight extension of his result that shows that when this criterion fails then

$$\lim_{n\to\infty} \frac{X_1^+ + \cdots + X_n^+}{X_1^- + \cdots + X_n^-} = 0 \text{ a.s.}$$

This means that one can deal with negative random variables in this case with no loss of generality. Some comparisons between the norming sequences are made in Section 9 and also some examples are given. An extension of the results of Klass and Teicher [15] for the case when the mean barely exists or barely does not exist is also included here. The one-sided Strassen converse is in Section 10 and some open problems are listed in the final section. Two problems which can be mentioned here are finding necessary and sufficient conditions analogous to (1.8) for centering at 0 or at ES_n. The first one has a different aspect since it may not be possible to find an appropriate norming sequence $\{\beta_n\}$ for $\beta_n^{-1} S_n$ even when $X \leq 0$ a.s. I do not believe that this phenomenon is adequately understood as yet. Ruling this case out some way, it is not hard to obtain separate necessary conditions and sufficient conditions. But for both problems what seemed to the author to be the most natural conjecture based on the other results in this paper proved to be incorrect.

The techniques used are all fairly standard and in most cases date back to Khintchine at least. Of course various improvements have been made by other authors including Kolmogorov, Lévy and Feller. I have made no effort to attribute these techniques to particular authors except in a couple of recent cases where some important new observation was made.

Most of the results in this paper were obtained during a sabbatical leave in 1977–78 spent at Cornell University. During the year I benefitted from many discussions with Harry Kesten. In particular, after I had obtained the two-sided results he suggested that I consider the one-sided problem. The question of what could be said when one only assumed the lim sup finite in Strassen's converse was suggested to me by Henry Teicher.

2. Preliminaries. For a given nondegenerate random variable X with distribution function F, we introduce three basic functions defined for $x > 0$:

$$G(x) = P\{|X| > x\}, \qquad K(x) = x^{-2}\int_{|y|\leq x} y^2 dF(y), \qquad M(x) = x^{-1}\int_{|y|\leq x} y\, dF(y).$$

Note that G and K depend only on the distribution of $|X|$. It will also be convenient to have a special notation for these functions corresponding to the random variables X^+ and X^-. For this we will use + and − subscripts. Thus, for example,

$$G_+(x) = P\{X > x\}, \qquad M_-(x) = -x^{-1}\int_{-x\leq y\leq 0} y\, dF(y).$$

In order to make our integrals compatible with G, we adopt the convention that the upper limit is to be included in the interval of integration iff it is in $[0, \infty)$ while the lower limit is to be included iff it is in $(-\infty, 0]$.

Define the function

$$f(x) = G(x) + K(x) = E\{(x^{-1}X)^2 \wedge 1\}, \quad x > 0 \tag{2.1}$$
$$= P\{X \neq 0\}, \quad x = 0.$$

It is easy to verify that f is positive, continuous, decreasing, and zero at infinity. Also f is strictly decreasing on $[a, \infty)$ where a is the infinum of the support of the distribution of $|X|$. Thus f has an inverse function uniquely defined on $(0, f(0))$. We will also use the function

$$g(x) = x^{-1}\int_0^x G(y)\,dy,\ x > 0; \qquad g(0) = P\{X \neq 0\}. \tag{2.2}$$

The function g has the same properties as f.

LEMMA 2.1. *For* $\lambda > 0$,

$$x^\lambda f(x) = \int_0^x y^{\lambda-1}\{\lambda G(y) - (2-\lambda)K(y)\}\,dy.$$

PROOF. It is sufficient to prove this for nonnegative random variables since G, K, and f are the same for X and $|X|$. We have

$$\int_0^x \lambda y^{\lambda-1}G(y)\,dy = \int_0^x \lambda y^{\lambda-1}\int_{(y,x]} dF(z)\,dy + \int_0^x \lambda y^{\lambda-1}G(x)\,dy$$

$$= \int_{[0,x]}\int_0^z \lambda y^{\lambda-1}dy\,dF(z) + x^\lambda G(x) \tag{2.3}$$

$$= \int_{[0,x]} z^\lambda dF(z) + x^\lambda G(x)$$

and

$$\int_0^x (2-\lambda)y^{\lambda-1}K(y)\,dy = \int_0^x (2-\lambda)y^{\lambda-3}\int_{[0,y]} z^2 dF(z)\,dy$$

$$= \int_{[0,x]}\int_z^x (2-\lambda)y^{\lambda-3}dy\,z^2 dF(z) \tag{2.4}$$

$$= \int_{[0,x]} (z^{\lambda-2} - x^{\lambda-2})z^2 dF(z) = \int_{[0,x]} z^\lambda dF(z) - x^\lambda K(x).$$

Subtracting the two equations completes the proof.

LEMMA 2.2. *For a nonnegative random variable*

$$\int_0^x G(y)\,dy = x\{M(x) + G(x)\}, \qquad \int_0^x K(y)\,dy = x\{M(x) - K(x)\}.$$

PROOF. Set $\lambda = 1$ in (2.3) and (2.4).

At this point it may be useful to point out some further properties of f and g. First we note that for $x > 0$, using Lemma 2.2 for $|X|$ yields

$$g(x) = E\{|x^{-1}X| \wedge 1\}. \tag{2.5}$$

It is clear from the definitions (or Lemma 2.1) that

$$x^2 f(x) \text{ and } xg(x) \text{ are nondecreasing.} \tag{2.6}$$

By Lemma 2.1,

$$\int_0^x \{G(y) - K(y)\}\, dy = xf(x) \tag{2.7}$$

and so

$$f(x) \le g(x). \tag{2.8}$$

Alternatively,

$$f(x) = E\{|x^{-1}X|^2 \wedge 1\} \le E\{|x^{-1}X| \wedge 1\} = g(x).$$

It is easy to check that for $\lambda > 0$, $E|X|^\lambda < \infty$ implies that $x^\lambda G(x) \to 0$ as $x \to \infty$ and also $x^\lambda K(x) \to 0$ as $x \to \infty$ provided that $\lambda < 2$. Thus we have

$$E|X|^\lambda < \infty \quad \text{implies} \quad \lim_{x\to\infty} x^\lambda f(x) = 0, \qquad 0 < \lambda < 2, \tag{2.9}$$

$$E|X|^\lambda < \infty \quad \text{implies} \quad \lim_{x\to\infty} x^\lambda g(x) = 0, \qquad 0 < \lambda < 1. \tag{2.10}$$

For $\lambda = 1$ we need the more precise result:

LEMMA 2.3. *For any random variable* X

$$E|X| < \infty \quad \text{iff} \quad \int_0^\infty f(y)\, dy < \infty \quad \text{iff} \quad \int_0^\infty G(y)\, dy < \infty \quad \text{iff} \quad \int_0^\infty K(y)\, dy < \infty.$$

In this case

$$E|X| = \int_0^\infty G(y)\, dy = \int_0^\infty K(y)\, dy = \frac{1}{2}\int_0^\infty f(y)\, dy.$$

PROOF. If $E|X| < \infty$, then by (2.5)

$$\int_0^x G(y)\, dy = xg(x) \le E|X|$$

and by (2.7)

$$\int_0^x K(y)\, dy \le \int_0^x G(y)\, dy.$$

Thus the three only if statements are clear. Now suppose that $\int K(y)\, dy$ converges. Although K need not be monotone, we do know that $x^2 K(x)$ increases so that

$$\int_x^\infty K(y)\, dy \ge x^2 K(x) \int_x^\infty y^{-2}\, dy = xK(x).$$

Thus $xK(x) \to 0$ and then by Lemma 2.2 $E|X| = \int_0^\infty K(y)\, dy$.

Next we need more information about the rate of decay of f. This is given in

LEMMA 2.4. *If* $K(y) \ge \eta G(y)$ *on some interval then* $y^\lambda f(y)$ *decreases on that interval where* $\lambda = 2\eta/(1+\eta)$. *If* $K(y) \le \eta G(y)$ *then* $y^\lambda f(y)$ *increases.*

PROOF. First note that $2 - \lambda = \lambda\eta^{-1}$. Then by Lemma 2.1,

$$z^{\lambda}f(z) - x^{\lambda}f(x) = \int_x^z \lambda y^{\lambda-1}\{G(y) - \eta^{-1}K(y)\}\,dy$$

and the integrand is nonpositive in the first case and nonnegative in the second.

Much of the previous work in this area has been for the case of a regularly varying G. We will not stress this case but it may be worth pointing out some of the implications. First we have ([4], pages 303 and 544)

$$x^{\lambda}f(x) \text{ is slowly varying iff } \lim_{x\to\infty}\frac{K(x)}{G(x)} = \frac{\lambda}{2-\lambda}$$

and these are equivalent to $x^{\lambda}K(x)$ slowly varying if $\lambda > 0$ and $x^{\lambda}G(x)$ slowly varying if $\lambda < 2$. These equivalences are not hard to prove using Lemma 2.4 and standard facts about regularly varying functions. Of course these conditions are exactly that of $|X|$ being in the domain of attraction of the stable law of index λ. The situation is somewhat special when $\lambda = 0$ or 2 since then either G or K dominates. We will use this fact when $\lambda = 0$ so we now prove what we need.

LEMMA 2.5. *For a nonnegative random variable*

$$\lim_{x\to\infty}\frac{M(x)}{G(x)} = 0 \quad \text{iff} \quad \lim_{x\to\infty}\frac{K(x)}{G(x)} = 0 \quad \text{iff} \quad G \textit{ is slowly varying.}$$

PROOF. For a nonnegative random variable, $K(x) \le M(x)$ so the first only if statement is clear. Next assume that $K(x)/G(x) \to 0$ and $c > 1$. Then for a given $\eta > 0$, there is an x such that $K(y) \le \eta G(y)$ for $y \ge x$. By Lemma 2.4

$$y^{\lambda}f(y) \le (cy)^{\lambda}f(cy), \qquad y \ge x.$$

But then

$$G(y) \le f(y) \le c^{\lambda}f(cy) \le c^{\lambda}(1+\eta)G(cy), \qquad y \ge x.$$

Since we can make $c^{\lambda}(1+\eta)$ arbitrarily close to one by a proper choice of η (recall that $\lambda = 2\eta/(1+\eta)$) and since $G(cy) \le G(y)$, this proves the second only if statement. If G is slowly varying,

$$\int_0^x G(y)\,dy \sim xG(x)$$

and the final implication then follows from Lemma 2.2.

REMARK. The second and third statements of Lemma 2.5 are equivalent and imply the first statement for an arbitrary random variable. This observation follows by applying the lemma to $|X|$. The second and third statements are also equivalent to f slowly varying and to g slowly varying.

We require one more lemma concerning the asymptotic behavior of these functions.

LEMMA 2.6. *For an arbitrary random variable* X,

$$\liminf_{x\to\infty}\frac{G(x)}{f(x) + |M(x)|} = 0 \tag{2.11}$$

if and only if at least one of the following conditions holds:

$$\liminf_{x\to\infty}\frac{G(x) + |M(x)|}{f(x)} = 0$$

or

$$\liminf_{x\to\infty} \frac{f(x)}{|M(x)|} = 0.$$

PROOF. The sufficiency of either of the two conditions is clear. If they both fail then we have for some positive C

$$f(x) \le C\{|M(x)| + G(x)\}, \qquad |M(x)| \le Cf(x) \qquad \text{for all} \quad x. \tag{2.12}$$

For $c > 1$, note that

$$K(cx) = c^{-2}x^{-2}\int_{|y|\le x} y^2\,dF + c^{-2}x^{-2}\int_{x<|y|\le cx} y^2\,dF \le c^{-2}K(x) + G(x)$$

and

$$|M(x)| = \left| x^{-1}\int_{|y|\le cx} y\,dF - x^{-1}\int_{x<|y|\le cx} y\,dF \right| \le c|M(cx)| + cG(x).$$

Thus

$$\begin{aligned} |M(x)| &\le c|M(cx)| + cG(x) \le cCf(cx) + cG(x) \\ &\le cC\{c^{-2}K(x) + G(x) + G(cx)\} + cG(x) \\ &\le c^{-1}C^2|M(x)| + \{c^{-1}C^2 + 2cC + c\}G(x). \end{aligned}$$

Taking $c = 2C^2 \vee 2$ shows that $M(x) = O(G(x))$ and then by (2.12) $f(x) = O(G(x))$ also. But this means that (2.11) fails as well.

Next we introduce some sequences which are defined in terms of f and are used in the definitions of the norming sequences $\{\beta_n\}$. We define $\{a_n\}$, $\{b_n\}$ by

$$f(a_n) = n^{-1}\log\log n, \qquad f(b_n) = \gamma n^{-1}. \tag{2.13}$$

γ is a parameter whose value will vary depending on the context. Of course, b_n depends on γ but this dependence will be suppressed when possible to simplify the notation. When centering at the median in the "nice" case, the norming sequence is defined by

$$\beta_n = a_n \log\log n + n\int_{a_n}^{b_n} f(x)\,dx. \tag{2.14}$$

β_n will also depend on γ. When centering at the mean in the "nice" case we will use

$$\beta_n = a_n \log\log n + n\int_{a_n}^{\infty} f(x)\,dx. \tag{2.15}$$

The integral converges by Lemma 2.3. We note that the latter definition does not depend on γ; in general, it gives a larger value for β_n than (2.14).

We now collect some facts about these sequences. Since f decreases, both a_n and b_n increase. In fact by (2.6)

$$a_n^2 n^{-1}\log\log n \quad \text{and} \quad b_n^2 n^{-1} \quad \text{are nondecreasing.} \tag{2.16}$$

Since $a_n \le b_n$ for large n, using (2.6) again, we see that

$$a_n^2 \log\log n \le \gamma b_n^2 \quad \text{for large } n. \tag{2.17}$$

Thus $a_n b_n^{-1} \to 0$. Note that (2.14) may also be written

$$\beta_n = n\int_0^{b_n} \{f(x) \wedge n^{-1}\log\log n\}\,dx \ge nb_n f(b_n) = \gamma b_n. \tag{2.18}$$

Also by using (2.6) again we have

$$n \int_{a_n}^{b_n} f(x)\,dx \ge n a_n^2 f(a_n) \int_{a_n}^{b_n} x^{-2}\,dx = n a_n f(a_n)(1 - a_n b_n^{-1}) \tag{2.19}$$
$$= a_n \log\log n(1 - a_n b_n^{-1}) \sim a_n \log\log n,$$

the last step being a consequence of (2.17). This means that in both (2.14) and (2.15) the $a_n \log\log n$ term could be omitted without changing the order of magnitude of β_n. However, it is useful to keep these terms in order to provide β_n with the desired monotonicity properties. For example, note that if β_n is as in (2.15) it can be written as in (2.18) but with the upper limit of integration being infinite. This shows that for this β_n

$$n^{-1}\beta_n \text{ decreases.} \tag{2.20}$$

Another monotonicity property is the subject of the next lemma.

LEMMA 2.7. *If β_n is defined by either* (2.14) *or* (2.15) *then $n^{-1/2}\beta_n$ is increasing for large n.*

PROOF. With β_n as in (2.14),

$$\begin{aligned}
\frac{\beta_{n+1}}{(n+1)^{1/2}} - \frac{\beta_n}{n^{1/2}} &\ge a_{n+1}\frac{\log\log(n+1)}{(n+1)^{1/2}} - a_n \frac{\log\log n}{n^{1/2}} \\
&\quad + \{(n+1)^{1/2} - n^{1/2}\} \int_{a_{n+1}}^{b_n} f(x)\,dx \\
&\quad + (n+1)^{1/2}\frac{\gamma}{n+1}(b_{n+1} - b_n) - n^{1/2}\frac{\log\log n}{n}(a_{n+1} - a_n) \\
&= a_{n+1}\left(\frac{\log\log(n+1)}{(n+1)^{1/2}} - \frac{\log\log n}{n^{1/2}}\right) \\
&\quad + \{(n+1)^{1/2} - n^{1/2}\} \int_{a_{n+1}}^{b_n} f(x)\,dx \\
&\quad + \frac{\gamma}{(n+1)^{1/2}}(b_{n+1} - b_n).
\end{aligned} \tag{2.21}$$

If β_n is given by (2.15) we have the same lower bound with b_n replaced by ∞ as the upper limit on the integral and the last term omitted. By the mean value theorem, the first term is

$$-a_{n+1}\left\{\frac{\log\log n}{2n^{3/2}} - \frac{1}{n^{3/2}\log n} + O\left(\frac{\log\log n}{n^{5/2}}\right)\right\}.$$

For the second term we use the fact that $x^2 f(x)$ is nondecreasing as in (2.19) to obtain a lower bound of

$$\left\{\frac{1}{2n^{1/2}} + O\left(\frac{1}{n^{3/2}}\right)\right\} a_{n+1}\frac{\log\log(n+1)}{n+1}\left(1 - \frac{a_{n+1}}{b_n}\right).$$

Combining these two estimates gives a lower bound for the first two terms of

$$a_{n+1}\left\{\frac{1}{n^{3/2}\log n}-\frac{a_{n+1}}{b_n}\frac{\log\log(n+1)}{2n^{1/2}(n+1)}+O\left(\frac{\log\log n}{n^{5/2}}\right)\right\}. \tag{2.22}$$

This is sufficient for β_n given by (2.15) since the middle term above is missing. For β_n as in (2.14), we consider two possibilities. If $b_n^{-1}a_{n+1} \le n^{-1/2}$, the lower bound in (2.22) will be positive for large n and we can throw away the third term in (2.21) since it is nonnegative. If, on the other hand, we have $b_n^{-1}a_{n+1} \ge n^{-1/2}$, then we use (2.16) and argue as in (2.17) to see that

$$\begin{aligned}\frac{\gamma}{(n+1)^{1/2}}(b_{n+1}-b_n) &\ge \gamma b_n\left\{\frac{1}{n^{1/2}}-\frac{1}{(n+1)^{1/2}}\right\}=\frac{\gamma b_n}{2n^{3/2}}+O\left(\frac{b_n}{n^{5/2}}\right)\\ &\ge \frac{1}{2n^{1/2}b_n}\frac{a_{n+1}^2\log\log(n+1)}{n+1}+O\left(\frac{a_{n+1}}{n^2}\right).\end{aligned}$$

The sum of this bound and the bound in (2.22) is then positive for large n.

We conclude this section by introducing the analogous sequences for the "nice" case when $E|X| = \infty$ and we center at zero. First define $\{c_n\}$, $\{d_n\}$ by

$$g(c_n) = \delta n^{-1}\log\log n, \qquad g(d_n) = \gamma n^{-1}. \tag{2.23}$$

As before, the dependence on δ and γ will usually be suppressed. By (2.8) we have $b_n \le d_n$ and, if $\delta \le 1$, $a_n \le c_n$. In this case the norming sequence is defined by

$$\beta_n = c_n \log\log n. \tag{2.24}$$

By (2.6)

$$n^{-1}\beta_n \quad \text{and} \quad n^{-1}d_n \quad \text{are nondecreasing} \tag{2.25}$$

and for large n

$$\delta\beta_n \le \gamma d_n. \tag{2.26}$$

3. Probability estimates. In this section we will derive the necessary probability estimates and also state some standard results that will be used. The letters r and a will denote arbitrary positive constants and it will be convenient to let $u = ra^{-1}$. $\{X_i\}$ will be a sequence of independent, identically distributed random variables with common distribution function F and $S_n = \sum_{i=1}^n X_i$. We introduce the truncated variables

$$Y_i = X_i \wedge a, \qquad Z_i = -a \vee Y_i \tag{3.1}$$

and their sums

$$T_n = \textstyle\sum_{i=1}^n Y_i, \qquad V_n = \sum_{i=1}^n Z_i. \tag{3.2}$$

The usual practice of letting X, Y, Z denote random variables with the same distributions as X_1, Y_1, Z_1 will be used.

The first three lemmas are closely related to what are usually called exponential bounds. The main differences are that the truncation used here is one-sided and in Lemma 3.2 we work in the range where the usual lower bound fails to apply. In Lemma 3.2 we assume that $EX = 0$ if $EX^2 < \infty$. A bound of this type is still valid when $EX \ne 0$ but then the constants in the lemma must depend on the distribution. This will also be the case for the results which use Lemma 3.2; for example, in Lemma 4.2 the lower bound may be smaller although it will still be positive if $EX \ne 0$ and $EX^2 < \infty$. Since the final results for $EX \ne 0$ are easily obtained by considering the summands $X_i - EX$ we thought it best to make this simplifying assumption where necessary. The functions f and g appearing in these lemmas are defined in (2.1) and (2.2).

LEMMA 3.1. *For any* $s > 0$,

$$P\{V_n \ge EV_n + \tfrac{1}{2}e^r rna\, f(a) + sar^{-1}\} \le e^{-s}.$$

PROOF. Note that

$$
\begin{aligned}
Ee^{uZ} &= \int_{-a}^{a} e^{ux} dF + e^{ua}G_+(a) + e^{-ua}G_-(a) \\
&\le \int_{-a}^{a} (1 + ux + \tfrac{1}{2} e^r u^2 x^2)\, dF + e^r G_+(a) + e^{-r}G_-(a) \\
&\le 1 - G(a) + uaM(a) + \tfrac{1}{2} e^r u^2 a^2 K(a) + (1 + r + \tfrac{1}{2} e^r r^2)G_+(a) \qquad (3.3)\\
&\quad + (1 - r + \tfrac{1}{2} r^2)G_-(a) \\
&\le 1 + r\{M(a) + G_+(a) - G_-(a)\} + \tfrac{1}{2} e^r r^2 f(a) \\
&\le \exp\{uEZ + \tfrac{1}{2} e^r r^2 f(a)\}.
\end{aligned}
$$

The result now follows from the elementary inequality

$$P\{V_n \ge t\} \le Ee^{uV_n - ut} = \{Ee^{uZ}\}^n e^{-ut}.$$

LEMMA 3.2. *If $EX^2 < \infty$, suppose that $EX = 0$. Then if $C_1 < .1795$ and both a and $nf(a)$ are sufficiently large*

$$P\{T_n \ge EV_n + C_1 naf(a)\} \ge e^{-35nf(a)}.$$

REMARK. Since $T_n \le V_n$, Lemmas 3.1 and 3.2 both apply to both T_n and V_n. Also letting $r = 1$ and $s = nf(a)$ in Lemma 3.1 we see that they give very similar upper and lower bounds.

PROOF. We start with the inequality

$$
\begin{aligned}
Ee^{uY} &= \int_{-\infty}^{a} e^{ux}\, dF + e^{ua}G_+(a) \ge \int_{-a}^{a} (1 + ux + \tfrac{1}{2} e^{-r}u^2x^2)\, dF + e^r G_+(a) \\
&\ge 1 - G(a) + uaM(a) + \tfrac{1}{2} e^{-r}u^2a^2K(a) + (1 + r + \tfrac{1}{2} r^2)G_+(a) \\
&\ge 1 + uEZ + \tfrac{1}{2} e^{-r} r^2\{K(a) + G_+(a)\} + (r - 1)G_-(a).
\end{aligned}
$$

Now let $r = r_0 = 1.2195$ and $u_0 = r_0 a^{-1}$. This has the effect of making the coefficients of the last two terms equal. Thus

$$Ee^{u_0 Y} \ge 1 + u_0 EZ + (r_0 - 1)f(a) = 1 + \epsilon, \tag{3.4}$$

say. Now $\epsilon \to 0$ as $a \to \infty$ but we also need to know that $\epsilon^2 = o(f(a))$. Since

$$|a^{-1}EZ| = |M(a) + G_+(a) - G_-(a)| \le |M(a)| + f(a)$$

and $M(a) \to 0$ as $a \to \infty$ we only need to show that $\{M(a)\}^2 = o(f(a))$. If $EX^2 < \infty$, then $M(a) = o(a^{-1})$ since we have assumed that $EX = 0$ in this case and $f(a) \sim a^{-2}EX^2$. If $EX^2 = \infty$, let $\eta > 0$ be given and $b = \eta a\{K(a)\}^{1/2}$. Then

$$
\begin{aligned}
|M(a)| &= \left| a^{-1} \int_{|x| \le b} x\, dF + a^{-1} \int_{b < |x| \le a} x\, dF \right| \\
&\le \eta\{K(a)\}^{1/2} + a^{-1}\left\{\int_{b < |x| \le a} x^2\, dF\right\}^{1/2} \{G(b)\}^{1/2} \\
&\le \{K(a)\}^{1/2}(\eta + \{G(b)\}^{1/2}).
\end{aligned}
$$

Since $EX^2 = \infty$ implies that $b \to \infty$ as $a \to \infty$, we have that $\{M(a)\}^2 = o(f(a))$ in this case also. Now we use the inequality $1 + \epsilon \ge \exp\{\epsilon - \epsilon^2\}$ which is valid in a neighborhood of zero in conjunction with (3.4) to obtain for sufficiently large a

$$Ee^{u_0 Y} \geq \exp\{u_0 EZ + .219\, f(a)\}.$$

Here we have decreased the coefficient of $f(a)$ to allow for the ϵ^2 term. Then

$$Ee^{u_0(T_n - EV_n)} \geq \exp\{.219 n f(a)\} \tag{3.5}$$

for a sufficiently large. Since $T_n - EV_n \leq 2na$, we can integrate by parts to obtain

$$Ee^{u_0(T_n - EV_n)} = \int_{-\infty}^{\infty} u_0 e^{u_0 x} P\{T_n - EV_n \geq x\}\, dx. \tag{3.6}$$

Now we let $\xi_1 = C_1 naf(a)$ and $\xi_2 = 28naf(a)$. Since $C_1 r_0 < .219$ we have by (3.5) that

$$\int_{-\infty}^{\xi_1} u_0 e^{u_0 x} P\{T_n - EV_n \geq x\}\, dx \leq e^{u_0 \xi_1} = e^{C_1 r_0 n f(a)} = o(Ee^{u_0(T_n - EV_n)}) \tag{3.7}$$

as $nf(a) \to \infty$. Next, using (3.3)

$$\begin{aligned} \int_{\xi_2}^{\infty} u_0 e^{u_0 x} P\{T_n - EV_n \geq x\}\, dx &\leq \int_{\xi_2}^{\infty} u_0 e^{u_0 x} E e^{2u_0(T_n - EV_n) - 2u_0 x}\, dx \\ &= Ee^{2u_0(T_n - EV_n)} e^{-u_0 \xi_2} \\ &\leq Ee^{2u_0(V_n - EV_n)} e^{-u_0 \xi_2} \\ &\leq \exp\{(2e^{2r_0} r_0^2 - 28 r_0) n f(a)\}. \end{aligned}$$

Since the coefficient of $nf(a)$ is negative this term is also small compared to (3.6) when $nf(a)$ is large. Thus we have by (3.6) and (3.7)

$$Ee^{u_0(T_n - EV_n)} \leq \eta E e^{u_0(T_n - EV_n)} + \int_{\xi_1}^{\xi_2} u_0 e^{u_0 x} P\{T_n - EV_n \geq x\}\, dx$$

or, using (3.5),

$$1 \leq (1 - \eta)\exp\{.219 n f(a)\} \leq P\{T_n - EV_n \geq \xi_1\} e^{u_0 \xi_2}.$$

This completes the proof since $u_0 \xi_2 = 28 r_0 n f(a) < 35\, nf(a)$.

In Section 8 we will need a different lower bound which is somewhat easier to obtain.

LEMMA 3.3. *Suppose that $X \leq 0$ and $C_2 > 1$, $C_3 > 1$. Then if both a and $ng(a)$ are sufficiently large*

$$P\{S_n \geq -C_2 nag(a)\} \geq e^{-C_3 ng(a)}.$$

PROOF. We let $u = a^{-1}$ and apply Lemma 2.2:

$$\begin{aligned} Ee^{uX} &= \int_{-\infty}^{0} e^{ux}\, dF \geq \int_{-a}^{0} (1 + ux)\, dF = 1 - G_-(a) - M_-(a) \\ &= 1 - a^{-1} \int_0^a G_-(x)\, dx = 1 - g(a). \end{aligned}$$

Then

$$Ee^{uS_n} \geq e^{-ng(a)\{1 + g(a)\}} \geq e^{-(1+\epsilon)ng(a)} \tag{3.8}$$

for large a. But

$$Ee^{uS_n} = \int_{-\infty}^{0} ue^{ux}P\{S_n \geq x\}\,dx \leq e^{-u\xi} + P\{S_n \geq -\xi\}$$

which, together with (3.8), gives

$$P\{S_n \geq -C_2nag(a)\} \geq e^{-(1+\epsilon)ng(a)} - e^{-C_2ng(a)}.$$

Choosing ϵ so that $1 + \epsilon < C_2 \wedge C_3$ yields the result.

Now we will derive some simple probability estimates which are consequences of Chebyshev's inequality. In addition to the sums of truncated variables introduced in (3.2) it will be convenient to let

$$W_n = \textstyle\sum_{i=1}^{n} -b \vee Y_i \tag{3.9}$$

where we will always assume that $a \leq b$. However, the specific values of a and b will vary with the context. When $E|X| < \infty$,

$$\begin{aligned} EW_n - ES_n &= naG_+(a) - nbG_-(b) - n\int_a^{\infty} x\,dF - n\int_{-\infty}^{-b} x\,dF \\ &= -n\int_a^{\infty} G_+(x)\,dx + n\int_b^{\infty} G_-(x)\,dx. \end{aligned} \tag{3.10}$$

Since

$$\begin{aligned} E(-b \vee Y)^2 &= \int_{-b}^{a} x^2\,dF + a^2G_+(a) + b^2G_-(b) \\ &\leq \int_{-b}^{b} x^2\,dF + b^2G_+(b) + b^2G_-(b) = b^2f(b), \end{aligned}$$

we have

$$P\{|W_n - EW_n| \geq Cb\} \leq C^{-2}nf(b). \tag{3.11}$$

We will typically use this with $b = b_n$ as defined in (2.13). Then

$$P\{T_n \neq W_n\} \leq 1 - \{1 - G_-(b)\}^n \leq 1 - (1 - \gamma n^{-1})^n \sim 1 - e^{-\gamma} \tag{3.12}$$

so that for large n and $\eta > 0$

$$P\{|T_n - EW_n| \geq Cb_n\} \leq 1 - e^{-\gamma} + \eta + C^{-2}\gamma. \tag{3.13}$$

If $\gamma < \log 2$ the right side of (3.13) is less than ½ if we take η small enough and C large enough. Thus for large n

$$\operatorname{med} S_n \geq \operatorname{med} T_n \geq EW_n - Cb_n, \qquad \gamma < \log 2. \tag{3.14}$$

If, in addition, we take $a = b$ in the definition of W_n then $P\{S_n \neq W_n\}$ can be bounded as in (3.12) and

$$\operatorname{med} S_n \leq EW_n + Cb_n, \qquad \gamma < \log 2.$$

However, we will need to use a slightly different approach in obtaining the upper bound for med S_n. We will find later that a modified version of this bound will be valid even if $\gamma \geq \log 2$ provided that we have an added condition which will be available when we need the upper bound. In Section 8 we will be working with summands X_i which are nonpositive. In this case $S_n = T_n$ and we will take $b = c_m$ as defined in (2.23). Since $f(x) \leq g(x)$, (3.11) leads to

$$P\{|S_n - EW_n| \ge c_m\} \le nG(c_m) + nf(c_m) \le 2\delta nm^{-1}\log\log m. \tag{3.15}$$

We will also use $b = d_n$ as defined in (2.23). Then, by Lemma 2.2,

$$\begin{aligned} EW_n &= n\int_{-d_n}^{0} x\,dF - nd_nG_-(d_n) = -nd_n\{M_-(d_n) + G_-(d_n)\} \\ &= -nd_ng(d_n) = -\gamma\, d_n. \end{aligned} \tag{3.16}$$

Since $f(x) \le g(x)$, (3.12) is also valid in this case with $b = d_n$ so that

$$P\{|S_n - EW_n| \ge Cd_n\} \le 1 - e^{-\gamma} + \eta + C^{-2}\gamma.$$

Thus, for any given $\gamma > 0$ we can choose C large enough so this probability will be less than one. Recalling (3.16) we have

$$P\{S_n \ge -Cd_n\} \ge \epsilon > 0 \tag{3.17}$$

for an appropriate C and ϵ.

We conclude this section with the statements of two well-known lemmas which will be used extensively.

LEMMA 3.4. (Borel-Cantelli). *Let $\{A_n\}$, $\{B_n\}$ be two sequences of events such that the events $\{A_n\}$ are independent and for each n the pair A_n, B_n are independent. Suppose that $\Sigma P(A_n)$ diverges and that $P(B_n) \ge c > 0$ for all n. Then $P\{A_nB_n$* i.o.$\} > 0$. *If $\{A_nB_n$* i.o.$\}$ *is also a tail event for the sequence $\{S_n\}$ then*

$$P\{A_nB_n \text{ i.o.}\} = 1.$$

PROOF. Let $E_n = A_nB_n$. Since $P(E_n) \ge cP(A_n)$, $\Sigma P(E_n)$ diverges and for $j \ne k$

$$P(E_jE_k) \le P(A_jA_k) = P(A_j)P(A_k) \le c^{-2}P(E_j)P(E_k).$$

Then the result follows from one of the standard Borel-Cantelli lemmas, e.g., [16]. The Hewitt-Savage 0–1 law [8] gives the last statement.

LEMMA 3.5. (Skorokhod). *Suppose that*

$$P\{S_n - S_j \ge -\xi\} \ge c > 0 \qquad \textit{for all} \quad j \le n.$$

Then

$$P\{\max_{j\le n} S_j \ge \lambda + \xi\} \le c^{-1}P\{S_n \ge \lambda\}.$$

This is a slight extension of Lévy's inequality which is quite useful. It was used in essentially this form by Skorokhod in [21]. The proof is the same as for Lévy's inequality.

4. Fundamental convergence lemmas. In this section we will prove the three fundamental lemmas. The first will lead readily to the necessity of the various conditions in the theorems of Sections 5 and 6 and also will be used in proving the lim sup infinite in the divergent cases of Sections 7 and 8. The two-sided case is essentially due to Heyde [9]. We introduce the truncated variables and their sums

$$Z_i = -\beta_i \vee (X_i \wedge \beta_i), \qquad R_n = \textstyle\sum_{i=1}^n Z_i. \tag{4.1}$$

LEMMA 4.1. (a) *Suppose that*

$$f(x) \le CG(x) \qquad \textit{for all} \quad x. \tag{4.2}$$

For any monotone sequence $\{\beta_n\}$, if $\Sigma P\{|X| > \beta_n\} < \infty$ then

(4.3) $$\lim_{n\to\infty} \frac{S_n - ER_n}{\beta_n} = 0 \quad \text{a.s.}$$

with R_n as in (4.1). *On the other hand, if* $\Sigma P\{|X| > \beta_n\} = \infty$, *then*

$$\limsup_{n\to\infty} \frac{|S_n - \alpha_n|}{\beta_n} = \infty \quad \text{a.s.}$$

for any centering sequence $\{\alpha_n\}$ *such that either* (1) $\alpha_n - \alpha_{n-1} = O(\beta_n)$ *or* (2) $P\{S_n \ge \alpha_n\} \ge \epsilon$ *and* $P\{S_n \le \alpha_n\} \ge \epsilon$ *for some* $\epsilon > 0$. *The divergent case is also valid without* (4.2) *provided that* $n^{-\lambda}\beta_n$ *increases for some* $\lambda > 0$.

(b) *Suppose that*

(4.4) $$f(x) \le CG_+(x) \qquad \textit{for all} \quad x.$$

For any monotone sequence $\{\beta_n\}$, *if* $\Sigma P\{X > \beta_n\} < \infty$ *then* (4.3) *holds. On the other hand, if* $\Sigma P\{X > \beta_n\} = \infty$, *then*

$$\limsup_{n\to\infty} \frac{S_n - \alpha_n}{\beta_n} = \infty \quad \text{a.s.}$$

for any centering sequence $\{\alpha_n\}$ *such that* $P\{S_n \ge \alpha_n\} \ge \epsilon > 0$. *The divergent case is also valid without* (4.4) *provided that* $n^{-\lambda}\beta_n$ *increases for some* $\lambda > 0$.

REMARK. The result is valid even if $\{\beta_n\}$ is not monotone if one changes the criterion to the convergence or divergence of

$$\Sigma\, 2^k \max_{2^{k-1} < n \le 2^k} P\{|X| > \beta_n\}$$

in the first case and the analogous one-sided condition in the second case. However, a somewhat different proof is required.

PROOF. The two cases will be proved together. If the series converges, note first that since f is positive for all x so is $G(G_+)$. Thus $\beta_n \to \infty$. Now $\beta_n^{-2}EZ_n^2 = f(\beta_n)$ which is summable in either case. Then $\Sigma\, \beta_n^{-1}(Z_n - EZ_n)$ converges a.s. ([19], page 236) and so

$$\lim_{n\to\infty}\beta_n^{-1} \textstyle\sum_{i=1}^{n} (Z_i - EZ_i) = 0 \quad \text{a.s.}$$

by Kronecker ([19], page 238). This is sufficient since $\Sigma P\{Z_i \ne X_i\}$ also converges (note that $G_-(x) \le CG_+(x)$ in case (b)) and so $P\{Z_i \ne X_i \text{ i.o.}\} = 0$. If the series diverges we have by (2.6) that for $M > 1$

(4.5) $$x^2G(x) \le x^2f(x) \le M^2x^2f(Mx) \le CM^2x^2G(Mx).$$

Thus $\Sigma P\{|X| > M\beta_n\}$ also diverges and so

(4.6) $$\limsup_{n\to\infty}\beta_n^{-1}|X_n| = \infty \quad \text{a.s.}$$

(To see that this is still true without assuming (4.2) when $n^{-\lambda}\beta_n$ increases note that this implies $M^\lambda\beta_n \le \beta_{Mn}$ for any integer $M > 1$. Then $\Sigma P\{|X| > M^\lambda\beta_n\} \ge \Sigma P\{|X| > \beta_{Mn}\} = \infty$.) The same argument with G replaced by G_+ shows that lim sup $\beta_n^{-1}X_n = \infty$ a.s. in case (b). Writing

$$X_n = (S_n - \alpha_n) - (S_{n-1} - \alpha_{n-1}) + (\alpha_n - \alpha_{n-1})$$

we see that if $\alpha_n - \alpha_{n-1} = O(\beta_n)$ then (4.6) implies that

$$\limsup_{n\to\infty} \beta_n^{-1}\,|(S_n - \alpha_n) - (S_{n-1} - \alpha_{n-1})| = \infty \quad \text{a.s.}$$

which is sufficient. Under the second assumption on the centering sequence we choose M so that $P\{|X| > M\beta_n\} \le \frac{1}{2}\epsilon$ for all n. Either $\Sigma P\{X > 2M\beta_n\}$ or $\Sigma P\{X < -2M\beta_n\}$ diverges. We assume the former; otherwise the same argument applies to $-X$. From this point on the proof in case (b) is the same. Note that

$$P\{S_{n-1} \ge \alpha_n - M\beta_n\} \ge P\{S_n \ge \alpha_n, X_n \le M\beta_n\}$$
$$\ge P\{S_n \ge \alpha_n\} - P\{X_n > M\beta_n\} \ge \tfrac{1}{2}\epsilon.$$

Now let

$$E_n = \{X_n \ge 2M\beta_n, S_{n-1} \ge \alpha_n - M\beta_n\}.$$

By Lemma 3.4 we have $P\{E_n \text{ i.o.}\} = 1$ and thus

$$\limsup_{n\to\infty} \frac{S_n - \alpha_n}{\beta_n} = \infty \quad \text{a.s.}$$

The other two lemmas will be used in proving the sufficiency of the various conditions in the theorems of Sections 5 and 6 and also in proving that the lim sup is positive and finite in the "nice" cases when centering at the mean or the median in Section 7. Since they are to be used in a variety of situations, we have been forced to keep the assumptions and notation rather general. We will use an increasing sequence of truncation points $\{u_k\}$. The actual values will vary with the context. Then the sums of truncated variables are as in (3.2) and (3.9):

$$T_{nk} = \sum_{i=1}^n X_i \wedge u_k, \quad V_{nk} = \sum_{i=1}^n -u_k \vee (X_i \wedge u_k), \tag{4.7}$$
$$W_{nk} = \sum_{i=1}^n (-b_n \wedge -u_k) \vee (X_i \wedge u_k),$$

where $\{b_n\}$ is defined in (2.13). We will also have a subsequence of times $\{n_k\}$ and we let

$$T_n = T_{nk}, \qquad V_n = V_{nk}, \qquad W_n = W_{nk}, \qquad n_{k-1} < n \le n_k. \tag{4.8}$$

The norming sequence $\{\beta_n\}$ will be defined by either

$$\beta_n = u_k \log k, \qquad n_{k-1} < n \le n_k, \tag{4.9}$$

or

$$\beta_n = u_k \log k + n \int_{u_k}^{b_n \vee u_k} f(x)\, dx, \qquad n_{k-1} < n \le n_k, \tag{4.10}$$

depending on whether $\alpha_n = EV_n$ or EW_n.

LEMMA 4.2. *Assume that $EX = 0$ if $EX^2 < \infty$. Suppose that*

$$f(u_k) \sim n_k^{-1} \log k \tag{4.11}$$

and that $n_{k+1} \ge 40 n_k$. Then with T_n, V_n as in (4.7) *and* (4.8) *and β_n as in* (4.9),

$$\frac{1}{240} \le \limsup_{n\to\infty} \frac{T_n - EV_n}{\beta_n} \le 4 \quad \text{a.s.}$$

The lower bound is still valid if the values of n used in computing the lim sup *are restricted to the intervals $(n_k/40, n_k]$. (The requirement that $n_{k+1} \ge 40n_k$ is only needed in the lower bound.)*

PROOF. Take $C > 1$. By (3.11), for $n \le n_k$

$$P\{|V_{nk} - EV_{nk}| \ge (2C \log k)^{1/2} u_k\} \le (2C \log k)^{-1} n f(u_k) \le \tfrac{1}{2} \tag{4.12}$$

for large k. Then by Lemmas 3.5 and 3.1

$$P\{\max_{n\le n_k}(V_{nk}-EV_{nk})\ge \tfrac{1}{2}\, en_k u_k f(u_k)+2u_k\log k+(2C\log k)^{1/2}u_k\}$$
$$\le 2P\{V_{n_k}-EV_{n_k}\ge \tfrac{1}{2}\, en_k u_k f(u_k)+2u_k\log k\}\le 2k^{-2}.$$

Therefore we have for $n_{k-1}<n\le n_k$ and k sufficiently large

$$T_n-EV_n\le V_{nk}-EV_{nk}\le 4u_k\log k.$$

For the lower bound we need to consider a subsequence of $\{n_k\}$. Let $k_1=1$, $k_2=2$, and

$$k_j=2+\textstyle\sum_{i=3}^{j}\{[\log\log i]+1\}, \qquad j\ge 3.$$

Then for large j

$$n_{k_j}n_{k_{j-1}}^{-1}\ge 40^{k_j-k_{j-1}}\ge 40^{\log\log j}\ge 5C\log j. \tag{4.13}$$

Furthermore,

$$\log j\le\log k_j\le\log(j+j\log\log j)\sim\log j. \tag{4.14}$$

Now we introduce the sequences

$$m_j=[n_{k_j}/40], \qquad \nu_j=\textstyle\sum_{i=1}^{j}m_i.$$

Note that for large j, $m_j\ge m_{j-1}\,4\,C\log j$ and so $\nu_j\sim m_j$. This means that $\nu_j\le n_{k_j}$ and we also have

$$\nu_j\ge 2+m_j\ge 1+n_{k_j}/40>n_{k_j-1}$$

so that ν_j is in the same "block" as n_{k_j}. Now by (4.11), (4.13), and (4.14)

$$P\{T_{\nu_{j-1}k_j}\ne V_{\nu_{j-1}k_j}\}\le \nu_{j-1}G_-(u_{k_j})\le n_{k_{j-1}}Cn_{k_j}^{-1}\log k_j\le \tfrac{1}{4}$$

and then by (4.12)

$$P\{T_{\nu_{j-1}k_j}\ge EV_{\nu_{j-1}k_j}-(2\,C\log k_j)^{1/2}u_{k_j}\}\ge \tfrac{1}{4}.$$

By Lemma 3.2, (4.11), and (4.14), for large j

$$P\{T_{\nu_jk_j}-T_{\nu_{j-1}k_j}\ge EV_{m_jk_j}+C_1m_ju_{k_j}f(u_{k_j})\}\ge\exp\{-35m_jf(u_{k_j})\}$$
$$\ge j^{-35C/40}$$

for any $C>1$. Utilizing the last two bounds in Lemma 3.4 we have infinitely often with probability one

$$T_{\nu_j}\ge EV_{\nu_j}+C_1\,m_j\,u_{k_j}f(u_{k_j})-(2\,C\log k_j)^{1/2}u_{k_j}$$
$$\ge EV_{\nu_j}+u_{k_j}\log k_j/240.$$

The final statement of the lemma is due to the fact that $\nu_j\in(n_{k_j}/40,\,n_{k_j}]$.

LEMMA 4.3. *Assume that $EX=0$ if $EX^2<\infty$. Suppose that*

$$f(u_k)\sim n_k^{-1}\log k \tag{4.15}$$

and that $n_{k+1}\ge 40\,n_k$. Furthermore, suppose that at least one of the following two conditions holds:

$$\lim_{k\to\infty}n_k\,G_+(u_k)=0 \tag{4.16}$$

or

$$\lim_{k\to\infty} \sup_{n\in(n_k/40,n_k]} n\beta_n^{-1} \int_{u_k}^{b_n} G_+(x)\,dx = 0 \tag{4.17}$$

where β_n is given by (4.10). *Then with T_n, W_n as in* (4.7), (4.8) *and β_n as in* (4.10)

$$\frac{1}{480} \le \limsup_{n\to\infty} \frac{T_n - EW_n}{\beta_n} \le 4 \quad \text{a.s.}$$

The lower bound is still valid if the values of n used in computing the lim sup are restricted to the intervals $(n_k/40, n_k]$. (The requirement that $n_{k+1} \ge 40\, n_k$ and the assumption (4.16) *or* (4.17) *are only needed in the lower bound. Also the value of γ involved in the definition of b_n, β_n, and W_n may be allowed to depend on n as long as* (4.17) *is satisfied. In fact, b_n may even be replaced by an arbitrary sequence provided that* (4.17) *is true.)*

PROOF. First we note that for $n \le n_k$

$$n\beta_n^{-1} \int_{u_k}^{b_n} G_+(x)\,dx \le n_k\, \beta_n^{-1}\, G_+(u_k)\, b_n.$$

Also if $n_k/40 < n \le n_k$ then $u_k < b_n$ and we have by (4.10) for large k

$$\beta_n \ge u_k \log k + n(b_n - u_k) f(b_n) \ge n b_n f(b_n) = \gamma b_n. \tag{4.18}$$

Thus (4.16) implies (4.17). Next we observe that if $b_n \le u_k$ for any $n \in (n_{k-1}, n_k]$ then $EV_n = EW_n$ and the β_n in (4.10) is the same as in (4.9) so that the upper bound for these n follows immediately from Lemma 4.2. Therefore in the remainder of the proof we will assume that $b_n \ge u_k$. Then for $n_{k-1} < n \le n_k$

$$\begin{aligned} EV_n - EW_n &= -n \int_{-b_n}^{-u_k} x\,dF - n u_k G_-(u_k) + n b_n G_-(b_n) \\ &= n \int_{u_k}^{b_n} G_-(x)\,dx \le n \int_{u_k}^{b_n} f(x)\,dx \end{aligned} \tag{4.19}$$

by Lemma 2.2. Furthermore, for $n_k/40 < n \le n_k$ and $\eta > 0$ by (2.7)

$$\begin{aligned} n \int_{u_k}^{b_n} f(x)\,dx &= n \int_{u_k}^{b_n} \{K(x) - G(x)\}dx + 2n \int_{u_k}^{b_n} G(x)\,dx \\ &= -n\, b_n f(b_n) + n u_k f(u_k) + 2n \int_{u_k}^{b_n} G_+(x)\,dx + 2n \int_{u_k}^{b_n} G_-(x)\,dx \\ &\le (1+\eta) u_k \log k + \eta\beta_n + 2n \int_{u_k}^{b_n} G_-(x)\,dx \end{aligned} \tag{4.20}$$

where at the last step we have used (4.15) and (4.17). Using this in conjunction with (4.19) shows that for $n_k/40 < n \le n_k$

$$\begin{aligned} EV_n - EW_n &\ge \frac{1}{240} n \int_{u_k}^{b_n} G_-(x)\,dx \\ &\ge \frac{1}{480} n \int_{u_k}^{b_n} f(x)\,dx - \frac{1+\eta}{480} u_k \log k - \frac{\eta}{480} \beta_n. \end{aligned} \tag{4.21}$$

Adding the bounds in (4.19) and (4.21) to the result of Lemma 4.2 completes the proof.

5. Two-sided results. In this section we will prove Kesten's theorem and also obtain necessary and sufficient conditions to be able to find $\{\beta_n\}$ so that

$$0 < \limsup_{n\to\infty} \frac{|S_n - \alpha_n|}{\beta_n} < \infty \quad \text{a.s.}$$

when $\alpha_n = 0$ and $\alpha_n = ES_n$. Since we view these as existence results only, we have been content to use a $\{\beta_n\}$ sequence which simplifies the proof. It is possible to find norming sequences $\{\beta_n\}$ which satisfy additional conditions. Kesten [11], pages 716–718, gives some information on this for the case of centering at the median of S_n. We use Lemma 4.2 in the proofs but we do not need to assume $EX = 0$ when $EX^2 < \infty$ since the results are clear if $EX \neq 0$, $EX^2 < \infty$.

THEOREM 5.1. (Kesten). *Suppose that* $\{\alpha_n\}$ *satisfies*

$$P\{S_n \geq \alpha_n\} \geq \epsilon, \qquad P\{S_n \leq \alpha_n\} \geq \epsilon$$

for some $\epsilon > 0$. *Then there is a nondecreasing sequence* $\{\beta_n\}$ *such that*

$$0 < \limsup_{n\to\infty} \frac{|S_n - \alpha_n|}{\beta_n} < \infty \quad \text{a.s.}$$

if and only if X *is in the domain of partial attraction of the normal distribution*, i.e.,

$$\liminf_{x\to\infty} \frac{G(x)}{f(x)} = 0. \tag{5.1}$$

If (5.1) *fails and* $\{\beta_n\}$ *is nondecreasing then*

$$\limsup_{n\to\infty} \frac{|S_n - \alpha_n|}{\beta_n} = 0 \quad \textit{or} \quad \infty \quad \text{a.s.}$$

according as $\Sigma\, P\{|X| > \beta_n\}$ *converges or diverges.*

PROOF. First suppose that (5.1) fails. The case when $\Sigma\, P\{|X| > \beta_n\}$ diverges follows immediately from Lemma 4.1. For the convergent case we only need to show that $\alpha_n - ER_n = o(\beta_n)$. But (4.3) implies that for any $\eta > 0$

$$ER_n - \eta\beta_n \leq \alpha_n \leq ER_n + \eta\beta_n$$

for large n. Now suppose that (5.1) is satisfied. We can find $\{u_k\}$ such that $f(u_{k+1}) \leq f(u_k)/40$ and

$$\sum \log k \frac{G(u_k)}{f(u_k)} < \infty.$$

We let $n_k = [\log k/f(u_k)]$. Then $f(u_k) \sim n_k^{-1} \log k$ and $n_{k+1} \geq 40\, n_k$. Thus Lemma 4.2 applies so that

$$0 < \limsup_{n\to\infty} \frac{T_n - EV_n}{\beta_n} < \infty \quad \text{a.s.} \tag{5.2}$$

Since

$$\textstyle\sum_k P\{S_n \neq T_n \quad \text{for some} \quad n_{k-1} < n \leq n_k\} \leq \sum_k n_k G_+(u_k) < \infty \tag{5.3}$$

we will have $S_n = T_n$ for sufficiently large n. By (3.11)

$$P\{|V_n - EV_n| \geq (2\epsilon^{-1} \log k)^{1/2}\, u_k\} \leq \tfrac{1}{2}\epsilon, \qquad n_{k-1} < n \leq n_k \tag{5.4}$$

and since $P\{S_n \neq V_n\} \to 0$ this means for large k

$$EV_n - (2\epsilon^{-1} \log k)^{1/2} u_k \leq \alpha_n \leq EV_n + (2\epsilon^{-1} \log k)^{1/2} u_k, \qquad n_{k-1} < n \leq n_k.$$

Then $\alpha_n - EV_n = o(\beta_n)$ and so T_n can be replaced by S_n and EV_n by α_n in (5.2). We can use the same sequences $\{u_k\}$, $\{n_k\}$ and thus $\{\beta_n\}$ if we consider $-X$ instead of X so that the same proof shows that the lim inf is negative and finite.

The next theorem solves a problem posed by Kesten in [11]. Actually the statement is slightly different as he requires that

$$-\infty < \liminf_{n\to\infty} \frac{S_n}{\beta_n} < \limsup_{n\to\infty} \frac{S_n}{\beta_n} < \infty \quad \text{a.s.}$$

The difference is that our formulation allows the limit to exist so long as it is not zero, which seems reasonable. However, the criterion (5.5) is the solution to both problems because it is easy to check that the given construction makes the lim inf and lim sup distinct. Thus, for example, if $EX = 1$ instead of using $\{n\}$ as the norming sequence as in the strong law one could use $\beta_n = 2^k$ for $2^{k-1} < n \leq 2^k$ and obtain

$$\liminf_{n\to\infty} \frac{S_n}{\beta_n} = \frac{1}{2}, \qquad \limsup_{n\to\infty} \frac{S_n}{\beta_n} = 1 \quad \text{a.s.}$$

THEOREM 5.2. *There is a nondecreasing sequence* $\{\beta_n\}$ *such that*

$$0 < \limsup_{n\to\infty} \frac{|S_n|}{\beta_n} < \infty \quad \text{a.s.}$$

if and only if

$$\liminf_{x\to\infty} \frac{G(x)}{f(x) + |M(x)|} = 0. \tag{5.5}$$

If (5.5) *fails and* $\{\beta_n\}$ *is nondecreasing then*

$$\limsup_{n\to\infty} \frac{|S_n|}{\beta_n} = 0 \quad \text{or} \quad \infty \quad \text{a.s.}$$

according as $\Sigma P\{|X| > \beta_n\}$ *converges or diverges.*

PROOF. First suppose that (5.5) fails so that there is a positive constant C such that

$$f(x) + |M(x)| \leq C\, G(x)$$

for all x. The divergent case follows immediately from Lemma 4.1 and for the convergent case we only need to show that $ER_n = o(\beta_n)$. With Z_n as in (4.1),

$$|\beta_n^{-1} EZ_n| = |M(\beta_n) + G_+(\beta_n) - G_-(\beta_n)| \leq CG(\beta_n)$$

and so $\Sigma \beta_n^{-1} EZ_n$ converges. Since β_n must tend to infinity as in the proof of Lemma 4.1 this implies that $ER_n = o(\beta_n)$ by Kronecker. Now we suppose that (5.5) is satisfied. By Lemma 2.6 we must have

$$\liminf_{x\to\infty} \frac{G(x) + |M(x)|}{f(x)} = 0 \tag{5.6}$$

or

$$\liminf_{x\to\infty} \frac{f(x)}{|M(x)|} = 0. \tag{5.7}$$

If we have (5.6) we proceed as in the proof of Theorem 5.1 except that when we choose $\{u_k\}$ we also insist that $M(u_k)/f(u_k) \to 0$. But then for $n_{k-1} < n \leq n_k$

$$|EV_n| = |nu_k\{M(u_k) + G_+(u_k) - G_-(u_k)\}| \leq \beta_n \frac{|M(u_k)| + G(u_k)}{f(u_k)} = o(\beta_n).$$

The rest of the proof is as before. Now suppose that (5.7) is true. Choose $\{u_k\}$ so that $|M(u_{k+1})| \le |M(u_k)|/2$ and

$$\sum_k \frac{f(u_k)}{|M(u_k)|} < \infty \tag{5.8}$$

and define $n_k = [1/|M(u_k)|]$,

$$\beta_n = u_k, \qquad n_{k-1} < n \le n_k.$$

Then with V_n as in (4.7), (4.8) we have by Kolmogorov's inequality

$$P\{\max_{n_{k-1}<n\le n_k} |V_n - EV_n| \ge \epsilon\beta_n\} \le \epsilon^{-2} n_k f(u_k)$$

and

$$P\{S_n \ne V_n \quad \text{for some} \quad n_{k-1} < n \le n_k\} \le n_k G(u_k).$$

Since both of these are summable by (5.8),

$$\lim_{n\to\infty} \frac{S_n - EV_n}{\beta_n} = 0 \quad \text{a.s.} \tag{5.9}$$

But for $n_{k-1} < n \le n_k$,

$$|EV_n| = |nu_k\{M(u_k) + G_+(u_k) - G_-(u_k)\}| \sim nu_k\,|M(u_k)| \sim nu_k n_k^{-1}$$

which, in conjunction with (5.9), shows that

$$\limsup_{n\to\infty} \frac{|S_n|}{\beta_n} = 1 \quad \text{a.s.}$$

THEOREM 5.3. *Suppose that $E|X| < \infty$. Then there is a nondecreasing sequence $\{\beta_n\}$ such that*

$$0 < \limsup_{n\to\infty} \frac{|S_n - ES_n|}{\beta_n} < \infty \quad \text{a.s.}$$

if and only if

$$\liminf_{x\to\infty} \frac{G(x)}{f(x) + |M_\infty(x)|} = 0 \tag{5.10}$$

where

$$M_\infty(x) = x^{-1} \int_{|y|>x} y\, dF(y).$$

If (5.10) *fails and* $\{\beta_n\}$ *is nondecreasing then*

$$\limsup_{n\to\infty} \frac{|S_n - ES_n|}{\beta_n} = 0 \quad or \quad \infty \quad \text{a.s.}$$

according as $\Sigma\, P\{|X| > \beta_n\}$ converges or diverges.

PROOF. Note first that if $EX = 0$ then $|M(x)| = |M_\infty(x)|$ and Theorem 5.2 implies Theorem 5.3 immediately. To complete the proof it is sufficient to show that the condition (5.10) and the convergence of $\Sigma\, P\{|X| > \beta_n\}$ when (5.10) fails are equivalent to the analogous conditions for the random variable $X - EX$. Thus we suppose that

$$f(x) + |M_\infty(x)| \le CG(x) \tag{5.11}$$

for all $x \ge 1$. Then as in (4.5) we have $G(x) = O(G(2x))$ so that for any fixed μ, $P\{|X| > x\}$ and $P\{|X - \mu| > x\}$ are comparable for large x. Similarly, for large x,

$$E(X-\mu)^2 1\{|X-\mu| \le x\} \le EX^2\, 1\{|X| \le 2x\} + O(1)$$
$$\le x^2K(x) + 4x^2G(x) + O(1) = O(x^2G(x))$$

and

$$E(X-\mu)1\{|X-\mu| > x\} = EX1\{|X| > x\} + O(xG(\tfrac{1}{2}x)) = O(xG(x)).$$

Thus (5.11) is also valid for the random variable $X - \mu$ and the series $\Sigma P\{|X| > \beta_n\}$ and $\Sigma P\{|X-\mu| > \beta_n\}$ will converge or diverge together. (Once more we are using the fact that under (5.11) the convergence of $\Sigma G(\beta_n)$ implies $\beta_n \to \infty$.)

6. One-sided results: necessary and sufficient conditions. In this section we will give necessary and sufficient conditions for finding norming sequences when centering at the median of S_n. We have not tried to make the norming sequence satisfy any conditions other than monotonicity. A very general situation where there are "nice" norming sequences when one centers at the median of S_n is discussed in the next section. As in the last section, the result is clear when $EX \neq 0$, $EX^2 < \infty$ so that we may use Lemma 4.3.

THEOREM 6.1. *Suppose that* $\{\alpha_n\}$ *satisfies*

$$P\{S_n \ge \alpha_n\} \ge \epsilon, \qquad P\{S_n \le \alpha_n\} \ge \epsilon \tag{6.1}$$

for some $\epsilon > 0$. *Then there is a nondecreasing sequence* $\{\beta_n\}$ *such that*

$$0 < \limsup_{n\to\infty} \frac{S_n - \alpha_n}{\beta_n} < \infty \quad \text{a.s.}$$

if and only if

$$\liminf_{x\to\infty} \frac{G_+(x)}{f(x)} = 0. \tag{6.2}$$

If (6.2) *fails and* $\{\beta_n\}$ *is nondecreasing then*

$$\limsup_{n\to\infty} \frac{S_n - \alpha_n}{\beta_n} = 0 \text{ or } \infty \quad \text{a.s.} \tag{6.3}$$

according as $\Sigma P\{X > \beta_n\}$ *converges or diverges.*

REMARKS. 1. If (6.2) holds, then one can find a norming sequence $\{\beta_n\}$ provided the first condition in (6.1) is satisfied. But if $\{\alpha_n\}$ is sufficiently negative that it dominates S_n, then one can find a sequence $\{\beta_n\}$ even without (6.2).

2. In case (6.2) fails and $\Sigma P\{X > \beta_n\}$ converges, the lim sup in (6.3) can be replaced by lim.

PROOF. First suppose that (6.2) is not true. The case when $\sum P\{X > \beta_n\}$ diverges follows immediately from Lemma 4.1. For the convergent case we only need to show that $\alpha_n - ER_n = o(\beta_n)$ and this is an immediate consequence of (4.3) as in the proof of Theorem 5.1. Now suppose that (6.2) is true and choose $\{u_k\}$ so that $f(u_{k+1}) \le f(u_k)/40$ and

$$\sum \log k \frac{G_+(u_k)}{f(u_k)} < \infty. \tag{6.4}$$

We let $n_k = [\log k/f(u_k)]$ so that $f(u_k) \sim n_k^{-1} \log k$ and $n_{k+1} \ge 40 n_k$. Fix a value of $\gamma \ge 2 \cdot 10^6 \epsilon^{-1}$ and use b_n corresponding to this γ in Lemma 4.3. Since (4.16) is a consequence of (6.4) we have all the conditions satisfied. In addition, it follows from (6.4) that $P\{S_n \neq T_n \text{ i.o.}\} = 0$ so that with β_n as in (4.10)

$$.002 \le \limsup_{n\to\infty} \frac{S_n - EW_n}{\beta_n} \le 4 \quad \text{a.s.} \tag{6.5}$$

If $n_k/40 < n \le n_k$ then $u_k < b_n$ so by (3.11)

$$P\{|W_n - EW_n| \ge (2\gamma\epsilon^{-1})^{1/2}b_n\} \le (2\gamma\epsilon^{-1})^{-1}nf(b_n) = \tfrac{1}{2}\epsilon.$$

Also $P\{S_n \ne T_n\} \le nG_+(u_k) = o(1)$ so that for large n

$$\begin{aligned} P\{S_n \ge EW_n + (2\gamma\epsilon^{-1})^{1/2}b_n\} &= P\{T_n \ge EW_n + (2\gamma\epsilon^{-1})^{1/2}b_n\} + o(1) \\ &\le P\{W_n \ge EW_n + (2\gamma\epsilon^{-1})^{1/2}b_n\} + o(1) < \epsilon. \end{aligned}$$

Thus by (6.1) and (4.18) for $n_k/40 < n \le n_k$

(6.6) $$\alpha_n \le EW_n + (2\gamma\epsilon^{-1})^{1/2}b_n \le EW_n + (2\gamma^{-1}\epsilon^{-1})^{1/2}\beta_n \le EW_n + 10^{-3}\beta_n$$

by the choice of γ. Now the idea is that (6.5) shows that $\{\beta_n\}$ will serve as a norming sequence if we center at $EW_n + 10^{-3}\beta_n$ and (6.6) shows that α_n is smaller in the range where it matters so that we can also center at α_n. To make this precise and show that we can make the norming sequence monotone we let

$$\beta_n' = \beta_n + \max_{m\le n}\{EW_m - \alpha_m\}.$$

It is not hard to check that since $n_k f(u_k) \le \log k$,

$$u_k \log k + n_k \int_{u_k}^{u_{k+1}} f(x)\,dx \le u_k \log k + n_k f(u_k)(u_{k+1} - u_k) \le u_{k+1}\log k$$

and then the sequence $\{\beta_n\}$ is increasing and so $\{\beta_n'\}$ is also. It is clear from (6.5) that

$$\limsup_{n\to\infty} \frac{S_n - \alpha_n}{\beta_n'} \le 4 \quad \text{a.s.}$$

For the lower bound there are two cases. First suppose that for some positive constant C

(6.7) $$\beta_n' \le C\beta_n \qquad \text{for all} \qquad n \in (n_k/40, n_k]$$

and all large k. Since the lower bound in (6.5) is still valid if we restrict n to the intervals $(n_k/40, n_k]$ we have by (6.6)

$$\limsup_{n\to\infty} \frac{S_n - \alpha_n}{\beta_n'} = \limsup_{n\to\infty} \frac{(S_n - EW_n) + (EW_n - \alpha_n)}{\beta_n} \cdot \frac{\beta_n}{\beta_n'} \ge 10^{-3}C^{-1} \quad \text{a.s.}$$

If (6.7) fails then we can find a subsequence $\{m_k\}$ which is in $\cup(n_j/40, n_j]$ and such that $\beta_{m_k}/\beta'_{m_k} \to 0$. Let $\nu_k \le m_k$ be such that

$$EW_{\nu_k} - \alpha_{\nu_k} = \max_{m\le m_k}\{EW_m - \alpha_m\}.$$

Since for $m_k \in (n_{j_k}/40, n_{j_k}]$ we have $b_{m_k} > u_{j_k}$ and since both $\{b_n\}$ and $\{u_j\}$ are monotone it follows that both the upper and lower truncation points for W_{ν_k} are contained in $[-b_{m_k}, b_{m_k}]$. This means that the second moment of the summands in W_{ν_k} is bounded by $b_{m_k}^2 f(b_{m_k})$ and as in (3.11)

$$P\{|W_{\nu_k} - EW_{\nu_k}| \ge Cb_{m_k}\} \le C^{-2}\nu_k f(b_{m_k}) \le C^{-2}\gamma.$$

Also, even if the lower truncation point is less than $-b_{\nu_k}$ we still have

$$P\{T_{\nu_k} \ne W_{\nu_k}\} \le 1 - \{1 - G_-(b_{\nu_k})\}^{\nu_k} \le 1 - (1 - \gamma\nu_k^{-1})^{\nu_k} \sim 1 - e^{-\gamma}.$$

Thus for $\eta > 0$ and k large

$$P\{T_{\nu_k} \ge EW_{\nu_k} - Cb_{m_k}\} \ge e^{-\gamma} - \eta - C^{-2}\gamma$$

and we can choose η small enough and C large enough to make this positive. Since $P\{S_n \ne T_n\} \to 0$ this will still be the case if T_{ν_k} is replaced by S_{ν_k}. Then by (4.18) we have infinitely often with probability one

$$S_{\nu_k} - \alpha_{\nu_k} \geq EW_{\nu_k} - Cb_{m_k} - \alpha_{\nu_k} = \beta'_{m_k} - \beta_{m_k} - Cb_{m_k}$$
$$\geq \beta'_{m_k} - (C\gamma^{-1} + 1)\beta_{m_k} \sim \beta'_{m_k} \geq \beta'_{\nu_k}$$

and so the lim sup will be at least one in this case.

7. One-sided results: the nice case. In this section we will show that a nice sequence of norming constants works under quite general conditions. This will be done for the cases of centering at the median of S_n and at ES_n when EX exists. The latter case in conjunction with the strong law also takes care of centering at zero when EX exists. Centering at zero when $E|X| = \infty$ is discussed in the next section.

Our work on centering at ES_n is a different approach to the results of Michael Klass [13, 14]. The norming sequence looks different but is comparable to his. We were led to the other definition because we considered the problem of centering at the median of S_n first. The methods used by Klass give very much tighter bounds on the actual value of the lim sup. We include this case here for purposes of comparison and since it requires very little extra work.

The first four lemmas are needed to handle the positive tail of the distribution. They are essentially due to Klass [13, 14]. Note that Lemmas 7.1–7.3 apply to fairly general norming sequences $\{\beta_n\}$ and not only the specific sequences considered thus far.

LEMMA 7.1. *For $\rho > 1$ let $n_k = [\rho^k]$ and*

$$Z_k = X1\{0 < X \leq \beta_{n_k}\}.$$

Suppose that $n^{-1/2}\beta_n$ increases and $\Sigma\, P\{X > \beta_n\}$ converges. Then

$$\Sigma\, n_k \beta_{n_k}^{-3} EZ_k^3 < \infty.$$

PROOF. First we note that $n_k\beta_{n_k}^{-3}$ is dominated by a convergent geometric series:

$$n_{k+1}\beta_{n_{k+1}}^{-3} n_k^{-1} \beta_{n_k}^3 \leq n_k^{1/2} n_{k+1}^{-1/2} \sim \rho^{-1/2}.$$

Then

$$\textstyle\sum_k n_k \beta_{n_k}^{-3} EZ_k^3 = \sum_k n_k \beta_{n_k}^{-3} \sum_{j\leq k} \displaystyle\int_{\beta_{n_{j-1}}}^{\beta_{n_j}} x^3\, dF = \textstyle\sum_j \displaystyle\int_{\beta_{n_{j-1}}}^{\beta_{n_j}} x^3\, dF \textstyle\sum_{k\geq j} n_k \beta_{n_k}^{-3}$$

$$\leq C \textstyle\sum_j n_j \beta_{n_j}^{-3} \displaystyle\int_{\beta_{n_{j-1}}}^{\beta_{n_j}} x^3\, dF \leq C \textstyle\sum_j n_j P\{X > \beta_{n_{j-1}}\} < \infty.$$

LEMMA 7.2. *For $\rho > 1$ let $n_k = [\rho^k]$ and*

$$A = \left\{ n : n \int_{a_n}^{\beta_n} x\, dF \geq 2\beta_n / \log\log n \right\}$$

where a_n is defined by (2.13). *Suppose that $n^{-1/2}\beta_n$ increases and $\sum P\{X > \beta_n\}$ converges. Then*

$$\operatorname{Card}\{k \colon 2^j < k \leq 2^{j+1} \text{ and } n_k \in A\} = o(j^5).$$

PROOF. First note that letting $r_n = a_n \vee \beta_n/(\log\log n)^2$

$$n \int_{a_n}^{r_n} x\, dF \leq \frac{n\beta_n}{(\log\log n)^2} \frac{\log\log n}{n} = \frac{\beta_n}{\log\log n}$$

so that for $n \in A$

$$n \int_{r_n}^{\beta_n} x\, dF \geq \beta_n/\log\log n.$$

With Z_k as in Lemma 7.1, we have for $n_k \in A$

$$EZ_k^3 \geq \int_{r_{n_k}}^{\beta_{n_k}} x^3\, dF \geq r_{n_k}^2 \beta_{n_k}/n_k \log\log n_k \geq \beta_{n_k}^3/n_k(\log\log n_k)^5.$$

For $2^j < k \leq 2^{j+1}$, $\log\log n_k \sim j \log 2$ so that if we sum $n_k\beta_{n_k}^{-3}EZ_k^3$ for k in this range we will obtain at least j^{-5} times the cardinality of the set of interest. This goes to zero by Lemma 7.1.

LEMMA 7.3. *For $\rho > 1$ let $n_k = [\rho^k]$ and*

$$Z_{jk} = (X_j - a_{n_k})1\{a_{n_k} < X_j \leq \beta_{n_{k-1}}\}$$

$$U_n = \textstyle\sum_{j=1}^n Z_{jk}, \qquad n_{k-1} < n \leq n_k,$$

where a_n is defined by (2.13). *Suppose that*

$$n^{-1/2}\beta_n \nearrow, \qquad EZ_{jk} = O(n_k^{-1}\beta_{n_{k-1}}) \tag{7.1}$$

and $\sum P\{X > \beta_n\}$ converges. Then

$$\lim_{n\to\infty} \frac{U_n - EU_n}{\beta_n} = 0 \quad \text{a.s.}$$

REMARK. Condition (7.1) is satisfied if β_n is defined by either (2.14) or (2.15). The first part follows from Lemma 2.7 while for the second part we have by Lemma 2.2, with either definition of β_n,

$$EZ_{jk} \leq \int_{a_{n_k}}^{\beta_{n_{k-1}}} x\, dF = \int_{a_{n_k}}^{\beta_{n_{k-1}}} G_+(x)\, dx - \beta_{n_{k-1}}G_+(\beta_{n_{k-1}}) + a_{n_k}G_+(a_{n_k})$$

$$\leq a_{n_k}f(a_{n_k}) + \int_{a_{n_k}}^{b_{n_{k-1}}} f(x)\, dx + \int_{b_{n_{k-1}}}^{b_{n_{k-1}} \vee \beta_{n_{k-1}}} f(x)\, dx$$

$$\leq n_{k-1}^{-1}\beta_{n_{k-1}} + \gamma n_{k-1}^{-1}\beta_{n_{k-1}}.$$

PROOF. If $\nu_k = \min\{j: \beta_{n_j} > a_{n_k}\}$ then

$$EZ_{jk}^2 \leq \textstyle\sum_{i=\nu_k}^{k-1} \displaystyle\int_{\beta_{n_{i-1}}}^{\beta_{n_i}} x^2\, dF \leq \textstyle\sum_{i=\nu_k}^{k-1} \beta_{n_i}^2 P\{X > \beta_{n_{i-1}}\} \leq n_{k-1}^{-1}\beta_{n_{k-1}}^2 \sum_{i=\nu_k}^{k-1} n_i P\{X > \beta_{n_{i-1}}\}$$

so that $EZ_{jk}^2 = o(n_k^{-1}\beta_{n_{k-1}}^2)$. Thus for $n \leq n_k$

$$P\{|\textstyle\sum_{j=1}^n (Z_{jk} - EZ_{jk})| \geq \epsilon\beta_{n_{k-1}}\} \leq \epsilon^{-2}\beta_{n_{k-1}}^{-2} nEZ_{1k}^2 = o(1).$$

By Lemma 3.5

$$P\{\max_{n_{k-1}<n\leq n_k} |U_n - EU_n| \geq 2\epsilon\beta_{n_{k-1}}\} \leq 2P\{|U_{n_k} - EU_{n_k}| \geq \epsilon\beta_{n_{k-1}}\}$$
$$\leq 2\epsilon^{-4}\beta_{n_{k-1}}^{-4} E(U_{n_k} - EU_{n_k})^4.$$

To estimate this, with Z_k as in Lemma 7.1,

$$\begin{aligned} E(U_{n_k} - EU_{n_k})^4 &= n_k E(Z_{1k} - EZ_{1k})^4 + n_k(n_k - 1)\{E(Z_{1k} - EZ_{1k})^2\}^2 \\ &\le Cn_k EZ_{1k}^4 + n_k^2\{EZ_{1k}^2\}^2 \\ &\le Cn_k EZ_{k-1}^4 + n_k^2 EZ_{1k} EZ_{1k}^3 \\ &\le Cn_k \beta_{n_{k-1}} EZ_{k-1}^3 + Cn_k \beta_{n_{k-1}} EZ_{k-1}^3. \end{aligned}$$

Substituting this estimate in the previous bound we obtain a convergent series by Lemma 7.1 so that by Borel-Cantelli we have for $n_{k-1} < n \le n_k$ and k sufficiently large

$$|U_n - EU_n| < 2\epsilon\beta_{n_{k-1}} < 2\epsilon\beta_n.$$

LEMMA 7.4. *Suppose that* $E|X| < \infty$ *and* β_n *is given by* (2.15). *If* $\sum P\{X > \beta_n\}$ *converges then*

$$n\int_{\beta_n}^{\infty} x\,dF = o(\beta_n)$$

as $n \to \infty$.

PROOF. We let $n_k = 2^k$ and suppose that $n_k < n \le n_{k+1}$. Then by (2.20),

$$\begin{aligned} \int_{\beta_n}^{\infty} x\,dF &\le \textstyle\sum_{j=k}^{\infty} \int_{\beta_{n_j}}^{\beta_{n_{j+1}}} x\,dF \le \sum_{j=k}^{\infty} \beta_{n_{j+1}} P\{X > \beta_{n_j}\} \\ &\le n^{-1}\beta_n \textstyle\sum_{j=k}^{\infty} n_{j+1} P\{X > \beta_{n_j}\} = o(n^{-1}\beta_n). \end{aligned}$$

Now we are ready to prove the two main results of this section. We will prove the theorem for centering at the mean first since it now requires very little work. In case $EX^2 < \infty$, it is easy to check that $\beta_n \sim 2(EX^2 n \log\log n)^{1/2}$. Thus considering $X - EX$ instead of X only changes the constant and so we need not assume that $EX = 0$ when $EX^2 < \infty$.

THEOREM 7.5. (Klass). *Suppose that* $E|X| < \infty$ *and* β_n *is given by* (2.15). *If* $\Sigma\, P\{X > \beta_n\}$ *converges then*

$$0 < \limsup_{n\to\infty} \frac{S_n - ES_n}{\beta_n} < \infty \quad \text{a.s.}$$

On the other hand, if $\sum P\{X > \beta_n\}$ *diverges then*

$$\limsup_{n\to\infty} \frac{S_n - ES_n}{\beta_n} = \infty \quad \text{a.s.}$$

PROOF. For the first part we fix $\rho > 1$ and let $n_k = [\rho^k]$ and $u_k = a_{n_k}$. Then by Lemma 4.2

$$\limsup_{n\to\infty} \frac{T_n - EV_n}{\beta'_n} \le 4 \quad \text{a.s.}$$

where $\beta'_n = a_{n_k} \log k \sim a_{n_k} \log\log n_k$ for $n_{k-1} < n \le n_k$. By Lemma 7.3

$$\lim_{n\to\infty} \frac{U_n - EU_n}{\beta_n} = 0 \quad \text{a.s.}$$

Also $P\{S_n \ne T_n + U_n$ for some $n_{k-1} < n \le n_k\} \le n_k P\{X > \beta_{n_{k-1}}\}$ which is summable. Thus for large n

$$S_n - ES_n \le EV_n + EU_n - ES_n + (4+\eta)\beta'_n + \eta\beta_n \tag{7.2}$$

and for $n_{k-1} < n \le n_k$, by Lemma 2.2

$$
\begin{aligned}
EV_n + EU_n - ES_n &= -n \int_{-\infty}^{-a_{n_k}} x\, dF - n \int_{\beta_{n_{k-1}}}^{\infty} x\, dF \\
&\quad + n a_{n_k} G_+(\beta_{n_{k-1}}) - n a_{n_k} G_-(a_{n_k}) \\
&\le n \int_{a_{n_k}}^{\infty} G_-(x)\, dx + o(a_{n_k}) \\
&\le n \int_{a_{n_k}}^{\infty} f(x)\, dx + \eta \beta_n'.
\end{aligned} \tag{7.3}
$$

Now (7.2) and (7.3) give the upper bound since by (2.20)

$$\beta_n' + n \int_{a_{n_k}}^{\infty} f(x)\, dx \sim a_{n_k} \log\log n_k + n \int_{a_{n_k}}^{\infty} f(x)\, dx \le \beta_{n_k} \le n_k n^{-1} \beta_n \le (\rho + \eta)\beta_n.$$

In fact 4 will still serve as an upper bound for the lim sup. For the lower bound we consider the sequence 40^i and form a subsequence $\{n_k\}$ of this sequence by including 40^i in the subsequence if and only if $40^{i-1} \notin A$ where A is defined in Lemma 7.2. We let $u_k = a_{n_k}$. Now $n_k \ge 40^k$ so that

$$\log\log n_k \ge \log k + \log\log 40.$$

On the other hand, by Lemma 7.2 if we choose j so that

$$2^{j-1} - (j-1)^5 < k \le 2^j - j^5$$

then $n_k \le 40^{2^{j+1}}$. This leads to

$$\log\log n_k \le (j+1)\log 2 + \log\log 40 \le \log k + 3\log 2 + \log\log 40$$

so that $\log\log n_k \sim \log k$ and (4.11) is satisfied. Thus by Lemma 4.2

$$\limsup_{n\to\infty} \frac{T_n - EV_n}{\beta_n'} \ge \frac{1}{240} \quad \text{a.s.} \tag{7.4}$$

Letting $m_k = n_k/40$ we have for $m_k < n \le n_k$ by (3.10)

$$
\begin{aligned}
EV_n - ES_n &= n \int_{a_{n_k}}^{\infty} G_-(x)\, dx - n \int_{a_{n_k}}^{\infty} G_+(x)\, dx \\
&= n \int_{a_{n_k}}^{\infty} G(x)\, dx - 2n \int_{a_{n_k}}^{\infty} G_+(x)\, dx.
\end{aligned} \tag{7.5}
$$

Using the fact that $m_k \notin A$ and Lemma 7.4,

$$n \int_{a_{n_k}}^{\infty} G_+(x)\, dx \le n \int_{a_{m_k}}^{\infty} x\, dF \le 40\, m_k \int_{a_{m_k}}^{\beta_{m_k}} x\, dF + 40 m_k \int_{\beta_{m_k}}^{\infty} x\, dF = o(\beta_{m_k}). \tag{7.6}$$

Next observe that by (2.7) and Lemma 2.3

$$\int_{a_{n_k}}^{\infty} K(x)\, dx = \int_{a_{n_k}}^{\infty} G(x)\, dx + a_{n_k} f(a_{n_k}). \tag{7.7}$$

Thus by (7.5)–(7.7) we have for $m_k < n \le n_k$

$$EV_n - ES_n = \frac{1}{2} n \left\{ \int_{a_{n_k}}^{\infty} f(x)\, dx - a_{n_k} f(a_{n_k}) \right\} + o(\beta_n)$$

$$\geq \frac{1}{12} n \left\{ \int_{a_{n_k}}^{\infty} f(x)\, dx - a_{n_k} f(a_{n_k}) \right\} + o(\beta_n)$$

$$\geq \frac{1}{480} n_k \int_{a_{n_k}}^{\infty} f(x)\, dx - \frac{1}{480} a_{n_k} \log \log n_k + o(\beta_n).$$

Since $S_n \geq T_n$ and since (7.4) is valid even when n is restricted to the intervals $(m_k, n_k]$ we obtain the result by adding this last statement to (7.4). For the divergent case we fix a value of γ and choose η and C so that by (3.13)

$$P\{T_n \geq EW_n - Cb_n\} \geq e^{-\gamma} - \eta - C^{-2}\gamma > 0.$$

Then since $S_n \geq T_n$ and by (3.10)

$$EW_n - ES_n = n \int_{b_n}^{\infty} G_-(x)\, dx - n \int_{a_n}^{\infty} G_+(x)\, dx \geq -n \int_{a_n}^{\infty} f(x)\, dx \geq -\beta_n$$

we have by (2.18)

$$P\{S_n \geq ES_n - (C\gamma^{-1} + 1)\beta_n\} \geq c > 0.$$

Then by Lemmas 4.1 and 2.7

$$\limsup_{n\to\infty} \frac{S_n - ES_n + (C\gamma^{-1} + 1)\beta_n}{\beta_n} = \infty \quad \text{a.s.}$$

and this implies the final result of the theorem.

REMARK. Note that when $\Sigma\, P\{X > \beta_n\}$ converges, if it happens to be the case that $\Sigma\, P\{|X| > \beta_n\}$ diverges then we may apply the divergent case of the theorem to the random variable $-X$ to obtain

$$\liminf_{n\to\infty} \frac{S_n - ES_n}{\beta_n} = -\infty \quad \text{a.s.} \tag{7.8}$$

Although the series $\Sigma\, P\{|X| > \beta_n\}$ does diverge somewhat generally it does not when K is sufficiently dominant as in the classical case. However, it will diverge whenever

$$K(x) = O\left(x^{-1} \int_{|y|>x} |y|\, dF \right). \tag{7.9}$$

To see this, we note that if $\Sigma\, P\{|X| > \beta_n\}$ converges then by Lemmas 7.2 and 7.4 applied to $|X|$,

$$n \int_{|x|>a_n} |x|\, dF < \eta\beta_n \tag{7.10}$$

for infinitely many n. Then by Lemma 2.2

$$\beta_n = na_n f(a_n) + n \int_{a_n}^{\infty} f(x)\, dx = 2n \int_{|x|>a_n} |x|\, dF + 2na_n K(a_n)$$

$$= O\left(n \int_{|x|>a_n} |x|\, dF \right)$$

by (7.9). But this is small compared to β_n infinitely often, a contradiction.

Now we will prove the main result for centering at the median of S_n in the nice case. We assume that G is not slowly varying in this theorem. The case of slowly varying G is treated at the end of the section. The reason for the separation is that when G is slowly varying the median of S_n may grow so rapidly that it is difficult to estimate its exact rate of growth. This means that β_n must be defined directly in terms of the median. The comment about the case when $EX^2 < \infty$ which preceded the statement of Theorem 7.5 also applies here.

THEOREM 7.6. *Suppose that G is not slowly varying and β_n is defined by* (2.14). *There is a value γ_0 such that if $\gamma_0 < \gamma < \log 2$ and $\Sigma P\{X > \beta_n\}$ converges then*

$$0 < \limsup_{n\to\infty} \frac{S_n - \text{med } S_n}{\beta_n} < \infty \quad \text{a.s.}$$

On the other hand, if $\Sigma P\{X > \beta_n\}$ diverges then

$$\limsup_{n\to\infty} \frac{S_n - \text{med } S_n}{\beta_n} = \infty \quad \text{a.s.}$$

REMARK. The divergent case is valid for any γ. Since β_n decreases as γ increases we have in the convergent case that the upper bound improves as γ increases while the lower bound improves as γ decreases. Example 9.6 shows that the upper bound may fail when $\gamma = \log 2$ even though the series converges. Furthermore, it should be noted that the series $\Sigma P\{X > \beta_n\}$ might converge for some values of γ and diverge for others.

PROOF. The divergent case follows immediately from Lemmas 4.1 and 2.7. The upper bound in the convergent case works for any $\gamma < \log 2$. We let $n_k = 2^k$, $u_k = a_{n_k}$. Then by Lemma 4.3

$$\limsup_{n\to\infty} \frac{T_n - EW_n}{\beta_n'} \le 4 \quad \text{a.s.}$$

where for $n_{k-1} < n \le n_k$

$$\beta_n' = a_{n_k} \log k + n \int_{a_{n_k}}^{b_n} f(x)\, dx \sim a_{n_k} \log\log n + n \int_{a_{n_k}}^{b_n} f(x)\, dx.$$

Since for $x \in [a_n, a_{n_k}]$,

$$n_k^{-1} \log\log n \le f(x) \le n^{-1} \log\log n$$

this means that

$$\beta_n \le \beta_n' \le (2 + \eta)\beta_n. \tag{7.11}$$

By Lemma 7.3

$$\lim_{n\to\infty} \frac{U_n - EU_n}{\beta_n} = 0 \quad \text{a.s.}$$

and $P\{S_n \ne T_n + U_n \text{ for some } n_{k-1} < n \le n_k\} \le n_k P\{X > \beta_{n_{k-1}}\}$ which is summable. Thus for large n we will have by (3.14), (7.1), and (2.18)

$$\begin{aligned} S_n - \text{med } S_n &\le EW_n + EU_n - \text{med } S_n + (8 + \eta)\beta_n \\ &\le Cb_n + O(\beta_{n_{k-1}}) + (8 + \eta)\beta_n = O(\beta_n). \end{aligned}$$

The lower bound is considerably more delicate. We start by choosing some constants. By Lemma 2.5 we can choose η_0 so that $\eta_0 < 1$ and

(7.12) $$0 < \eta_0 < \limsup_{x\to\infty} \frac{K(x)}{G(x)}.$$

Then we let $\lambda_0 = \eta_0/2(1 + \eta_0/4)$ and choose γ so that $2^{-\lambda_0} \log 2 < \gamma < \log 2$ and then γ_1 so that $\log 2 < \gamma_1 < \gamma 2^{\lambda_0}$. Next we choose η_1 to satisfy

(7.13) $$(1 + 3\eta_1)\gamma_1\gamma^{-1} < 2^{\lambda_0}, \quad e^{-\gamma_1} + 10\eta_1\gamma_1 < ½, \quad \eta_1(1 + 10\gamma_1) < e^{-\gamma_1}, \quad \eta_1 \le \eta_0/4.$$

Finally we let

(7.14) $$\eta_2 = 10^{-4}, \qquad \gamma_2 = \eta_2^{-2}.$$

There will be two parts to the proof of the lower bound. For the first part we assume that for all sufficiently large n we have at least one of the following conditions satisfied:

(7.15) $$b_n \le \eta_2\beta_n \qquad \text{for the } \gamma \text{ selected above}$$

or

(7.16) $$K(x) \ge \eta_1 G(x) \qquad \text{for all} \qquad x \in [b_n(\gamma_2), b_n(\gamma)].$$

Suppose, for the moment, that (7.16) is satisfied. Then by Lemma 2.4 with $\lambda = 2\eta_1/(1 + \eta_1)$

$$n\int_{b_n(\gamma_2)}^{b_n(\gamma)} f(x)\,dx \le n\{b_n(\gamma_2)\}^{\lambda}\gamma_2 n^{-1}\{b_n(\gamma)\}^{1-\lambda}(1-\lambda)^{-1} \le \gamma_2(1-\lambda)^{-1}(\gamma_2\gamma^{-1})^{-1+1/\lambda} b_n(\gamma_2)$$

since

$$\{b_n(\gamma_2)\}^{\lambda}\gamma_2 n^{-1} \ge \{b_n(\gamma)\}^{\lambda}\gamma n^{-1}.$$

But then by (2.18)

(7.17) $$\beta_n(\gamma) = \beta_n(\gamma_2) + n\int_{b_n(\gamma_2)}^{b_n(\gamma)} f(x)\,dx = O(\beta_n(\gamma_2)).$$

Now we choose n_k as in the proof of the lower bound in Theorem 7.5 and let $u_k = a_{n_k}$. With $m_k = n_k/40$ we have for $m_k \le n \le n_k$ since $m_k \notin A$ and $\Sigma P\{X > \beta_n\}$ converges (when the parameter in b_n or β_n is not specified it is to be the chosen value of γ)

$$n\int_{a_{n_k}}^{\beta_{m_k}} G_+(x)\,dx = n\int_{a_{n_k}}^{\beta_{m_k}} x\,dF + n\{\beta_{m_k}G_+(\beta_{m_k}) - a_{n_k}G_+(a_{n_k})\}$$

$$\le 40\, m_k \int_{a_{m_k}}^{\beta_{m_k}} x\,dF + 40\, m_k G_+(\beta_{m_k})\,\beta_{m_k} = o(\beta_{m_k}) = o(\beta_n).$$

Since $\log k \sim \log\log n_k$ we still have $\beta_n \le (1 + \eta)\beta_n'$ as in (7.11) so that if $b_n \le \beta_{m_k}$ this implies (4.17). If $b_n > \beta_{m_k}$, then by (2.18)

$$n\int_{\beta_{m_k}}^{b_n} G_+(x)\,dx \le 40\, m_k G_+(\beta_{m_k})\, b_n = o(b_n) = o(\beta_n).$$

Thus we can apply Lemma 4.3 and (7.11) to obtain

(7.18) $$\limsup_{n\to\infty} \frac{T_n - EW_n}{\beta_n} \ge \frac{1}{480} \quad \text{a.s.}$$

Note that since $b_n(\gamma_2) \le b_n(\gamma)$ we will still have (4.17) satisfied if we use $b_n(\gamma_2)$ and $\beta_n(\gamma)$. But if n satisfies (7.16) then (7.17) is true so that (4.17) remains valid with both b_n,

β_n replaced by $b_n(\gamma_2)$ and $\beta_n(\gamma_2)$. This means that (7.18) still holds if we use EW_n and β_n depending on γ_2 instead of γ for those n satisfying (7.16). Now apply Lemma 7.3 to the entire sequence $\{40^k\}$; the definition of U_n is to be in terms of $\beta_n(\gamma)$ for all n but for those n satisfying (7.16) we can also use $\beta_n(\gamma_2)$ to norm with in view of (7.17). Then, since (7.18) is valid for n restricted to the intervals $[m_k, n_k]$,

$$\limsup_{n\to\infty} \frac{S_n - EW_n - EU_n}{\beta_n} \geq \limsup_{n\to\infty} \frac{T_n + U_n - EW_n - EU_n}{\beta_n} \geq \frac{1}{480} \quad \text{a.s.} \tag{7.19}$$

By (3.11) it follows that

$$P\{T_n - EW_n \geq 2\gamma^{1/2}b_n\} \leq P\{W_n - EW_n \geq 2\gamma^{1/2}b_n\} \leq 1/4$$

and by Lemma 7.3, $U_n \leq EU_n + \eta_2\beta_n$ for large n a.s. Again this is true with $\beta_n(\gamma_2)$ for those n satisfying (7.16). Finally, for $m_k \leq n \leq n_k$, since the upper truncation point for the summands in U_n is $\beta_{m_k}(\gamma)$,

$$P\{S_n \neq T_n + U_n\} \leq nG_+(\beta_{m_k}) \to 0.$$

Thus we have for $m_k \leq n \leq n_k$ and large k

$$\text{med } S_n \leq EW_n + EU_n + 2\gamma^{1/2}b_n + \eta_2\beta_n.$$

If n satisfies (7.15) we use this with γ to obtain

$$\text{med } S_n \leq EW_n + EU_n + 3\eta_2\beta_n \tag{7.20}$$

while if n satisfies (7.16) we use this with γ_2 to obtain by (2.18)

$$\text{med } S_n \leq EW_n + EU_n + 2\gamma_2^{-1/2}\beta_n + \eta_2\beta_n$$

which by (7.14) is the same as (7.20) except that EW_n and β_n depend on γ_2. Since this was also the case in (7.19) and since $3\eta_2 < 10^{-3}$ this gives

$$\limsup_{n\to\infty} \frac{S_n - \text{med } S_n}{\beta_n} \geq 10^{-3}.$$

This is still under the provision that we use $\beta_n(\gamma_2)$ when n satisfies (7.16) but then by (7.17) we see that this lim sup is positive even if we norm with $\beta_n(\gamma)$ for all n. We are now ready for the second part of the proof. Thus we will have for infinitely many n that both

$$b_n \geq \eta_2\beta_n \qquad \text{for the selected } \gamma \tag{7.21}$$

and also that (7.16) fails. For the remainder of the proof we will work with this sequence of values of n. We define

$$x_n = \sup\{x \in [b_n(\gamma_2), b_n(\gamma)]: K(x) \leq \eta_1 G(x)\}$$

$$y_n = \inf\{x \geq x_n: K(x) \geq \eta_0 G(x)\}$$

$$w_n = \sup\{x \in [x_n, y_n]: K(x) \leq \eta_1 G(x)\}$$

with $w_n = x_n$ if the last set is empty. The first two sets are nonempty since (7.16) fails and by (7.12). Since G and K are right continuous,

$$K(y_n) \geq \eta_0 G(y_n). \tag{7.22}$$

We take $z_n \in [1/2\, w_n, w_n]$ with

$$K(z_n) \leq \eta_1 G(z_n). \tag{7.23}$$

We will need an estimate for $G_+(z_n)$. Let

$$\nu_n = [\gamma/f(z_n)], \qquad j_n = 1 + [\gamma_1/G_-(z_n)].$$

First we suppose that $n \le \nu_n$. This implies $z_n \ge b_n$. Since $K(x) < \eta_0 G(x)$ for $x \in [x_n, y_n)$ and $x_n \le b_n \le z_n \le y_n$ we have by Lemma 2.4 with $\lambda_1 = 2\eta_0/(1+\eta_0)$ and (2.18)

$$\begin{aligned}\nu_n^{-1}\beta_{\nu_n} &\le \int_0^{b_n} \{f(x) \wedge \nu_n^{-1} \log\log \nu_n\}\, dx + \int_{b_n}^{z_n} f(x)\, dx \\ &\le n^{-1}\beta_n + z_n^{\lambda_1} f(z_n) z_n^{1-\lambda_1}(1-\lambda_1)^{-1} \qquad (7.24)\\ &\le \eta_2^{-1} n^{-1} b_n + (1-\lambda_1)^{-1}\, \gamma \nu_n^{-1} z_n,\end{aligned}$$

where we have used (7.21) at the last step. Another application of Lemma 2.4 yields

$$\gamma n^{-1} b_n^{\lambda_1} = b_n^{\lambda_1} f(b_n) \le z_n^{\lambda_1} f(z_n) \le z_n^{\lambda_1} \gamma \nu_n^{-1}$$

and this means that

$$\beta_{\nu_n} \le \eta_2^{-1}(\nu_n n^{-1})^{1-1/\lambda_1} z_n + (1-\lambda_1)^{-1}\gamma z_n \le \{\eta_2^{-1} + \gamma(1-\lambda_1)^{-1}\} z_n. \qquad (7.25)$$

Since $n^{-1/2}\beta_n$ increases we have $\Sigma\, P\{X > C\beta_n\}$ converges for any $C > 0$ and thus

$$\frac{G_+(z_n)}{f(z_n)} \le \gamma^{-1}(\nu_n + 1) G_+(C\beta_{\nu_n}) \to 0. \qquad (7.26)$$

If, on the other hand, $\nu_n \le n$ then $K(x) \ge \eta_1 G(x)$ for $x \in [x_n, b_n]$ so that with $\lambda = 2\eta_1/(1+\eta_1)$,

$$x_n^{\lambda} \gamma_2 n^{-1} \ge x_n^{\lambda} f(x_n) \ge b_n^{\lambda} f(b_n) = b_n^{\lambda} \gamma n^{-1}.$$

Then by (7.21)

$$z_n \ge \tfrac{1}{2}\, x_n \ge \tfrac{1}{2}\, (\gamma\gamma_2^{-1})^{1/\lambda} b_n \ge \tfrac{1}{2}\, (\gamma\gamma_2^{-1})^{1/\lambda} \eta_2 \beta_n \qquad (7.27)$$

so that

$$\nu_n G_+(z_n) \le n G_+(C\beta_n) \to 0$$

and we have (7.26) in this case also. Next we need to show that

$$\beta_{j_n} = O(z_n). \qquad (7.28)$$

By the definitions, we have $\nu_n \le j_n$. If $j_n \le n$, then $\beta_{j_n} \le \beta_n = O(z_n)$ by (7.27). Otherwise, as in (7.24) we use

$$\begin{aligned}&\beta_{j_n} \le j_n \nu_n^{-1} \beta_{\nu_n} + j_n \int_{b_{\nu_n}}^{b_{j_n}} f(x)\, dx \quad \text{if} \quad n \le \nu_n, \\ (7.29)\qquad &\beta_{j_n} \le j_n n^{-1} \beta_n + j_n \int_{b_n}^{b_{j_n}} f(x)\, dx \quad \text{if} \quad \nu_n \le n.\end{aligned}$$

Since $f(z_n) \le (1+\eta_1) G(z_n)$ by (7.23), we have $G_+(z_n)/G_-(z_n) \to 0$ by (7.26) and then

$$j_n \le 1 + \frac{\gamma_1}{G_-(z_n)} \sim 1 + \frac{\gamma_1}{G(z_n)} \le 1 + \frac{\gamma_1(1+\eta_1)}{f(z_n)} \le 1 + \frac{\gamma_1(1+\eta_1)}{\gamma}(\nu_n + 1). \qquad (7.30)$$

This means that the first term in (7.29) is $O(z_n)$ in either case by (7.25) and (7.27). Next we observe that if $x \in [y_n, 2y_n]$ then by (7.22)

$$K(x) \ge x^{-2} y_n^2 K(y_n) \ge \tfrac{1}{4}\, \eta_0 G(y_n) \ge \tfrac{1}{4}\, \eta_0 G(x) \qquad (7.31)$$

and so with $\lambda_0 = \eta_0/2(1+\eta_0/4)$,

$$y_n^{\lambda_0} f(y_n) \ge (2y_n)^{\lambda_0} f(2y_n)$$

by Lemma 2.4. Thus, for large n,

$$f(2y_n) \le 2^{-\lambda_0} f(y_n) \le 2^{-\lambda_0} f(z_n) \le 2^{-\lambda_0}(1+2\eta_1) G_-(z_n)$$
$$\le 2^{-\lambda_0}(1+3\eta_1)\gamma_1 j_n^{-1} < \gamma j_n^{-1}$$

by (7.13). This implies that $b_{j_n} < 2y_n$. Next we have $K(x) \ge \eta_1 G(x)$ for all $x \in [w_n, 2y_n]$ by (7.31) and (7.13). With $\lambda = 2\eta_1/(1+\eta_1)$, this means that if $b_{j_n} \ge w_n$ then

$$\gamma j_n^{-1} b_{j_n}^\lambda = b_{j_n}^\lambda f(b_{j_n}) \le w_n^\lambda f(w_n) \le w_n^\lambda f(z_n) \le \gamma \nu_n^{-1} w_n^\lambda$$

so that $b_{j_n} \le (j_n \nu_n^{-1})^{1/\lambda} w_n$. Then by (7.30)

$$j_n \int_{w_n}^{b_{j_n}} f(x)\,dx \le j_n w_n^\lambda f(w_n) b_{j_n}^{1-\lambda}(1-\lambda)^{-1}$$
$$\le (1-\lambda)^{-1} j_n \gamma \nu_n^{-1} (j_n \nu_n^{-1})^{-1+1/\lambda} w_n = O(w_n) = O(z_n).$$

Finally, if $b_{\nu_n} \le w_n$ we have by (7.30)

$$j_n \int_{b_{\nu_n}}^{w_n} f(x)\,dx \le j_n \gamma \nu_n^{-1} w_n = O(w_n) = O(z_n)$$

with a similar bound holding for the second case in (7.29). This also covers the possibility that $b_{j_n} \le w_n$. Thus we have shown (7.28). Now we can complete the proof. Define

$$V_{j_n} = \textstyle\sum_{i=1}^{j_n} X_i\, 1\{|X_i| \le z_n\}.$$

Then

$$P\{|V_{j_n} - EV_{j_n}| \ge \tfrac{1}{3} z_n\} \le 9 j_n K(z_n) \le 9\eta_1 j_n G(z_n)$$
$$\sim 9\eta_1 j_n G_-(z_n) \sim 9\eta_1\gamma_1$$

and

$$P\{X_i > z_n \text{ for some } i \le j_n\} \le j_n G_+(z_n) = o(j_n G_-(z_n)) = o(1).$$

Also

(7.32) $$P\{X_i \ge -z_n,\ i \le j_n\} = \{1 - G_-(z_n)\}^{j_n} \le \exp\{-j_n G_-(z_n)\} \le e^{-\gamma_1}.$$

Then by (7.13)

$$P\{|V_{j_n} - EV_{j_n}| < \tfrac{1}{3} z_n,\ X_i \le z_n \text{ for } i \le j_n, \quad \text{at least one} \quad X_i < -z_n \text{ for } i \le j_n\} \ge \tfrac{1}{2}.$$

On this event, $S_{j_n} < EV_{j_n} + \tfrac{1}{3} z_n - z_n$ so that

(7.33) $$\text{med } S_{j_n} < EV_{j_n} - \tfrac{2}{3} z_n.$$

Since the probability in (7.32) tends to $e^{-\gamma_1}$ we also have

$$P\{|V_{j_n} - EV_{j_n}| < \tfrac{1}{3} z_n,\ |X_i| \le z_n \text{ for } i \le j_n\} \ge e^{-\gamma_1}(1-\eta_1) - 10\eta_1\gamma_1$$

for large n. This is positive by (7.13) so that we have

$$P\{S_{j_n} \ge EV_{j_n} - \tfrac{1}{3} z_n \text{ i.o.}\} = 1$$

by the Hewitt-Savage zero-one law. Then by (7.33)

$$\limsup_{n\to\infty} \frac{S_{j_n} - \text{med } S_{j_n}}{z_n} \ge \frac{1}{3} \quad \text{a.s.}$$

Recalling (7.28) completes the proof.

For the case where G is slowly varying we will first need to prove two lemmas.

LEMMA 7.7. *Suppose that G is slowly varying. Then if $\gamma < \log 2$, med $S_n/b_n \to 0$. If $n\,P\{X > b_n\} \to 0$ (this condition is independent of the value of γ) and $\gamma_1 > \log 2$ then* med $S_n/b_n(\gamma_1) \to -\infty$.

PROOF. Fix γ and let

$$V_n = \textstyle\sum_{i=1}^n X_i 1\{|X_i| \le b_n\}.$$

Then

$$P\{|V_n - EV_n| \ge \epsilon b_n\} \le \epsilon^{-2} n\,K(b_n) = o(nG(b_n)) = o(1)$$

by Lemma 2.5. Also $EV_n = nb_n M(b_n) = o(b_n)$ by the same lemma so that $b_n^{-1}V_n \to 0$ in probability. Now if $\gamma < \log 2$

$$P\{S_n = V_n\} \ge \{1 - G(b_n)\}^n \ge \exp\{-nG(b_n)(1 + G(b_n))\}$$
$$\ge \exp\{-\gamma(1 + G(b_n))\} > 1/2$$

for large n. Thus we can ensure that $|S_n| \le \epsilon b_n$ with probability at least one half so that $|\text{med } S_n| \le \epsilon b_n$. For the other part, take $\gamma_2 \in (\log 2, \gamma_1)$. Now

$$P\{X_i > b_n \text{ for some } i \le n\} \le n\,P\{X > b_n\} \to 0.$$

But

$$P\{X_i < -b_n(\gamma_2) \text{ for some } i \le n\} = 1 - \{1 - G_-(b_n(\gamma_2))\}^n \sim 1 - e^{-\gamma_2} > 1/2$$

for large n since $K(b_n) = o(G(b_n))$ by Lemma 2.5 and $nG_+(b_n) \to 0$ so we must have $nG_-(b_n) \to \gamma_2$ in this case. Since we showed above that $b_n^{-1}V_n \to 0$ in probability (for any γ) we will have

$$\text{med } S_n < -(1 - \epsilon)b_n(\gamma_2).$$

To complete the proof we note that $b_n(\gamma_1)/b_n(\gamma_2) \to 0$. This is so since G slowly varying implies that f is also and then if $b_n(\gamma_1) \ge \epsilon b_n(\gamma_2)$ for infinitely many n we would have for such n

$$1 \le \frac{f(\epsilon b_n(\gamma_2))}{f(b_n(\gamma_1))} \sim \frac{f(b_n(\gamma_2))}{f(b_n(\gamma_1))} = \frac{\gamma_2}{\gamma_1} < 1.$$

LEMMA 7.8. *Suppose that G is slowly varying. If $\sum P\{X > b_n\}$ converges, then* lim sup $b_n^{-1}S_n = 0$ a.s. (*The series either converges for all γ or diverges for all γ.*)

PROOF. Let V_n be as in the proof of the last lemma. We showed that $b_n^{-1}V_n \to 0$ in probability, $n\,P\{X > b_n\} \to 0$, and

$$P\{X_i \ge -b_n, i \le n\} = \{1 - G_-(b_n)\}^n \sim e^{-\gamma}.$$

Thus for large n

$$P\{S_n \ge -\epsilon b_n\} \ge 1/2\, e^{-\gamma}$$

and so we will have $S_n \ge -\epsilon b_n$ i.o. with probability one by the Hewitt-Savage zero-one law. This means that lim sup $b_n^{-1}S_n \ge 0$ a.s. For the upper bound we let $n_k = 2^k$ and for $n_k < n \le n_{k+1}$, let

$$T_n = \textstyle\sum_{i=1}^n X_i\, 1\{0 < X_i \le b_{n_k}\}.$$

Then

$$P\{\max_{n_k < n \le n_{k+1}} S_n > \epsilon b_{n_k}\} \le P\{T_{n_{k+1}} > \epsilon b_{n_k}\} + n_{k+1}P\{X > b_{n_k}\}$$
$$\le \epsilon^{-1} b_{n_k}^{-1} ET_{n_{k+1}} + n_{k+1}P\{X > b_{n_k}\}.$$

Since the last term will converge when summed on k, it will suffice to show that the first term does also. This is true since

$$\sum_k b_{n_k}^{-1} ET_{n_{k+1}} \le \sum_k b_{n_k}^{-1} n_{k+1} \sum_{j\le k} b_{n_j} P\{X > b_{n_{j-1}}\}$$
$$= \sum_j b_{n_j} P\{X > b_{n_{j-1}}\} \sum_{k=j}^{\infty} b_{n_k}^{-1} n_{k+1}$$
$$\le 2 \sum_j n_{j+1} P\{X > b_{n_{j-1}}\} < \infty,$$

where the last inequality follows from the fact that $n^{-2} b_n$ increases which results from using $\eta = 1/3$ in Lemma 2.4.

THEOREM 7.9. *Suppose that G is slowly varying. Then if $\sum P\{X > b_n\}$ converges we have*

$$\limsup_{n\to\infty} \frac{S_n - \text{med } S_n}{-\text{med } S_n} = 1 \quad \text{a.s.} \tag{7.34}$$

If $\sum P\{X > b_n\}$ diverges then

$$\limsup_{n\to\infty} \frac{S_n - \text{med } S_n}{b_n} = \infty \quad \text{a.s.}$$

for all γ.

REMARKS. 1. It is a consequence of Lemma 7.7 that the convergence of $\sum P\{X > b_n\}$ is equivalent to the convergence of $\sum P\{X > -\text{med } S_n\}$ but the condition in terms of b_n will usually be easier to check.

2. The norming sequence $\{-\text{med } S_n\}$ need not be monotone but it is easy to see that (7.34) is still true if one norms with $\max_{j\le n}(-\text{med } S_j)$ instead of $-\text{med } S_n$. The key fact here is that the proof of Lemma 7.8 actually shows that $\limsup b_n^{-1} S_n \ge 0$ a.s. along an arbitrary subsequence.

PROOF. In the convergent case this follows immediately from Lemmas 7.7 and 7.8 since

$$\frac{S_n}{-\text{med } S_n} = \frac{S_n}{b_n(\gamma_1)} \cdot \frac{b_n(\gamma_1)}{-\text{med } S_n}$$

and this has lim sup zero. The divergent case follows immediately from Lemma 4.1.

8. Centering at zero. In this section we will assume that $E|X| = \infty$. This is because the strong law implies that $\{n\}$ will serve as a norming sequence if $EX \neq 0$ while if $EX = 0$ centering at zero is the same as centering at ES_n and this was discussed in the last section.

The results of this section are of a fundamentally different nature than the earlier ones since $\limsup \beta_n^{-1} S_n$ is negative. This means that we are using centering constants which are "almost" outside the support of the distribution of S_n on the right side. Nevertheless, this may be a useful thing to do since in many cases the norming sequence $\{\beta_n\}$ used here will be smaller than the norming sequence for centering at the median of S_n.

Although the main theorem in this section follows easily from the results of Fristedt and the author [6] in conjunction with the integral test of Erickson [1], we believe that it is worth giving a new proof of the results in [6]. There are two reasons for this. One is that the main result in [6] is for continuous time subordinators with the observation being made at the end that the same method will work for sums of independent random variables. The other reason is that it fits in better with the present work if the norming sequence is defined in terms of the function G instead of in terms of the exponent of the Laplace transform of the distribution function F as is the case in [6]. The price one pays for this change is in losing some information about the precise value of the lim sup.

We start with a lemma which gives a slight extension of Erickson's result and also translates his condition into our notation. We will use the function g defined in (2.2) as well as the analogous g_+ and g_- defined in the same way for the random variables X^+ and X^-. In our notation the integral J_+ defined by Erickson is

$$J_+ = \int_0^\infty \frac{1}{g_-(x)}\, dF(x).$$

Erickson proves that

$$J_+ < \infty \quad \text{iff} \quad \lim_{n\to\infty} n^{-1}S_n = -\infty \quad \text{a.s.} \tag{8.1}$$

and in the alternative case $\limsup n^{-1}S_n = \infty$. He also proves that (8.1) is equivalent to

$$\limsup_{n\to\infty}(X_n^+/\textstyle\sum_{i=1}^n X_i^-) = 0 \quad \text{a.s.} \tag{8.2}$$

with this lim sup also being infinite in the alternative case. We will show that g_- may be replaced by g in the definition of J_+ without changing the criterion and that (8.2) still holds if X_n^+ is replaced by $\sum_{i=1}^n X_i^+$. The importance of this is that it allows us to work with negative summands.

LEMMA 8.1. *Assume* $E|X| = \infty$. *Then for any* $\gamma > 0$

$$J_+ < \infty \quad \text{iff} \quad \int_0^\infty \frac{1}{g(x)}\, dF(x) < \infty \quad \text{iff} \quad \sum P\{X > d_n\} < \infty \tag{8.3}$$

where d_n *is defined in* (2.23). *Furthermore, the conditions in* (8.3) *are equivalent to*

$$\limsup_{n\to\infty}(\textstyle\sum_{i=1}^n X_i^+/\sum_{i=1}^n X_i^-) = 0 \quad \text{a.s.} \tag{8.4}$$

When $J_+ = \infty$, *the* lim sup *in* (8.4) *is infinite.*

PROOF. The first only if statement is clear since $g(x) \ge g_-(x)$. Next we observe that

$$\int_0^\infty \frac{1}{g(x)}\, dF(x) = E\frac{1}{g(X)}1\{X \ge 0\}$$

and this is finite if and only if

$$\sum P\left\{\frac{1}{g(X)} > \gamma^{-1}n,\, X \ge 0\right\} = \sum P\{X > d_n\} < \infty.$$

Now suppose that $\sum P\{X > d_n\}$ converges. Let $n_k = 2^k$ and observe that by (2.25)

$$\int_{d_{n_i}}^{d_{n_k}} G_+(x)\, dx \le \textstyle\sum_{j=i+1}^k d_{n_j} P\{X > d_{n_{j-1}}\} \le n_k^{-1} d_{n_k} \sum_{j=i+1}^k n_j P\{X > d_{n_{j-1}}\}$$

and the sum may be made small by choice of i. Thus

$$g_+(d_{n_k}) \le d_{n_k}^{-1} d_{n_i} + \epsilon n_k^{-1} = n_k^{-1}\{n_k d_{n_k}^{-1} d_{n_i} + \epsilon\}$$

and so $n_k\, g_+(d_{n_k}) \to 0$ since $E|X| = \infty$ implies that

$$\gamma n^{-1} d_n = d_n g(d_n) = \int_0^{d_n} G(y)\, dy \to \infty$$

by Lemma 2.3. Now suppose that $d_{n_{k-1}} < x \le d_{n_k}$. Then

$$g_+(x) \le g_+(d_{n_{k-1}}) = o(n_{k-1}^{-1}) = o(n_k^{-1})$$

while

$$g(x) \geq g(d_{n_k}) = \gamma n_k^{-1}.$$

This implies that $g_+(x) = o(g(x))$ and thus $g(x) \sim g_-(x)$. This completes the proof of (8.3). To see that (8.3) implies (8.4) we observe that by (2.6) for any $C > 1$

$$g(Cx) \geq C^{-1} g(x)$$

and so the integral in (8.3) is still finite if the distribution of X^+ is changed to that of CX^+. Then (8.1) implies that

$$C \sum_{i=1}^n X_i^+ - \sum_{i=1}^n X_i^- \leq 0$$

for all sufficiently large n. This implies (8.4). The final statement follows since Erickson proved that even the lim sup in (8.2) is infinite when $J_+ = \infty$.

Now we will prove the fundamental convergence lemmas for this case.

LEMMA 8.2. *Suppose that* $X \leq 0$ a.s. *and* $\delta > 8$. *Then with* β_n *as defined in* (2.23) *and* (2.24),

$$\limsup_{n\to\infty} \beta_n^{-1} S_n < 0 \quad \text{a.s.}$$

PROOF. Let $n_k = [\rho^k]$ where $\rho > 1$. By Lemma 3.1 with $r = 1/2$

$$P\{S_{n_{k-1}} \geq EV_{n_{k-1}} + 1/4\, e^{1/2} n_{k-1} c_{n_k} f(c_{n_k}) + 4\, c_{n_k} \log\log n_k\} \leq \{\log n_k\}^{-2}$$

since $S_{n_{k-1}} = T_{n_{k-1}} \leq V_{n_{k-1}}$. Now by Lemma 2.2 and (2.8)

$$\begin{aligned} EV_{n_{k-1}} &+ \frac{1}{4} e^{1/2} n_{k-1} c_{n_k} f(c_{n_k}) + 4 c_{n_k} \log\log n_k \\ &\leq -\frac{1}{2} n_{k-1} c_{n_k} g(c_{n_k}) + 4 c_{n_k} \log\log n_k \\ &= c_{n_k} \log\log n_k \left\{ -\frac{1}{2} \delta \frac{n_{k-1}}{n_k} + 4 \right\} \\ &\sim \beta_{n_k} \left\{ -\frac{1}{2} \delta \rho^{-1} + 4 \right\}. \end{aligned}$$

Since $\delta > 8$ we can take ρ close enough to one to make the coefficient of β_{n_k} negative. We then have $S_n \leq S_{n_{k-1}}$ for $n_{k-1} < n \leq n_k$ and this proves the lemma.

REMARK. When the random variables are nonpositive the factor e^r in Lemma 3.1 is not needed. Then one may use $r = 1$ and $s = (1 + \epsilon)\log\log n_k$ to see that it is actually enough to assume that $\delta > 2$.

LEMMA 8.3. *Suppose that* $E|X|^\epsilon < \infty$ *for some* $\epsilon > 0$. *Let* δ *be given and* c_n *be defined by* (2.23). *Let* $n_k = [\rho^k]$ *for* $\rho > 1$ *and*

$$C = \rho^{4/\epsilon} \quad \text{if} \quad \epsilon \leq 1, \qquad C = \rho^4 \quad \text{if} \quad \epsilon > 1.$$

Then

$$N_j = \operatorname{card}\{k: 2^j \leq k < 2^{j+1} \quad \text{and} \quad c_{n_{k+1}} \geq C c_{n_k}\} \leq 2^{j-1}$$

for large j.

PROOF. There is no loss in assuming that $\epsilon \leq 1$. By (2.10) and Lemma 2.3 $c_n^\epsilon g(c_n)$ is bounded which implies that $c_n \leq n^{1/\epsilon}$ for large n. Let $m_j = n_k$ where $k = 2^j$. Then for j sufficiently large

$$C^{N_j} \le C^{N_j} c_{m_j} \le c_{m_{j+1}} \le m_{j+1}^{1/\epsilon}$$

or

$$N_j \log C \le \epsilon^{-1} \log m_{j+1} \le \epsilon^{-1} 2^{j+1} \log \rho$$

and substituting the given value of C yields the result.

LEMMA 8.4. *Suppose that* $X \le 0$ a.s. *and* $E|X|^\epsilon < \infty$ *for some* $\epsilon > 0$. *Then with* β_n *as defined in* (2.23) *and* (2.24),

$$\limsup_{n\to\infty} \beta_n^{-1} S_n > -\infty \quad \text{a.s.}$$

PROOF. We take $\rho > \delta \vee 1$, ρ an integer, and apply Lemma 8.3. For large j there must be at least 2^{j-1} values of $k \in [2^j, 2^{j+1})$ such that

$$c_{n_{k+1}} \le C c_{n_k}. \tag{8.5}$$

We form a subsequence $\{m_i\}$ by taking every jth one of these n_k for $j = 1, 2, \ldots$. There will be at least $j^{-1} 2^{j-1} - 1$ values of n_k with $k \in [2^j, 2^{j+1})$ in the subsequence for large j. Now let $\xi \in (\delta\rho^{-1}, 1)$ and $u_n = c_n(\xi)$. By Lemma 3.3 with $C_3 = \xi^{-1}$,

$$P\{S_n \ge -2\, n u_n g(u_n)\} \ge (\log n)^{-1}.$$

For n_k with $k \in [2^j, 2^{j+1})$,

$$(\log n_k)^{-1} = (k \log \rho)^{-1} \ge (\log \rho)^{-1} 2^{-j-1}.$$

If we sum this for those n_k in the subsequence we will have a contribution of j^{-1} from this range of k and thus a divergent series. Now we let

$$\nu_j = \sum_{i=1}^{j} m_i.$$

Since $m_i \ge \rho m_{i-1}$, we have $\nu_j \le m_j \rho/(\rho - 1)$. By the way the m_i subsequence was spaced we have for those i with $m_i = n_k$ where $k \in [2^j, 2^{j+1})$ that $m_i \ge m_{i-1}\rho^j$ and so

$$\nu_{i-1} m_i^{-1} \log\log m_i \le m_{i-1}\rho(\rho - 1)^{-1} m_i^{-1} \log\log m_i \le j/\rho^{j-1}(\rho - 1). \tag{8.6}$$

Then by (3.15),

$$P\{|S_{\nu_{i-1}} - EW_{\nu_{i-1}}| \ge u_{m_i}\} \le 2\xi\, \nu_{i-1} m_i^{-1} \log\log m_i \to 0$$

and by Lemma 2.2

$$EW_{\nu_{i-1}} = -\nu_{i-1} u_{m_i} g(u_{m_i}) = -\nu_{i-1}\xi\, m_i^{-1} u_{m_i} \log\log m_i = o(u_{m_i}).$$

Thus

$$P\{S_{\nu_{i-1}} \ge -2u_{m_i}\} \to 1$$

and so by Lemma 3.4,

$$P\{S_{\nu_i} - S_{\nu_{i-1}} \ge -2m_i u_{m_i} g(u_{m_i});\ S_{\nu_{i-1}} \ge -2u_{m_i} \quad \text{i.o.}\} = 1.$$

But this means that infinitely often

$$S_{\nu_i} \ge -2\xi u_{m_i} \log\log m_i - 2u_{m_i} \ge -2u_{m_i} \log\log m_i. \tag{8.7}$$

Now for large i,

$$g(c_{\rho m_i}) = \delta\rho^{-1} m_i^{-1} \log\log(\rho m_i) < \xi m_i^{-1} \log\log m_i = g(u_{m_i})$$

and then by (8.5)

$$u_{m_i} \le c_{\rho m_i} \le C c_{m_i}.$$

Recalling (8.7), we see that

$$S_{\nu_i} \geq -2C\beta_{m_i} \geq -2C\beta_{\nu_i} \quad \text{i.o.}$$

with probability one.

The main result of this section now follows easily from what we have proved. We will assume in the theorem that $E(X^-)^\epsilon < \infty$ for some $\epsilon > 0$. Example 9.7 shows that some condition of this sort is needed.

THEOREM 8.5. *Suppose that* $E|X| = \infty$ *and* $E(X^-)^\epsilon < \infty$ *for some* $\epsilon > 0$. *Define* d_n, β_n *as in* (2.23), (2.24) *with* $\delta > 8$. *If* $\Sigma P\{X > d_n\}$ *converges then*

$$-\infty < \limsup_{n\to\infty} \frac{S_n}{\beta_n} < 0 \quad \text{a.s.}$$

and

$$\liminf_{n\to\infty} \frac{S_n}{\beta_n} = \liminf_{n\to\infty} \frac{S_n}{d_n} = -\infty \quad \text{a.s.} \tag{8.8}$$

On the other hand, if $\Sigma P\{X > d_n\}$ *diverges then*

$$\limsup_{n\to\infty} \frac{S_n}{\beta_n} = \limsup_{n\to\infty} \frac{S_n}{d_n} = \infty \quad \text{a.s.} \tag{8.9}$$

(*The assumption about* $E(X^-)^\epsilon$ *is not needed for* (8.8) *or* (8.9).)

REMARKS. The convergence of $\Sigma P\{X > d_n\}$ is independent of γ by Lemma 8.1. Also this convergence is implied by the convergence of $\Sigma P\{X > \beta_n\}$ by (2.25) and (2.26). As in Lemma 8.2, it is actually enough to assume that $\delta > 2$.

PROOF. Suppose first that $\Sigma P\{X > d_n\}$ converges. We showed in the proof of Lemma 8.1 that this implies $g(x) \sim g_-(x)$. Then if $\xi \in (8, \delta)$ and we define $c_n^-(\delta)$ by

$$g_-(c_n^-(\delta)) = \delta n^{-1} \log \log n$$

we will have for large n

$$g(c_n^-(\xi)) \leq \delta n^{-1} \log \log n \leq g(c_n^-(\delta))$$

so that $c_n^-(\delta) \leq c_n \leq c_n^-(\xi)$. Then if $\beta_n^-(\delta) = c_n^-(\delta) \log \log n$ we obtain by Lemma 8.4

$$\limsup_{n\to\infty} \frac{S_n}{\beta_n} \geq \limsup_{n\to\infty} \frac{-\sum_{i=1}^n X_i^-}{\beta_n} \geq \limsup_{n\to\infty} \frac{-\sum_{i=1}^n X_i^-}{\beta_n^-(\delta)} > -\infty \quad \text{a.s.}$$

Also by Lemmas 8.1 and 8.2

$$\limsup_{n\to\infty} \frac{S_n}{\beta_n} = \limsup_{n\to\infty} \frac{-\sum_{i=1}^n X_i^-}{\beta_n} \leq \limsup_{n\to\infty} \frac{-\sum_{i=1}^n X_i^-}{\beta_n^-(\xi)} < 0 \quad \text{a.s.}$$

For the divergent case we can apply Lemma 4.1 by (2.25) and (3.17) if we let $\alpha_n = -Cd_n$. Thus we obtain

$$\limsup_{n\to\infty} \frac{S_n + Cd_n}{d_n} = \infty \quad \text{a.s.}$$

which gives the final result by (2.26). To obtain (8.8) we note first that in the proof of Lemma 8.1 we showed that the convergence of $\Sigma P\{X > d_n\}$ implies that $ng_+(d_n) \to 0$. Applying this argument to $|X|$ would lead to $ng(d_n) \to 0$ which is not true. Thus $\Sigma P\{|X| > d_n\}$ always diverges. Since we have assumed that $\Sigma P\{X > d_n\}$ converges this means that $\Sigma P\{-X > d_n\}$ diverges. Hence the divergent case applies to $-S_n$.

9. Some comparisons and examples. First we will look at the question of comparing the sizes of the various norming sequences. It is not hard to see by modifying Lemma 4.2 that if the positive tail is small enough then one can center at $\alpha_n = nE(-a_n \vee (X \wedge a_n))$ and use $\beta_n = a_n \log\log n$ provided that a_n is defined by $f(a_n) = \delta n^{-1} \log\log n$ and $\delta < 1/35$. (This possibility is discussed further in Section 11.) This is always the smallest of the norming sequences we consider. But since it is quite common to use other centering sequences it is of interest to compare the sizes of the other norming sequences. Although it is probably not possible to state any useful completely general comparison result there are some fairly general situations that will give some feeling for the problem.

First suppose that

$$y^\lambda f(y) \le C x^\lambda f(x) \qquad \text{for} \quad y \ge x \ge 1 \tag{9.1}$$

for some $C > 0$, $\lambda > 1$. This includes, for example, f regularly varying with exponent $\xi < -1$ or even f such that

$$\limsup_{x\to\infty} \frac{f(2x)}{f(x)} \le 2^\xi.$$

Under (9.1), $E|X| < \infty$ by Lemma 2.3 and with a_n as in (2.13)

$$n \int_{a_n}^\infty f(x)\,dx \le nCa_n^\lambda f(a_n) \int_{a_n}^\infty x^{-\lambda}\,dx = C(\lambda - 1)^{-1} a_n \log\log n.$$

Also, under (9.1), the a_n defined above in terms of δ are all comparable. Thus the norming sequences for centering at $nE(-a_n \vee (X \wedge a_n))$, ES_n, and median S_n are all comparable under (9.1). It is also possible to check that in this case

$$ES_n - \text{median } S_n = O(b_n) = o(a_n \log\log n).$$

However, the difference between $nE(-a_n \vee (X \wedge a_n))$ and ES_n will typically be as large as $a_n \log\log n$.

As the next case, suppose that

$$y^\lambda f(y) \ge c x^\lambda f(x) \qquad \text{for} \quad y \ge x \tag{9.2}$$

for some $c > 0$, $\lambda < 1$. This includes f regularly varying with exponent $\xi > -1$ or even f such that

$$\liminf_{x\to\infty} \frac{f(2x)}{f(x)} \ge 2^\xi.$$

Under (9.2), $E|X| = \infty$ and by (2.8)

$$\begin{aligned} f(x) \le g(x) \le x^{-1} \int_0^x f(y)\,dy &\le c^{-1} x^{\lambda-1} f(x)(1-\lambda)^{-1} x^{1-\lambda} \\ &= c^{-1}(1-\lambda)^{-1} f(x). \end{aligned} \tag{9.3}$$

Even with this comparison between f and g, $c_n \log\log n$ and $a_n \log\log n$ need not be comparable because if g decays very slowly then changing the value of δ may change c_n significantly. But both of these sequences will be much smaller than the norming sequence for centering at the median of S_n since by (9.2)

$$a_n \log\log n \le (\gamma c^{-1})^{1/\lambda} b_n (\log\log n)^{1-1/\lambda} = o(b_n)$$

and then we use (2.18). A similar argument using (9.3) as well shows that $c_n \log\log n = o(b_n)$. Typically the median of S_n in this case will be comparable to $-\beta_n$ with β_n as in (2.14).

More can be said if f is regularly varying with exponent 0, -1, or -2. These are the cases that have received more study in the literature. In the first case G dominates both K and M as we have seen in Lemma 2.5. One can show that in the last case K dominates both G and M (if $EX = 0$) while in the middle case the function M corresponding to $|X|$

dominates both G and K. Actually in this situation the dominance is true under a weaker condition which we will now consider. As mentioned in Section 2, the functions $xf(x)$, $xG(x)$, and $xK(x)$ are all slowly varying when any one of them is. But there is a weaker set of equivalent conditions which will be useful. For a nonnegative random variable X, the following are equivalent:

(9.4)
(1) $xg(x)$ is slowly varying; (2) $xM(x)$ is slowly varying;

(3) $\lim_{x\to\infty} \frac{G(x)}{M(x)} = 0;$ (4) $\lim_{x\to\infty} \frac{K(x)}{M(x)} = 0.$

Of course these are equivalent for a general random variable if M is replaced by the M corresponding to $|X|$. These conditions are implied by the slow variation of $xf(x)$ but the distribution with

$$P\{X = 2^n\} = \frac{1}{2^n}, \qquad n = 1, 2, \cdots$$

is an example for which $xg(x)$ is slowly varying but $xf(x)$ is not. (This is the Petersburg game. Incidentally, this shows the first sentence in the footnote on page 233 of [4] is incorrect.) The conditions in (9.4) are all trivially satisfied when $E|X| < \infty$; they are of interest when $E|X| = \infty$. There is an analogous set of conditions that are of interest when $E|X| < \infty$. In this case we define

$$h(x) = x^{-1} \int_x^\infty K(y)\, dy, \qquad M_\infty(x) = x^{-1} \int_{|y|>x} y\, dF(y).$$

Then the following are equivalent for a nonnegative random variable X with $EX < \infty$:

(9.5)
(1) $xh(x)$ is slowly varying; (2) $xM_\infty(x)$ is slowly varying;

(3) $\lim_{x\to\infty} \frac{G(x)}{M_\infty(x)} = 0;$ (4) $\lim_{x\to\infty} \frac{K(x)}{M_\infty(x)} = 0.$

These conditions are implied by the slow variation of $xf(x)$.

As an example of the usefulness of the domination of M or M_∞ in these cases we will derive a (slight) generalization of the results of Klass and Teicher [15]. This theorem is equivalent to (2.14) of [14].

THEOREM 9.1. *(Klass). Suppose $E|X| < \infty$ and $xh(x)$ is slowly varying. If*

$$\Sigma P\{X > \beta_n\} < \infty \tag{9.6}$$

where β_n is defined by (2.15) *then*

$$\limsup_{n\to\infty} \frac{S_n - ES_n}{\beta_n} = \frac{1}{2} \quad \text{a.s.,} \tag{9.7}$$

$$\liminf_{n\to\infty} \frac{S_n - ES_n}{\beta_n} = -\infty \quad \text{a.s.} \tag{9.8}$$

REMARK. Under (9.5), $\beta_n \sim 2n\bar{\mu}(a_n)$ where $\bar{\mu}$ is defined by Klass and Teicher. This is not comparable in general to the sequence $\{b_n\}$ used in their Theorem 3. However, under their supplementary hypothesis (19) it follows that $\beta_n \sim 2b_n$. The condition (9.6) is equivalent to their condition (18) when (19) is assumed. Thus their theorem is true with (19) replaced by the weaker assumption that $xh(x)$ is slowly varying provided that their sequence $\{b_n\}$ is changed to $\{\beta_n\}$ and (18) is changed to (9.6).

PROOF. First we use Lemma 4.2 with $n_k = 40^k$ and $u_k = a_{n_k}$. The result is that

$$\frac{1}{240} \le \limsup_{n\to\infty} \frac{T_n - EV_n}{\beta'_n} \le 4 \quad \text{a.s.} \tag{9.9}$$

where $\beta'_n = a_{n_k} \log k$ for $n_{k-1} < n \le n_k$. Next we note that by (2.7)

$$\int_x^\infty f(y)\, dy = 2 \int_x^\infty K(y)\, dy - \int_x^\infty \{K(y) - G(y)\}\, dy = 2xh(x) - xf(x).$$

Also we have for $|X|$

$$h(x) = M_\infty(x) + K(x) \sim M_\infty(x)$$

by Lemma 2.2 and (9.5) so that

$$\int_x^\infty f(y)\, dy \sim 2xh(x).$$

Thus

$$\beta_n = a_n \log\log n + n \int_{a_n}^\infty f(x)\, dx \sim na_n f(a_n) + 2na_n h(a_n) \sim 2na_n h(a_n)$$

and

(9.10) $\quad \beta'_n \sim a_{n_k} \log\log n_k = n_k a_{n_k} f(a_{n_k}) = o(n_k a_{n_k} h(a_{n_k})) = o(na_n h(a_n)) = o(\beta_n)$

since $xh(x)$ decreases. Thus

$$\limsup_{n\to\infty} \frac{T_n - EV_n}{\beta_n} = 0 \quad \text{a.s.}$$

Then by (9.6) and Lemma 7.3 we obtain

(9.11) $$\limsup_{n\to\infty} \frac{S_n - EV_n - EU_n}{\beta_n} = 0 \quad \text{a.s.}$$

By (7.3),

$$EV_n + EU_n - ES_n = n \int_{a_{n_k}}^\infty G_-(x)\, dx - n \int_{\beta_{n_{k-1}}}^\infty x\, dF + na_{n_k} G_+(\beta_{n_{k-1}}).$$

The second term is $o(\beta_n)$ by Lemma 7.4 and the third is $o(a_{n_k})$ by (9.6) and this is also $o(\beta_n)$ by (9.10). Thus

(9.12) $$EV_n + EU_n - ES_n = n \int_{a_{n_k}}^\infty G_-(x)\, dx + o(\beta_n).$$

Then we use

$$n \int_{a_n}^\infty G_-(x)\, dx \le n \int_{a_n}^\infty G(x)\, dx \le \frac{1}{2} n \int_{a_n}^\infty f(x)\, dx \le \frac{1}{2} \beta_n.$$

For the lower bound we note that as in the proof of Theorem 7.5 we have the lower bound in (9.9) even if we restrict to those n with $m_k < n \le n_k$ and $m_k \notin A$ where A is defined in Lemma 7.2 and $m_k = n_k/40$. Then (9.11) is still valid if we restrict to these n. But for $m_k < n \le n_k$ and $m_k \notin A$ we have by (7.6) that

$$n \int_{a_{n_k}}^\infty G_+(x)\, dx = o(\beta_{m_k}) = o(\beta_n)$$

and then by (9.12) and (9.10)

$$EV_n + EU_n - ES_n = n \int_{a_{n_k}}^\infty G(x)\, dx + o(\beta_n) = \frac{1}{2} n \int_{a_{n_k}}^\infty f(x)\, dx + o(\beta_n) = \frac{1}{2} \beta_n + o(\beta_n)$$

since

$$n \int_{a_n}^{a_{n_k}} f(x)\, dx \le a_{n_k} \log \log n.$$

The fact that the lim inf is $-\infty$ in this case is a consequence of the remark following Theorem 7.5 since by (9.5) we have (7.9) satisfied.

The analogous result for the case of an infinite mean is similar but slightly easier so we omit the proof.

THEOREM 9.2. *Suppose $E|X| = \infty$ and $xg(x)$ is slowly varying. If*

$$\Sigma P\{X > d_n\} < \infty \tag{9.13}$$

where d_n is defined by (2.23) *then with β_n as in* (2.24)

$$\limsup_{n\to\infty} \frac{S_n}{\beta_n} = -\delta \quad \text{a.s.,} \tag{9.14}$$

$$\liminf_{n\to\infty} \frac{S_n}{\beta_n} = \liminf_{n\to\infty} \frac{S_n}{d_n} = -\infty \quad \text{a.s.} \tag{9.15}$$

REMARK. In this case $\beta_n = \delta^{-1} n\mu(c_n)$ where μ is defined by Klass and Teicher. This is not comparable in general to the sequence $\{b_n\}$ used in their Theorem 4. However, under their supplementary hypothesis (34) it follows that $\beta_n \sim \delta^{-1} b_n$. The condition (9.13) is equivalent to their condition (33). Thus their theorem is true with (34) replaced by the weaker assumption that $xg(x)$ is slowly varying provided that their sequence $\{b_n\}$ is changed to $\{\beta_n\}$.

The apparent advantage of Theorems 9.1 and 9.2 as compared to Theorems 7.5 and 8.5 is that the constant value of the lim sup is obtained. But the reason this can be done in these cases is that the norming sequence for centering at the median of S_n is smaller and so in Theorem 9.1 one is simply picking up the difference between the median and expected value of S_n and in Theorem 9.2 simply the median of S_n. We now give an example to illustrate this point.

EXAMPLE 9.3. Consider the distribution F with density

$$\frac{1}{2x^2 \log^2 |x|}, \qquad x \le -e$$

and the rest of the mass placed at zero. (We could put on any positive tail having finite variance and it would only change the values of the expectation and the median of S_n in the obvious way.) Then it is easy to check that

$$G(x) \sim K(x) \sim \frac{1}{2x \log^2 x}, \qquad h(x) \sim \frac{1}{2x \log x},$$

$$a_n \sim \frac{n}{\log^2 n \log \log n}, \qquad b_n \sim \frac{n}{\gamma \log^2 n},$$

$$\beta_n(\text{mean}) \sim \frac{n}{\log n}, \qquad \beta_n(\text{median}) \sim \frac{n \log \log \log n}{\log^2 n}.$$

Furthermore, it is clear that $ES_n = -\frac{1}{2} n$ and by (3.14) and the sentence following it

$$\begin{aligned} \text{med}\, S_n = EW_n + O(n/\log^2 n) &= -\frac{1}{2} n(1 - 1/\log b_n) + O(n/\log^2 n) \\ &= -\frac{1}{2} n + \frac{1}{2} \frac{n}{\log n} + \frac{n \log \log n}{\log^2 n} + O\left(\frac{n}{\log^2 n}\right). \end{aligned}$$

Thus we see that Theorem 7.6 implies that

$$\limsup_{n\to\infty} \frac{S_n - ES_n}{n/\log n} = \frac{1}{2}$$

in this case as theorem 9.1 asserts. Since there is no positive tail in the present case we also have

$$0 < \limsup_{n\to\infty} \frac{S_n - EV_n}{a_n \log\log n} < \infty \quad \text{a.s.}$$

where we are letting $EV_n = nE(-a_n \vee X)$. Now

$$\begin{aligned} EV_n &= -\frac{1}{2} n(1 - 1/\log a_n) + O(n/\log^2 n) \\ &= -\frac{1}{2} n + \frac{1}{2}\frac{n}{\log n} + \frac{n \log\log n}{\log^2 n} + \frac{1}{2}\frac{n \log\log\log n}{\log^2 n} + O\left(\frac{n}{\log^2 n}\right) \end{aligned}$$

so that one can even obtain the constant in Theorem 7.6 in this case:

$$\limsup_{n\to\infty} \frac{S_n - \operatorname{med} S_n}{n \log\log\log n/\log^2 n} = \frac{1}{2} \quad \text{a.s.}$$

But this is due to the fact that this result is only picking up the difference between the median of S_n and the better centering sequence EV_n.

If $X \geq 0$ and G is slowly varying it is an immediate consequence of Lemma 2.5 and Theorem 6.1 that it is impossible to find a norming sequence for $S_n - \operatorname{med} S_n$. Teicher [23] has shown that it is still impossible in this case even if the centering is at zero and the criterion for (6.3) remains the same. This is basically due to the fact that $M(x)/G(x) \to 0$ in this case which allows comparison of the median of S_n with any norming sequence that might work.

When f is regularly varying with exponent -2, i.e., F is in the domain of attraction of the normal, most of the work has been for the two-sided problem [5, 11]. But even when there is a norming sequence for the two-sided problem there may be an advantage in considering the one-sided problem. To illustrate this we consider the following example.

EXAMPLE 9.4. Let F have density

$$\frac{1}{|x|^3}, \qquad x \leq -1,$$

with mass ½ at zero. Then $G(x) = ½\, x^{-2}$ and $K(x) = x^{-2} \log x$ so that F is in the domain of attraction of the normal. Thus even if we made F symmetric by spreading the mass at the origin with density x^{-3}, $x \geq 1$, it would be possible to find a norming sequence $\{\beta_n\}$ for $\limsup \beta_n^{-1} |S_n|$. But it would not be a nice norming sequence. On the other hand, if the positive tail is a little smaller then there will be a nice norming sequence for the one-sided problem when centering at either the mean or the median of S_n. The sequence $\beta_n = (n \log n \log\log n)^{1/2}$ will work whenever $\Sigma\, P\{X > \beta_n\}$ converges. Thus if the density on the positive axis is $x^{-3}(\log\log\log x)^{-1-\epsilon}$ for some $\epsilon > 0$ then one may use this nice norming sequence in the one-sided problem.

Next we will clarify the point made in the introduction that if the positive tail is smaller than the negative tail by an appropriate factor then the nice norming sequences will always work. First if $E(X^+)^2 < \infty$ then $\Sigma\, P\{X > \beta_n\}$ converges since for any of the norming sequences $\beta_n \geq \sqrt{n \log\log n}$ and even $\Sigma\, P\{X > \sqrt{n}\}$ converges when the positive tail has a finite second moment. If the negative tail barely has an infinite second moment this is about as much as one can say in general. But when the negative tail is fatter one can put less restrictive assumptions on the positive tail. For example, we have

THEOREM 9.5. *If*

(9.16) $$P\{X > x\} \le P\{|X| > x\}/\log x \log\log x\{\log\log\log x\}^{1+\epsilon}$$

for some $\epsilon > 0$ *then the condition on the positive tail* (*the convergence of* $\Sigma P\{X > \beta_n\}$, $\Sigma P\{X > b_n\}$, *or* $\Sigma P\{X > d_n\}$) *is satisfied in Theorems* 7.5, 7.6, 7.9 *and* 8.5.

PROOF. Since we have $b_n = O(\beta_n)$ and $b_n \le d_n$ and $n^{-1/2}b_n$ increases it is enough to show that $\Sigma P\{X > b_n\}$ converges. This is clear from (9.16) since $G(b_n) = O(n^{-1})$ and $b_n > cn^{1/2}$.

This result is given as an indication of the wide applicability of the various theorems. The conditions on the positive tail in terms of the convergence of $\Sigma P\{X > \beta_n\}$, $\Sigma P\{X > b_n\}$, or $\Sigma P\{X > d_n\}$ are precise and not hard to check.

EXAMPLE 9.6. This is the example mentioned in Section 7 that shows that the upper bound in Theorem 7.6 may fail if $\gamma = \log 2$. Let $n_k = 2^{2^k}$ and let the distribution F have mass

$$\frac{\log 2}{n_k + 1/10} - \frac{\log 2}{n_{k+1} + 1/10} \quad \text{at} \quad -n_{k+1}, \qquad k \ge 0,$$

with the remaining mass to be at zero. Then the median of S_{n_k} is no larger than $-n_{k+1}$ since the probability that all the summands are greater than $-n_{k+1}$ is less than one half. By truncating at $-n_k$ (include the mass at $-n_k$) and using Chebyshev it follows that

$$P\{S_{n_k} \ge -n_k^{3/2}\} \ge c > 0.$$

Thus

$$S_{n_k} - \text{med } S_{n_k} \ge -n_k^{3/2} + n_{k+1} \sim n_k^2 \quad \text{i.o.}$$

with probability one. On the other hand, with $\gamma = \log 2$ one can check that

$$b_{n_k} = O(n_k^{7/4}) \quad \text{and} \quad \beta_{n_k} = O(n_k^{7/4}).$$

EXAMPLE 9.7. This is the example mentioned in Section 8 that shows that some condition is needed on the negative tail in Theorem 8.5. An example of this type is given in [6] but it is easier to explain now that Erickson's test is available. Let F have density

$$\frac{1}{|x| \log^2 |x|}, \qquad x \le -10,$$

and place the remaining mass at zero. We will also use a distribution H with the same negative tail as F but also having density

$$\frac{1}{x \log^{5/2} x}, \qquad x \ge 10,$$

with the remaining mass placed at zero. We let $\{Y_k\}$ be a sequence of independent random variables having distribution H. The $\{X_k\}$ will have distribution F as usual. For either distribution we have

$$g(x) \sim G(x) \sim (\log x)^{-1}.$$

Then

$$\int_0^\infty \frac{1}{g(x)}\, dH(x) < \infty$$

so by Lemma 8.1

(9.17) $$\lim_{n\to\infty} \textstyle\sum_{i=1}^n Y_i^+ / \sum_{i=1}^n Y_i^- = 0 \quad \text{a.s.}$$

Now observe that for $x \ge 10$

$$P\{|Y| > x\} = \frac{1}{\log x} + \frac{2}{3\log^{3/2} x} > \frac{1}{\log x/2} \geq P\{2|X| > x\}.$$

This means that it is possible to construct the $\{X_i\}$, $\{Y_i\}$ sequences in such a way that $|Y_i| \geq 2|X_i|$. Then

$$-\textstyle\sum_{i=1}^n Y_i^- (1 + \sum_{i=1}^n Y_i^+ / \sum_{i=1}^n Y_i^-) \leq 2S_n.$$

Divide this by β_n and use (9.17) and the fact that $-Y^-$ has the same distribution as X. We obtain

$$\limsup_{n\to\infty} \frac{S_n}{\beta_n} \leq 2 \limsup_{n\to\infty} \frac{S_n}{\beta_n}$$

and so this lim sup must be nonnegative or minus infinity. Since the S_n are nonpositive this means that lim sup $\beta_n^{-1} S_n$ is either zero or minus infinity for any norming sequence $\{\beta_n\}$.

10. The one-sided Strassen converse. This result follows easily from a theorem of Kesten [10] and the work of Klass [14]. This result has been proved independently by Rosalsky [24]. His paper also contains some complementary results.

THEOREM 10.1. *If*

$$\limsup_{n\to\infty} \frac{S_n}{(2n \log\log n)^{1/2}} = 1 \quad \text{a.s.,} \tag{10.1}$$

then $EX = 0$, $EX^2 = 1$.

PROOF. Kesten proves that whenever $E|X| = \infty$ then

$$\limsup_{n\to\infty} n^{-1} S_n = \pm\infty \quad \text{a.s.}$$

Under (10.1), lim sup $n^{-1}S_n = 0$ a.s. so we must have $E|X| < \infty$ and then $EX = 0$ by the strong law. Thus we may consider that we have centered at ES_n in (10.1). Now by Theorem 7.5,

$$\limsup_{n\to\infty} \frac{S_n}{\beta_n} > 0 \quad \text{a.s.} \tag{10.2}$$

where β_n is given by (2.15); the lim sup may even be infinite. If $EX^2 = \infty$, then $x^2 f(x) \to \infty$ which implies

$$\frac{\beta_n^2}{n \log\log n} \geq \frac{a_n^2 (\log\log n)^2}{n \log\log n} = a_n^2 f(a_n) \to \infty$$

and then (10.2) contradicts (10.1). Thus $EX^2 < \infty$ and must equal one by the Hartman-Wintner theorem.

11. Open problems. The first two problems are those mentioned in the introduction:

PROBLEM 1. Find necessary and sufficient conditions for there to exist a monotone norming sequence $\{\beta_n\}$ such that

$$\limsup_{n\to\infty} \frac{S_n - ES_n}{\beta_n} = 1 \quad \text{a.s.}$$

PROBLEM 2. Find necessary and sufficient conditions for there to exist a monotone norming sequence $\{\beta_n\}$ such that

$$\limsup_{n\to\infty} \frac{S_n}{\beta_n} = c \quad \text{a.s.}$$

with c a finite nonzero constant. In this case c might be positive in some cases and negative in others.

Solving Problem 2 will presumably result in a better understanding of the phenomenon illustrated by Example 9.7 when there is no correct norming sequence even though the summands are negative. As an indication of the lack of knowledge about this case there is no known criterion to distinguish whether lim sup $\beta_n^{-1}S_n$ is minus infinity or zero for a given sequence $\{\beta_n\}$ even in the case of Example 9.7.

PROBLEM 3. For an appropriate centering sequence $\{\alpha_n\}$, which may need to be a slight modification of EV_n, find the right condition on the positive tail so that

$$0 < \limsup_{n\to\infty} \frac{S_n - \alpha_n}{a_n \log\log n} < \infty \quad \text{a.s.} \tag{11.1}$$

This problem is different from the ones for centering at the expectation or median of S_n since $a_n \log\log n$ may be much smaller than the norming sequences used in those problems. We can show that if

$$\Sigma\, P\{X > a_n(\log\log n)^{\lambda}\} < \infty$$

for some $\lambda < 1$ then (11.1) is true with $\alpha_n = nE(-a_n \vee (X \wedge a_n \log\log n))$ provided that a_n is defined by $f(a_n) = \delta n^{-1} \log\log n$ and $\delta < 1/35$. Although this is certainly not the right condition it is enough to show that (11.1) is valid when $E(X^+)^2 < \infty$ since in any case $a_n \geq cn^{1/2}\{\log\log n\}^{-1/2}$.

This is an important problem since $a_n \log\log n$ may be significantly smaller than the other norming sequences. Although there cannot really be any best place to center in the one-sided problem as we pointed out in the introduction, it seems that some truncated mean such as EV_n may be the best universal centering sequence.

Finally, it should be possible to tighten the bounds on the constant value of the lim sup in some of the problems. We did not consider this because we started working on the problem of centering at the median and it seems highly unlikely that very good bounds can be obtained in general for this problem because of the possible erratic behavior of the median. However, Klass obtained very tight bounds for the case of centering at the mean and this can probably also be done when centering at zero or EV_n.

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