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INVARIANT MEASURES AND LONG TIME BEHAVIOUR OF THE SMOOTHING PROCESS

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The invariant measures with finite first moment of the smoothing process are characterized in terms of the harmonic functions. It is also shown that under some restriction on the initial configuration, the difference between the mass at a given site and the mean of the masses of its neigbours converges in probability to zero.

1. Introduction. Spitzer (1981) introduced several Markov processes with an infinite number of interacting components. Among these is the smoothing process, which can be described in the following way: a finite or countable set S is given as well as a probability matrix $p(x, y)$ on S and a collection of independent Poisson processes N_x of parameter one indexed by S. The state space of the process is either $[0, \infty)^S$ or an appropriate subset of it and its evolution is given by this rule: when an event of the Poisson process N_x occurs the configuration ω becomes

$$\omega_x(y) = \begin{cases} \omega(y) & \text{if} \quad y \neq x \\ \sum_{z\in S} p(x, z)\omega(z) & \text{if} \quad y = x. \end{cases}$$

The existence of the process in the whole space $[0, \infty)^S$ offers no difficulty if S is finite. However when S is infinite some complications arise. Liggett and Spitzer (1981) constructed the process for an infinite S in a subspace Γ of $[0, \infty)^S$. We describe now their construction and state some related results that will be needed later. The subspace Γ is obtained through a strictly positive real-valued function γ on S satisfying the following property: there exists a constant M such that for all $y \in S$

$$\sum_{x\in S} \gamma(x)p(x, y) \leq M\gamma(y). \tag{1.1}$$

Now Γ is defined as the set of elements $\omega \in [0, \infty)^S$ which satisfy

$$\| \omega \| = \sum_{x\in S} \omega(x)\gamma(x) < \infty.$$

Note that if $M > 1$ and z_0 is an arbitrary element of S then

$$\gamma(x) = \sum_{n=0}^{\infty} M^{-n}p^{(n)}(z_0, x) \tag{1.2}$$

satisfies (1.1) Furthermore if $p(x, y)$ is irreducible $\gamma(x) > 0$ for all $x \in S$. Throughout this paper it will be assumed that p is irreducible and γ is given by (1.2)

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Once Γ is determined, it is endowed with the Borel σ-algebra induced by the product topology on $[0, \infty)^S$. Then $\mathscr{L}$ is defined as the set of real-valued functions f on Γ for which there exists a constant $L(f)$ such that

$$|f(\omega_1) - f(\omega_2)| \leq L(f) \sum_{x\in S} |\omega_1(x) - \omega_2(x)| \gamma(x)$$

for all $\omega_1, \omega_2 \in \Gamma$. Finally Liggett and Spitzer (1981) used finite approximations to construct the process by means of a semigroup of linear operators $S(t)$ acting on $\mathscr{L}$. They also proved that $S(t)$ enjoys the following desirable properties: For all $f \in \mathscr{L}$ and $\omega \in \Gamma$ we have

(1.3) (a).
$$\lim_{t\downarrow 0} \frac{S(t)f(\omega) - f(\omega)}{t} = \Omega f(\omega)$$

where Ω is the natural generator of the process (i.e. $\Omega f(\omega) = \sum_{x\in S}(f(\omega_x) - f(\omega)))$

(1.4) (b).
$$S(t)f(\omega) = f(\omega) + \int_0^t \Omega S(s)f(\omega)\, ds$$

(1.5) (c).
$$\Omega S(s)f(\omega) = S(s)\Omega f(\omega)$$

(1.6) (d). $S(s)f(\omega)$ is continuous in s

(1.7) (e). $\Omega S(s)f(\omega)$ is continuous in s.

Given a probability measure μ on Γ, $\mu S(t)$ is defined as the unique measure such that

$$\int f d(\mu S(t)) = \int S(t)f\, d\mu$$

for all bounded $f \in \mathscr{L}$. As usual $\mu S(t)$ represents the distribution of the process at time t if the initial distribution is μ. For this reason a probability measure μ will be called invariant if $\mu S(t) = \mu$ for all $t \geq 0$. The set of invariant probability measures will be denoted by $\mathscr{I}$. Liggett and Spitzer (1981) proved that a measure μ such that $\int \|\omega\|\, d\mu < \infty$ is in $\mathscr{I}$ if and only if $\int \Omega f\, d\mu = 0$ for all $f \in \mathscr{L}$.

One of the basic problems concerning infinite systems is to determine the set $\mathscr{I}$. When $S = \mathbb{Z}^d$, $p(x, y) = p(0, y - x)$ and p is irreducible it has been proved by Liggett and Spitzer (1981) that all the invariant measures which are also invariant under translations in $\mathbb{Z}^d$ concentrate on constant configurations. In this paper S will have no structure, and we will not require measures to be translation invariant. To exhibit a class of invariant measures we recall that a function β on S is called harmonic if

$$\beta(x) = \sum_{y\in S} p(x, y)\beta(y) \quad \text{for all} \quad x \in S.$$

Note that with our choice of γ all positive harmonic functions β are in Γ and the invariance of δ_β, the point mass at β, is an immediate consequence of the statement following the definition of $\mathscr{I}$. The question that arises is whether there are elements in $\mathscr{I}$ which are not mixtures of these measures. The following theorem says that there are not if we restrict ourselves to measures with finite first moment.

THEOREM 1.8. *If $p(x, y)$ is irreducible, $\mu \in \mathscr{I}$ and $\int \omega(z)\, d\mu(\omega) < \infty$ for some $z \in S$ then*

$$\int |\omega(x) - \textstyle\sum_y p(x, y)\omega(y)|\, d\mu(\omega) = 0 \quad \text{for all} \quad x \in S.$$

A natural question to ask is whether there are invariant measures that do not concentrate on the set of positive harmonic functions. Of course if μ is such a measure we must have $\int \omega(x)\, d\mu(\omega) = \infty$ for all $x \in S$. The coupling techniques used by Andjel (1982, Sections 4 and 6) give a negative answer to this question if $p(x, y)$ is recurrent, but the problem remains open if $p(x, y)$ is transient.

To prove the theorem above we need to define another process, the potlatch process, described in the next paragraph and introduced by Spitzer (1981). A proposition concerning this process is stated and proved in Section 2 while the proof of the theorem is in Section 3. In Section 2 we also obtain some information about the long time behaviour of the smoothing process. This as well as Theorem (1.8) follow from the proposition of Section 2 and a duality relation between the potlatch and smoothing processes stated at the end of this introduction.

As in the case of the smoothing process, a finite or countable set S is given as well as a probability matrix $p(x, y)$ on S and independent Poisson processes $N_x (x \in S)$ of parameter one. For the purposes of this paper it is convenient to construct the process on the following state space:

$$\bar{\Gamma} = \{\lambda \in [-1, 1]^S \colon \textstyle\sum_{x\in S} \lambda^+(x) \le 1 \text{ and } \sum_{x\in S} \lambda^-(x) \le 1\}$$

where $\lambda^+(x) = \max\{0, \lambda(x)\}$ and $\lambda^-(x) = \max\{0, -\lambda(x)\}$. The potlatch process evolves like this: when an event of the Poisson process N_x occurs, the configuration λ becomes

$$\lambda_x(y) = \begin{cases} \lambda(y) + \lambda(x)p(x, y) & \text{if } \ y \neq x \\ \lambda(x)p(x, x) & \text{if } \ y = x. \end{cases}$$

This process is a modified version of the one studied by Liggett and Spitzer (1981). It can be constructed using their method; more precisely we let

$$\delta(x) = \textstyle\sum_{n=0}^{\infty} M^{-n} p^n(x, x_0)$$

for some fixed $x_0 \in S$ and $M > 1$. Then $\mathscr{L}$ is defined as the set of real-valued functions f on $\bar{\Gamma}$ such that there exists a constant $L(f)$ satisfying

$$|f(\lambda_1) - f(\lambda_2)| \le L(f) \textstyle\sum_{x\in S} |\lambda_1(x) - \lambda_2(x)|\, \delta(x)$$

for all λ_1, λ_2 in $\bar{\Gamma}$. Finally using finite approximations one obtains a semigroup $\bar{S}(t)$ acting on $\mathscr{L}$ and such that for all $f \in \mathscr{L}$ and $\lambda \in \bar{\Gamma}$

(a) $$\lim_{t\downarrow 0} \frac{\bar{S}(t)f(\lambda) - f(\lambda)}{t} = \bar{\Omega}f(\lambda) \quad \text{where} \quad \bar{\Omega}f = \textstyle\sum_{x\in S} (f(\lambda_x) - f(\lambda))$$

(b) $$\bar{S}(t)f(\lambda) = f(\lambda) + \int_0^t \bar{\Omega}\bar{S}(s)f(\lambda)\, ds = f(\lambda) + \int_0^t \bar{S}(s)\bar{\Omega}f(\lambda)\, ds.$$

Given an initial distribution, one obtains the distribution of the process at time t in the same way as in the case of smoothing process. For both processes we adopt the following notation: if η is an initial configuration then η_t denotes the random configuration observed at time t. It follows that for the smoothing (potlatch) process $E^{\omega}f(\omega_t) = S(t)f(\omega)$ for all $f \in \mathscr{L}$ and all $\omega \in \Gamma$ ($E^{\lambda}f(\lambda_t) = \bar{S}(t)f(\lambda)$ for all $f \in \bar{\mathscr{L}}$ and all $\lambda \in \bar{\Gamma}$). This equality allows us to extend $S(t)$ ($\bar{S}(t)$) to all positive functions f on $\Gamma(\bar{\Gamma})$.

The duality statement connecting the two processes that will be needed in this paper is the next proposition. We recall that $\gamma(x)$ is defined by (1.2).

PROPOSITION 1.9. *If $\omega \in \Gamma$, $\lambda \in \bar{\Gamma}$ and $|\lambda(x)| \le c\gamma(x)$ for some constant c and all $x \in S$, then $\sum_{x\in S} \omega_t(x)\lambda(x)$ and $\sum_{x\in S} \omega(x)\lambda_t(x)$ converge absolutely with probability one and*

$$E^{\omega}|\sum_{x\in S} \omega_t(x)\lambda(x)| = E^{\lambda}|\sum_{x\in S} \omega(x)\lambda_t(x)|.$$

This duality relation was first mentioned by Spitzer (1981). It has been proved in similar contexts by Spitzer (1981, for finite S) and Holley and Liggett (1981, for generalized versions of these processes taking only positive values). Noting that $|\lambda_t(x)| \le (\lambda^+)_t(x) + (\lambda^-)_t(x)$ their proofs can be easily adapted to Proposition 1.9. For this reason we refer the reader to them.

2. A property of the potlatch process. The main result of this section, Proposition (2.8), is a key ingredient in the proofs of Theorem (1.8) and Corollary (2.15). Before stating it we introduce some notation and prove two preliminary lemmas.

Let S_ℓ be an increasing sequence of finite subsets of S such that $S_\ell \uparrow S$ and let

$$p_\ell(x, y) = \begin{cases} p(x, y) & \text{if } x, y \in S_\ell \\ 0 & \text{otherwise.} \end{cases}$$

To each ℓ we associate a potlatch process in the following way: we only consider the Poisson processes N_x, $x \in S_\ell$ and when an event of N_x ($x \in S_\ell$) occurs, the configuration λ (defined in S) becomes:

$$\lambda_x(y) = \begin{cases} \lambda(y) + \lambda(x)p_\ell(x, y) & \text{if } y \ne x \\ \lambda(x)p_\ell(x, x) & \text{if } y = x. \end{cases} \tag{2.1}$$

We denote by $\bar{S}_\ell(t)$ the semigroup of this process.

Let $\tau_x^1, \tau_x^2 \cdots$ be the successive times of the events of $N_x (x \in S)$. Fix $x_0 \in S$, $X = (x_1, \cdots, x_n) \in S^n$ and $t > 0$, then consider the following events

E_1: The sequence $i_1, \cdots, i_{n-1}$ defined by

(a) $i_1 = 1$

(*b*) $\tau_{x_j}^{i_j}$ is the time of the first event of N_{x_j} occuring after $\tau_{x_{j-1}}^{i_{j-1}}$ $j = 2, \cdots, n-1$ satisfies $\tau_{x_{n-1}}^{i_{n-1}} < t$.

E_2: There exists no sequence $i_1, \cdots, i_n$ such that

$$\tau_{x_1}^{i_1} < \tau_{x_2}^{i_2} < \cdots < \tau_{x_n}^{i_n} < t.$$

E_3: If $i_1, \cdots, i_{n-1}$ is the sequence constructed in the definition of E_1 then for each $j = 0, \cdots, n-2$ there exists no sequence $k_0, k_1, \cdots, k_j$ such that

$$\tau_{x_0}^{k_0} < \tau_{x_1}^{k_1} < \cdots < \tau_{x_j}^{k_j} < \tau_{x_{j+1}}^{i_{j+1}}.$$

E_4: There exists no sequence $k_0, k_1, \cdots, k_{n-1}$ such that

$$\tau_{x_0}^{k_0} < \tau_{x_1}^{k_1} < \cdots < \tau_{x_{n-1}}^{k_{n-1}} < t.$$

By $g(t, x_0, x)$ we denote the probability of $\bigcap_{i=1}^4 E_i$. Similarly $r(t, x_0, X)$ will be the probability of $E_1 \cap E_3$.

If $x_0, y_0 \in S$, $X = (x_1, \cdots, x_n) \subset S^n$, $Y = (y_1, \cdots, y_m) \in S^m$ and $q(x, y)$ is a subprobability matrix on S then we let

$$q(x_0, X) = \prod_{i=1}^n q(x_{i-1}, x_i), \quad XY = (x_1, \cdots, x_n, y_1, \cdots, y_m) \in S^{n+m}$$

and

$$Xy_0 = (x_1, \cdots x_n, y_0) \in S^{n+1}$$

It follows from the definitions of g and r that

$$g(t, x_0, X) \leq r(t, x_0, X) \tag{2.2}$$

and

$$g(t, x_0, X) + r(t, x_0, Xy_0) \leq r(t, x_0, X). \tag{2.3}$$

Finally if β is a strictly positive function on S such that

$$\textstyle\sum_y q(x, y)\beta(y) \leq \beta(x) \quad \forall x \in S$$

then we let

$$q^\beta(x, y) = q(x, y)\beta(y)/\beta(x).$$

The new matrix q^β is a subprobability matrix and it will be a probability matrix if β satisfies $\sum_y q(x, y)\beta(y) = \beta(x)$ for all $x \in S$.

Fix a positive harmonic function β and let

$$f(\lambda) = \textstyle\sum |\lambda(x)| \beta(x), \quad f^+(\lambda) = \sum \lambda^+(x)\beta(x), \quad f^-(\lambda) = \sum \lambda^-(x)\beta(x) \tag{2.4}$$

where $\lambda^+(x) = \max\{\lambda(x), 0\}$ and $\lambda^-(x) = \max\{-\lambda(x), 0\}$.

LEMMA (2.5). *If $x_0 \in S_\ell$ and*

$$\lambda_0(y) = \begin{cases} 1 - p(x_0, x_0) & \text{if } y = x_0 \\ -p(x_0, y) & \text{if } y \neq x_0, \end{cases}$$

then

$$\bar{S}_\ell(t)\, f^-(\lambda_0) \leq \textstyle\sum_{n=1}^\infty \sum_{X \in S^n} \beta(x_0)p^\beta(x_0, X)g(t, x_0, X) + \sum_{x \notin S_\ell} p(x_0, x)\beta(x).$$

PROOF. We call $\lambda(x)$ the mass at x; this mass may be positive or negative and when an event of N_x occurs ($x \in S_\ell$) it is distributed among the neighbours of x according to (2.1). Since S is finite, the number of events that occur up to time t is finite with probability one. Therefore, it is possible to follow the trajectories of all the masses jumping from the sites of S. If we start only with the negative masses, the quantity starting at $x_1 \in S$ that follows the path $x_1, x_2, \cdots, x_n$ ($x_i \in S_\ell$ $1 \le i \le n$) is $p(x_0, X)$ where $X = (x_1, \cdots, x_n)$. However, starting with the configuration λ_0, this negative mass may have been cancelled by (a part of) the positive mass starting at x_0. In our estimate of $\bar{S}_\ell(t)f^-(\lambda_0)$ we are only going to consider cancellings of the following type: for some sequence $x_1, \cdots, x_n$ in S the negative mass starting at x_1 and following the path $x_1, \cdots, x_n$ is still at x_n when the positive mass following the path $x_0, x_1, \cdots, x_n$ arrives at x_n. Note that these two masses have the same absolute value $p(x_0, X)$. Hence if this cancelling occurs both masses vanish. If other type of cancellings take place we consider the positive and negative masses to be at the same point of S_ℓ without cancellation. This applies also to the case $x_0 = x_1$: the initial configuration λ_0 is considered as having at x_0 a positive mass of absolute value one and a negative mass of absolute value $p(x_0, x_0)$.

Note that if $X = (x_1, \cdots, x_n)$, $g(t, x_0, X)$ is the probability that the negative mass starting at x_1 and following the path $(x_1, \cdots, x_n)$ is at x_n by time t without having been cancelled in the way described in the previous paragraph; therefore

$$\bar{S}_\ell(t)f^-(\lambda_0) \le \sum_{n=1}^\infty \sum_{X=(x_1,\cdots,x_n)\in(S_\ell)^n} p(x_0, x_1) \cdots p(x_{n-1}, x_n)\beta(x_n)g(t, x_0, X)$$
$$+ \sum_{x\notin S_\ell} p(x_0, x)\beta(x).$$

The right-hand side can be written as:

$$\sum_{n=1}^\infty \sum_{X\in(S_\ell)^n} \beta(x_0)p^\beta(x_0, X)g(t, x_0, X) + \sum_{x\notin S_\ell} p(x_0, x)\beta(x)$$

and the lemma follows.

LEMMA (2.6). *If $x_0 \in S$, $X \in S^n$ and x_n is the last coordinate of X then for any probability matrix q*

$$g(t, x_0, X) + \sum_{k=1}^\infty \sum_{Y\in S^k} q(x_n, Y)g(t, x_0, XY) \le r(t, x_0, X).$$

PROOF. By (2.2) it suffices to show that for all $m \ge 1$

$$\begin{aligned} F(m) = g(t, x_0, X) &+ \sum_{k=1}^{m-1} \sum_{Y\in S^k} q(x_n, Y)g(t, x_0, XY) \\ &+ \sum_{Y\in S^m} q(x_n, Y)r(t, x_0, XY) \le r(t, x_0, X). \end{aligned} \tag{2.7}$$

This is done by induction on m. If $m = 1$ the inequality follows from (2.3). Suppose now that (2.7) holds for $m = N$.

$$\begin{aligned} F(N+1) = g(t, x_0, X) &+ \sum_{k=1}^{N} \sum_{Y\in S^k} q(x_n, Y)g(t, x_0, XY) \\ &+ \sum_{Y\in S^N} \sum_{y\in S} q(x_n, Yy)r(t, x_0, XYy). \end{aligned}$$

By (2.3) this is bounded above by

$$\begin{aligned} &g(t, x_0, X) + \textstyle\sum_{k=1}^{N} \sum_{Y\in S^k} q(x_n, Y)g(t, x_0, XY) \\ &\qquad + \textstyle\sum_{Y\in S^N} \sum_{y\in S} q(x_n, Yy)[r(t, x_0, XY) - g(t, x_0, XY)] \\ &\quad = g(t, x_0, X) + \textstyle\sum_{k=1}^{N} \sum_{Y\in S^k} q(x_n, Y)g(t, x_0, XY) \\ &\qquad + \textstyle\sum_{Y\in S^N} q(x_n, Y)[r(t, x_0, XY) - g(t, x_0, XY)] = F(N) \end{aligned}$$

and the lemma follows by the inductive hypothesis.

PROPOSITION (2.8). *For all $x_0 \in S$ and all positive harmonic function β*

$$\lim_{t\to\infty} \bar{S}(t)f(\lambda_0) = 0$$

where f and λ_0 are as in (2.4) *and* (2.5) *respectively.*

PROOF. Liggett and Spitzer (1981) used different approximations of p and showed that $\lim \bar{S}_\ell(t)g = \bar{S}(t)g$ for all $g \in \mathscr{L}$. However their technique applies to our approximations also. Hence if

$$f_k^-(\lambda) = \textstyle\sum_{x\in S_k} \lambda^-(x)\beta(x), \quad \text{then} \quad \bar{S}(t)f_k^-(\lambda) = \lim_\ell \bar{S}_\ell(t)f_k^-(\lambda).$$

Therefore by Lemma (2.5)

$$\bar{S}(t)f_k^-(\lambda_0) \le \textstyle\sum_{n=1} \sum_{X\in S^n} \beta(x_0)p^\beta(x_0, X)g(t, x_0, X)$$

and by monotone convergence

$$\bar{S}(t)f^-(\lambda_0) \le \textstyle\sum_{n=1}^{\infty} \sum_{X\in S^n} \beta(x_0)p^\beta(x_0, X)g(t, x_0, X) \tag{2.9}$$

where f^- is as in (2.4). The right-hand side of (2.9) can be written as

$$\begin{aligned} &\beta(x_0) \textstyle\sum_{n=1}^{N-1} \sum_{X\in S^n} p^\beta(x_0, X)g(t, x_0, X) + \beta(x_0) \sum_{X=(x_1,\cdots,x_N)\in S^N} p^\beta(x_0, X) \\ &\qquad \cdot [g(t, x_0, X) + \textstyle\sum_{k=1}^{\infty} \sum_{Y\in S^k} p^\beta(x_N, Y)g(t, x_0, XY)] \\ &\quad \le \beta(x_0) \textstyle\sum_{n=1}^{N-1} \sum_{X\in S^n} p^\beta(x_0, X)g(t, x_0, X) \\ &\qquad + \beta(x_0) \textstyle\sum_{X\in S^N} p^\beta(x_0, X)r(t, x_0, X) \end{aligned} \tag{2.10}$$

where the inequality follows from Lemma (2.6).

Fix $\varepsilon > 0$ and let k be such that the hitting time of the origin for a simple symmetric random walk starting at one is less than $k - 1$ with probability larger than $1 - \varepsilon$. Then choose N in such a way that a random walk on S starting at x_0 and with probability transitions given by p^β has probability smaller than ε of visiting at most k different sites after N steps. Fix $X = (x_1, \cdots, x_n)$ and note that E_3 occurs if and only if the negative mass starting at x_1, and following the path $x_1, \cdots, x_n$ is ahead of the positive mass starting at x_0 and following the path $x_0, x_1, \cdots, x_n$ at any time $s \le \tau_{x_{n-1}}^{i_{n-1}}$. Now observe that the number of steps it is ahead is one at time 0 and evolves like this: it remains unchanged as long as the two masses are at the same site. (If X has two equal coordinates this can occur even if the negative mass is ahead of the positive mass.) When the two masses are at different sites it increases by one with probability ½ and decreases by one with the same probability. It follows that the probability of E_3 is less than ε whenever $|X| > k$. Hence

$$\sum_{X\in S^N;|X|>k} p^{\beta}(x_0, X)r(t, x_0, X) \leq \varepsilon \sum_{X\in S^N} p^{\beta}(x_0, X) = \varepsilon.$$

Furthermore from our choice of k and N it follows that

$$\sum_{X\in S^N;|X|\leq k} p^{\beta}(x_0, X) < \varepsilon.$$

Therefore

$$\sum_{X\in S^N} p^{\beta}(x_0, X)r(t, x_0, X) < 2\varepsilon. \tag{2.11}$$

Since $g(t, x_0, X) \to 0$ as $t \to \infty$ we have

$$\lim_{t\to\infty} \sum_{X\in S^n} p^{\beta}(x_0, X)g(t, x_0, X) = 0 \quad \text{for all} \quad n. \tag{2.12}$$

Now (2.9), (2.10), (2.11) and (2.12) combine into $\limsup_{t\to\infty}\bar{S}(t)f^-(\lambda_0) \leq 2\varepsilon\,\beta(x_0)$. Since ε is arbitrary

$$\lim_{t\to\infty} \bar{S}(t)f^-(\lambda_0) = 0. \tag{2.13}$$

Since λ_0 satisfies the condition of Proposition (1.9),

$$E^{\lambda_0}|\sum \lambda_t(x)\beta(x)| = E^{\beta}|\sum \lambda_0(x)\beta_t(x)| = |\sum \lambda_0(x)\beta(x)| = 0. \tag{2.14}$$

Where the evolution of $\lambda_t(\beta_t)$ is the one of the potlatch (smoothing) process and the second equality is a consequence of $\delta_\beta \in \mathscr{I}$. Since $\sum \lambda_t(x)\beta(x)$ converges absolutely we can write

$$\sum \lambda_t(x)\beta(x) = \sum [\lambda_t(x)]^+\beta(x) - \sum [\lambda_t(x)]^-\beta(x).$$

Taking expectations it follows from (2.14) that

$$\bar{S}(t)f^+(\lambda_0) = \bar{S}(t)f^-(\lambda_0)$$

and since $f = f^+ + f^-$ the proposition follows from (2.13).

The following corollary is an immediate consequence of (1.9) and (2.8).

COROLLARY (2.15). Let $p(x, y)$ be irreducible and ω an initial configuration for the smoothing process bounded above by a harmonic function, then $\forall x \in S$

$$|\omega_t(x) - \sum_y p(x, y)\omega_t(y)| \to 0$$

in probability as t goes to infinity.

3. Invariant measures for the smoothing process.

LEMMA 3.1. *If $\mu \in \mathscr{I}$, $p(x, y)$ is irreducible and $\int \omega(z)\, d\mu(\omega) < \infty$ for some $z \in S$, then $g(x) = \int \omega(x)\, d\mu(\omega)$ is finite for all $x \in S$ and g is harmonic.*

PROOF. From the definition of $\mu S(t)$ and the invariance of μ it follows that $\int S(t)f\, d\mu = \int f\, d\mu$ for all bounded $f \in \mathscr{L}$. A standard truncation argument shows that the same equality holds for all positive $f \in \mathscr{L}$. In particular

$$\int S(t)h_z(\omega)\, d\mu(\omega) = \int h_z(\omega)\, d\mu(\omega)$$

where $h_z(\omega) = \omega(z)$, hence by (1.4) and (1.5)

$$\int \left[\int_0^t S(s)\Omega h_z(\omega)\, ds\right] d\mu(\omega) = 0.$$

Therefore

$$\int \left(\int_0^t S(s)(\textstyle\sum_x p(z, x)\omega(x))\, ds\right) d\mu(\omega) = \int \left(\int_0^t S(s)\omega(z)\, ds\right) d\mu(\omega).$$

Using Tonelli's theorem and the invariance of μ on both sides of the equality we get

$$t \textstyle\sum_x p(z, x)g(x) = tg(z).$$

Hence

$$\textstyle\sum_x p(z, x)g(x) = g(z) < \infty.$$

This shows that $g(x) < \infty$ for all x such that $p(z, x) > 0$ and the lemma follows from a standard inductive argument that uses the irreducibility of p.

PROOF OF THEOREM (1.8). Fix $x_0 \in S$ and let

$$v(\omega) = \omega(x_0) - \textstyle\sum_{y \in S} p(x_0, y)\omega(y).$$

Then

$$\int |v(\omega)|\, d\mu(\omega) = \int S(t)\,|v(\omega)|\, d\mu(\omega) = \int E^{\omega}|v(\omega_t)|\, d\mu(\omega)$$

$$= \int E^{\omega}|\textstyle\sum \omega_t(y)\lambda_0(y)|\, d\mu(\omega)$$

where $\lambda_0(y)$ is as in Proposition (2.8). As a consequence of (1.9) this last expression is equal to

$$\int E^{\lambda_0}|\textstyle\sum \omega(y)\lambda_t(y)|\, d\mu(\omega) \leq \int \textstyle\sum \omega(y)E^{\lambda_0}|\lambda_t(y)|\, d\mu(\omega).$$

By Lemma (3.1) the right-hand side is equal to $\sum E^{\lambda_0}|\lambda_t(y)|\beta(y)$ for some positive harmonic function β. By Proposition (2.8) this last expression converges to zero as t goes to infinity. Since $\int |v(\omega)|\, d\mu(\omega)$ is independent of t we must have

$$\int |v(\omega)|\, d\mu(\omega) = 0$$

and the theorem is proved.

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