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ON THE DISTRIBUTIONS OF L_p NORMS OF WEIGHTED UNIFORM EMPIRICAL AND QUANTILE PROCESSES

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The asymptotic distributions of L_p functionals of weighted uniform quantile and empirical processes are studied. The asymptotic laws obtained are represented in terms of Gaussian and Poisson integrals.

1. Introduction. Let $U_1, U_2, \ldots$ be independent uniform-(0, 1) random variables (r.v.'s) with the corresponding order statistics $U_{1,n} \le U_{2,n} \le \cdots \le U_{n,n}$ of the first n of these, and define their uniform empirical quantile function by $U_n(0) = 0$,

$$U_n(s) = U_{k,n}, \qquad (k-1)/n < s \le k/n, \qquad k = 1, \ldots, n,$$

and the uniform quantile process

$$u_n(s) = n^{1/2}(s - U_n(s)), \qquad 0 < s \le 1.$$

Define also the uniform empirical distribution function

$$E_n(s) = \begin{cases} 0, & U_{1,n} > s, \\ k/n, & U_{k,n} \le s < U_{k+1,n}, \qquad k = 1, \ldots, n-1, \\ 1, & U_{n,n} \le s, \end{cases}$$

and the uniform empirical process

$$e_n(s) = n^{1/2}(E_n(s) - s), \qquad 0 \le s \le 1.$$

We assume, without loss of generality, that the underlying probability space $(\Omega, \mathscr{A}, P)$ is so rich that it accomodates all the r.v.'s and processes introduced so far and also later on.

It was shown in M. Csörgő, S. Csörgő, Horváth and Mason (1986) that the optimal Chibisov (1964)–O'Reilly (1974) weak convergence of e_n and u_n in weighted metrics does not even imply the convergence in distribution of the supremum of u_n and e_n for all possible weight functions. Hence the asymptotic behaviour of different functionals of these processes requires separate investigations. In this spirit the notion of weighted supremum convergence of the uniform empirical and quantile processes in terms of the Erdös–Feller–Kolmogorov–Petrovski upper class functions for Brownian motion was introduced in M. Csörgő, S. Csörgő, Horváth and Mason (1986). A complete characterization of

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the asymptotic behaviour of weighted supremum functions of u_n and e_n can be found in Mason (1983, 1985), M. Csörgő and Horváth (1985a), M. Csörgő, Horváth and Steinebach (1987) and M. Csörgő and Mason (1985). When giving a complete characterization of the asymptotic behavior of weighted supremum functionals of the general (not necessarily uniform) quantile process, M. Csörgő and Horváth (1985b) and Horváth (1987) showed that the limit laws obtained describing the tail behaviour of these functionals are governed by the three domains of attraction of extreme value distributions.

In this paper we study the asymptotics of the L_p norms, $1 \le p < \infty$, of weighted u_n and e_n processes. The asymptotic distribution of functionals of the form $\int |u_n(s)|^p/q(s)\,ds$ and $\int |e_n(s)|^p/q(s)\,ds$ will be studied with various limits of integration. Our results are also new for L_1 and L_2. All in all they amount to dealing with weight functions that are the boundary of what is possible in forming Cramér–von Mises type L_2-weighted statistics for empirical distributions and quantiles and their L_p analogs.

Due to symmetry of the processes e_n and u_n, our results will be proven only on the intervals $[1/(n+1), 1/2]$, $[1/(n+1), k_n/n]$ and $[k_n/n, 1/2]$. It will be clear that they can be also formulated in the following forms:

$$a_1(n)\int_{1/(n+1)}^{k_n/n} |Z_n(t)|^p/q(t)\,dt - b_1(n) \to_{\mathscr{D}} \xi_1,$$

$$a_2(n)\int_{k_n/n}^{m_n/n} |Z_n(t)|^p/q(t)\,dt - b_2(n) \to_{\mathscr{D}} \xi_2,$$

$$a_3(n)\int_{m_n/n}^{n/(n+1)} |Z_n(t)|^p/q(t)\,dt - b_3(n) \to_{\mathscr{D}} \xi_3,$$

as $n \to \infty$, where Z_n is either e_n or u_n, $a_i(n)$, $b_i(n)$, $i = 1, 2, 3$, are normalizing sequences, and the ξ_i are nondegenerate r.v.'s, $i = 1, 2, 3$. The various limits of integration are needed to show which parts of the random sample will determine the asymptotic behaviour of weighted L_p norms of u_n and e_n. Intuitively, it is clear that extreme value order statistics will govern the limits in the heavy weighted case, while the middle of the random sample will dominate in the light weighted case. We will usually assume that $\{k_n\}$ and $\{m_n\}$ are sequences of positive numbers such that as $n \to \infty$,

$$1 \le k_n \le n, \qquad k_n \to \infty, \tag{1.1}$$

$$k_n/n \to 0, \tag{1.2}$$

and

$$1 \le m_n \le n, \qquad n - m_n \to \infty, \tag{1.3}$$

$$m_n/n \to 1. \tag{1.4}$$

Assuming (1.1)–(1.4), the independence of the r.v.'s ξ_1, ξ_2 and ξ_3 can be discussed on the basis of a theorem of Rossberg (1967) as it was done by M. Csörgő and Mason (1985) in the sup-norm case. Hence we will omit these details here. The methodology of the present paper was applied to similar studies of the general quantile process in M. Csörgő and Horváth (1985b).

Our results are stated in Section 2. Section 3 examines preliminaries. In Section 4 we prove our L_p results.

2. Results. Let q be a positive function on $(0, 1/2]$, i.e., $\inf_{\delta \le s \le 1/2} q(s) > 0$ for each $0 < \delta < 1/2$. The process $\{B(s);\ 0 \le s \le 1\}$ denotes a Brownian bridge and $\{W(s);\ 0 \le s < \infty\}$ stands for a standard Wiener process throughout.

THEOREM 2.1. *Let q be a positive function on $(0, 1/2]$, $1 \le p < \infty$, and assume*

$$\int_0^{1/2} s^{p/2}/q(s)\, ds < \infty. \tag{2.1}$$

Then with $\{k_n\}$ as in (1.1) *and* (1.2) *we have, as $n \to \infty$,*

$$\int_{k_n/n}^{1/2} |u_n(s)|^p/q(s)\, ds \to_{\mathscr{D}} \int_0^{1/2} |B(s)|^p/q(s)\, ds, \tag{2.2}$$

$$\int_{1/(n+1)}^{k_n/n} |u_n(s)|^p/q(s)\, ds \to_{\mathscr{D}} 0, \tag{2.3}$$

and

$$\int_{U_n(k_n/n)}^{U_n(1/2)} |e_n(s)|^p/q(s)\, ds \to_{\mathscr{D}} \int_0^{1/2} |B(s)|^p/q(s)\, ds, \tag{2.4}$$

$$\int_{U_n(1/n)}^{U_n(k_n/n)} |e_n(s)|^p/q(s)\, ds \to_{\mathscr{D}} 0. \tag{2.5}$$

REMARK 2.1. Combining (2.2) and (2.3) we obtain convergence in distribution with limits of integration from $1/(n+1)$ to $1/2$ immediately. Now we show that

$$\int_0^{1/2} |u_n(s)|^p/q(s)\, ds \to_{\mathscr{D}} \int_0^{1/2} |B(s)|^p/q(s)\, ds, \tag{2.6}$$

if and only if

$$\int_0^{\delta} (1/q(s))\, ds < \infty, \quad \text{for some } \delta > 0. \tag{2.7}$$

We have

$$\begin{aligned} &\int_0^{1/(n+1)} |u_n(s)|^p/q(s)\, ds \\ &\quad \ge n^{p/2} \int_0^{U_{1,n}} (U_{1,n} - s)^{1/2}/q(s)\, ds\, I(U_{1,n} \le 1/(n+1)) \\ &\quad \ge n^{p/2} \int_0^{U_{1,n}/2} (U_{1,n}/2)^p/q(s)\, ds\, I(U_{1,n} \le 1/(n+1)), \end{aligned}$$

where $I(A)$ is the indicator function of the set A. Hence it follows that if (2.7) does not hold, then

$$\liminf_{n \to \infty} P\left\{ \int_0^{1/(n+1)} |u_n(s)|^p/q(s)\, ds = \infty \right\} \ge 1 - e^{-1}. \tag{2.8}$$

The sufficiency part is immediate, since

$$\int_0^{1/(n+1)} |u_n(s)|^p/q(s)\,ds = O_P(1)\int_0^{1/(n+1)} (1/q(s))\,ds = o_P(1),$$

by (2.7).

REMARK 2.2. Assuming (2.1) we have

$$\begin{aligned}\int_0^{U_n(1/n)} |e_n(s)|^p/q(s)\,ds &= n^{p/2}\int_0^{U_n(1/n)} s^p/q(s)\,ds\\ &= (nU_{1,n})^{p/2}\int_0^{U_n(1/n)} s^{p/2}/q(s)\,ds = o_P(1),\end{aligned}$$

and hence by (2.4), (2.5) and the latter line we get

$$\int_0^{U_n(1/2)} |e_n(s)|^p/q(s)\,ds \to_{\mathscr{D}} \int_0^{1/2} |B(s)|^p/q(s)\,ds.$$

We note also that condition (2.1) is equivalent to

$$E\int_0^{1/2} |B(s)|^p/q(s)\,ds < \infty,$$

which implies that the asymptotic r.v. of Theorem 2.1 is well defined. Shepp (1966) showed that if $p = 2$, then (2.1) is equivalent to the almost sure finiteness of $\int_0^{1/2} B^2(s)/q(s)\,ds$.

From now on we assume that the weight function q is regularly varying at zero. This means that $q(s) = s^\nu L(s)$, $-\infty < \nu < \infty$, where L is a slowly varying function, i.e., $L(s)$ is positive on $(0, 1/2]$, Lebesgue measurable and

$$\lim_{s \downarrow 0} L(\lambda s)/L(s) = 1, \quad \text{for all } \lambda > 0. \tag{2.9}$$

We note that condition (2.1) holds true for all q, regularly varying at zero with exponent $\nu < 1 + p/2$.

THEOREM 2.2. *Let L be slowly varying at zero and $\{k_n\}$ be as in* (1.1) *and* (1.2). *If* $-\infty < \nu < 1 + p/2$, *then as* $n \to \infty$,

$$\begin{aligned}&\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{1/(n+1)}^{k_n/n} \frac{|u_n(s)|^p}{s^\nu L(s)}\,ds\\ &\qquad \to_{\mathscr{D}} \int_0^1 s^{-\nu}|W(s)|^p\,ds,\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}&\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{U_n(1/n)}^{U_n(k_n/n)} \frac{|e_n(s)|^p}{s^\nu L(s)}\,ds\\ &\qquad \to_{\mathscr{D}} \int_0^1 s^{-\nu}|W(s)|^p\,ds.\end{aligned} \tag{2.11}$$

REMARK 2.3. If $-\infty < \nu < 1$, then

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right) \int_0^{k_n/n} \frac{|u_n(s)|^p}{s^\nu L(s)}\, ds \to_{\mathscr{D}} \int_0^1 s^{-\nu}|W(s)|^p\, ds. \tag{2.12}$$

If $1 \leq \nu < p/2 + 1$ and

$$\int_0^\delta 1/(s^\nu L(s))\, ds = \infty, \quad \text{for some } \delta > 0, \tag{2.13}$$

then

$$\liminf_{n\to\infty} P\left\{\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right) \int_0^{k_n/n} \frac{|u_n(s)|^p}{s^\nu L(s)}\, ds = \infty\right\} \geq 1 - e^{-1}. \tag{2.14}$$

We also note that when $1 < \nu < p/2 + 1$, then (2.13) holds for any slowly varying function L.

REMARK 2.4. If $-\infty < \nu < p/2 + 1$, then

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right) \int_0^{U_n(k_n/n)} \frac{|e_n(s)|^p}{s^\nu L(s)}\, ds \to_{\mathscr{D}} \int_0^1 s^{-\nu}|W(s)|^p\, ds. \tag{2.15}$$

The theorems and remarks introduced thus far are concerned with light weight functions. In M. Csörgő and Horváth (1985a) we pointed out that there is a distinct borderline between light and heavy weights in the supremum norm. In the latter case the weight function $(s(1-s))^{1/2}$, the standard deviation of a Brownian bridge, is the separating line, first studied by Eicker (1979) and Jaeschke (1979). This weight function cannot be replaced by $s^{1/2}L(s)$, as it was noted in Remark 2.3 of M. Csörgő and Horváth (1985a). The next theorem is an L_p-approximations version of the Eicker–Jaeschke theorems.

THEOREM 2.3. *As $n \to \infty$ we have*

$$\int_{1/(n+1)}^{n/(n+1)} \frac{\left||u_n(s)|^p - |B_n(s)|^p\right|}{s(1-s)^{p/2+1}}\, ds = O_P(1), \tag{2.16}$$

and

$$\int_{\lambda/(n+1)}^{1-\lambda/(n+1)} \frac{\left||e_n(s)|^p - |B_n(s)|^p\right|}{s(1-s)^{p/2+1}}\, ds = O_P(1), \quad \textit{for all } \lambda > 0, \tag{2.17}$$

$$\int_{U_{1,n}}^{U_{n,n}} \frac{\left||e_n(s)|^p - |B_n(s)|^p\right|}{s(1-s)^{p/2+1}}\, ds = O_P(1), \tag{2.18}$$

where the $\{B_n\}$ are Brownian bridges as in Theorem 3.1.

This theorem enables one to study the corresponding L_p norm of a Brownian bridge for the sake of deducing limit theorems for the L_p norms of weighted empirical processes. In particular, we obtain the following interesting central limit theorems.

COROLLARY 2.1. *Let $n/(n+1) \le k_n < m_n \le n^2/(n+1)$, and assume that as $n \to \infty$,*

$$\frac{m_n}{k_n}\frac{n-k_n}{n-m_n} \to \infty. \tag{2.19}$$

Then we have

$$\left(2D\log\left(\frac{m_n}{k_n}\frac{n-k_n}{n-m_n}\right)\right)^{-1/2}\left\{\int_{k_n/n}^{m_n/n}\frac{|u_n(s)|^p}{(s(1-s))^{p/2+1}}\,ds - m\log\left(\frac{m_n}{k_n}\frac{n-k_n}{n-m_n}\right)\right\} \to_{\mathscr{D}} N(0,1), \tag{2.20}$$

where $D = D(p)$ is a positive constant, $m = m(p) = E|N(0,1)|^p$ and $N(0,1)$ stands for the standard normal r.v.

COROLLARY 2.2. *Let $0 \le k_n < m_n \le n$ and assume* (2.19). *Then, with $k_n^* = \max(k_n, 1)$ and $m_n^* = \min(m_n, n-1)$, we have*

$$\left(2D\log\left(\frac{m_n^*}{k_n^*}\frac{n-k_n^*}{n-m_n^*}\right)\right)^{-1/2}\left\{\int_{U_n(k_n/n)}^{U_n(m_n/n)}\frac{|e_n|^p}{(s(1-s))^{p/2+1}}\,ds - m\log\left(\frac{m_n^*}{k_n^*}\frac{n-k_n^*}{n-m_n^*}\right)\right\} \to_{\mathscr{D}} N(0,1), \tag{2.21}$$

where $D = D(p)$, $m = m(p)$ and $N(0,1)$ are as in Corollary 2.1.

We note, for example, that if $p = 2$, then $D = D(2) = 2$ and, of course, $m = m(2) = 1$.

Next we study the L_p convergence of u_n and e_n with heavier weights $[p/2 + 1 < \nu < \infty$ in our weight functions of the form $s^\nu L(s)]$. The first such result is concerned with integrating over the middle part of the unit interval for the empirical processes involved. This, in turn, will result in Gaussian asymptotic behaviour.

THEOREM 2.4. *Let L be slowly varying at zero and $\{k_n\}$ be as in* (1.1) *and* (1.2). *If $p/2 + 1 < \nu < \infty$, then as $n \to \infty$,*

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{k_n/n}^{1/2}\frac{|u_n(s)|^p}{s^\nu L(s)}\,ds \to_{\mathscr{D}} \int_0^1 s^{\nu-(p+2)}|W(s)|^p\,ds, \tag{2.22}$$

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{U_n(k_n/n)}^{U_n(1/2)}\frac{|e_n(s)|^p}{s^\nu L(s)}\,ds \to_{\mathscr{D}} \int_0^1 s^{\nu-(p+2)}|W(s)|^p\,ds. \tag{2.23}$$

We note that in all of our results so far $U_n(t)$ can be replaced by t in the limits of integration.

Let

$$[\![x]\!] = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ [x] - 1, & \text{if } x \text{ is an integer.} \end{cases}$$

We also introduce the partial sums $S_1, S_2, \dots$ of independent exponential r.v.'s with expectation 1 and the corresponding Poisson process

$$N(t) = \sum_{i=1}^{\infty} iI((S_i < t \le S_{i+1}]), \qquad 0 < t < \infty, \tag{2.24}$$

where $I(A)$ is the indicator function of set A.

When integrating over the tail parts of the unit interval, the L_p behaviour of u_n and e_n ceases to be Gaussian in the limit in the heavy weighted case.

THEOREM 2.5. *Let L be slowly varying at zero and $\{k_n\}$ be as in* (1.1) *and* (1.2). *If $p/2 + 1 < \nu < \infty$, then as $n \to \infty$,*

$$n^{(p/2+1)-\nu}L(1/n)\int_{\lambda/n}^{k_n/n}|u_n(s)|^p/(s^\nu L(s))\,ds \to_{\mathscr{D}} \int_{\lambda}^{\infty} t^{-\nu}\left|S_{[\![t]\!]+1} - t\right|^p dt, \tag{2.25}$$

for all $\lambda > 0$.

It is clear from the form of this limiting distribution that λ cannot be replaced by $\lambda_n \to 0$.

THEOREM 2.6. *Let L be slowly varying at zero and $\{k_n\}$ be as in* (1.1) *and* (1.2). *If $p/2 + 1 < \nu < \infty$, then as $n \to \infty$,*

$$n^{(p/2+1)-\nu}L(1/n)\int_{U_n(1/n)}^{U_n(k_n/n)}|e_n(s)|^p/(s^\nu L(s)) \to \int_{S_1}^{\infty} t^{-\nu}|N(t) - t|^p\,dt, \tag{2.26}$$

and for all $\lambda > 0$,

$$\begin{aligned} &n^{(p/2+1)-\nu}L(1/n)\int_{\lambda/n}^{U_n(k_n/n)}|e_n(s)|^p/(s^\nu L(s))\,ds \\ &\qquad \to_{\mathscr{D}} \int_{\lambda}^{\infty} t^{-\nu}|N(t) - t|^p\,dt. \end{aligned} \tag{2.27}$$

REMARK 2.5. If $p/2 + 1 < \nu < p + 1$, then

$$\begin{aligned} &n^{(p/2+1)-\nu}L(1/n)\int_{0}^{U_n(k_n/n)}|e_n(s)|^p/(s^\nu L(s))\,ds \\ &\qquad \to_{\mathscr{D}} \int_{0}^{\infty} t^{-\nu}|N(t) - t|^p\,dt, \end{aligned} \tag{2.28}$$

and if $p + 1 \le \nu < \infty$, then

$$n^{(p/2+1)-\nu}L(1/n)\int_{0}^{U_n(k_n/n)}|e_n(s)|^p/(s^\nu L(s))\,ds \to_{\mathscr{D}} \infty. \tag{2.29}$$

In (2.26)–(2.29) we can replace $U_n(k_n/n)$ by k_n/n.

3. Tools and background. One of the main tools in this paper is the following weighted approximation of M. Csörgő, S. Csörgő, Horváth and Mason (1986).

THEOREM 3.1. *We can define a sequence of Brownian bridges* $\{B_n(s);\ 0 \le s \le 1\}$ *such that as* $n \to \infty$,

$$\sup_{1/(n+1) \le s \le n/(n+1)} n^{\alpha}|u_n(s) - B_n(s)|/(s(1-s))^{1/2-\alpha} = O_P(1), \tag{3.1}$$

for every $0 \le \alpha < 1/2$, *and for all* $\lambda > 0$,

$$\sup_{\lambda/(n+1) \le s \le 1-\lambda/(n+1)} n^{\beta}|e_n(s) - B_n(s)|/(s(1-s))^{1/2-\beta} = O_P(1), \tag{3.2}$$

for every $0 \le \beta < 1/4$.

For a simple proof of Theorem 3.1 we refer to M. Csörgő and Horváth (1986) and Mason (1986).

The next useful results are due to Wellner (1978).

THEOREM 3.2. *For any* $0 < \varepsilon < 1$ *there exist positive constants* λ_1 *and* λ_2 *such that we have for all* n,

$$P\{\lambda_1 s \le U_n(s) \le \lambda_2 s;\ 1/(n+1) \le s \le n/(n+1)\} \ge 1 - \varepsilon \tag{3.3}$$

and

$$P\{\lambda_1 s \le E_n(s) \le \lambda_2 s;\ 0 \le s \le 1\} \ge 1 - \varepsilon. \tag{3.4}$$

Now using the well-known representation of uniform order statistics in terms of partial sums S_k of exponential r.v.'s

$$\{U_{k,n};\ 1 \le k \le n\} =_{\mathscr{D}} \{S_k/S_{n+1};\ 1 \le k \le n\} \tag{3.5}$$

and the weak law of large numbers we easily get the following theorem of Smirnov (1949).

THEOREM 3.3. *Let* $\{k_n\}$ *be as in* (1.1). *Then as* $n \to \infty$,

$$(n/k_n)U_n(k_n/n) \to_P 1.$$

Let $\{V(t);\ -\infty < t < \infty\}$ be the Ornstein–Uhlenbeck process with $EV(t) = 0$ and covariance function $\exp(-|t-s|)$. This process is a stationary Gaussian Markov process. We need the following central limit theorem.

THEOREM 3.4. *As* $T \to \infty$,

$$(DT)^{-1/2}\left(\int_0^T |V(t)|^p\,dt - mT\right) \to_{\mathscr{D}} N(0,1), \tag{3.6}$$

where $D = D(p)$ *is a positive constant*,

$$m = m(p) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} |x|^p \exp(-x^2/2)\,dx,$$

and $N(0,1)$ *stands for the standard normal r.v.*

A long elementary calculation gives $D = D(2) = 2$.

Mandl (1968) proved (3.6) when the starting point of V is fixed. Following the proof of Theorem 9 on page 94 in Mandl (1968) we obtain (3.6).

Additional tools needed in this paper are the results for regularly varying functions. The following results are from de Haan (1975) (cf. Corollary 1.2.1 and Theorem 1.2.1).

THEOREM 3.5. *Let L be a slowly varying function at zero. Then*

$$\lim_{s\downarrow 0}\sup_{a\le\lambda\le b}\left|\frac{L(s)}{L(\lambda s)}-1\right| = 0, \qquad 0<a\le b<\infty. \tag{3.7}$$

Also, if $\gamma < 2$, then

$$K_{1-\gamma}(s) = \frac{s^{2-\gamma}/L(s)}{\int_0^s t^{1-\gamma}/L(t)\,dt} \to 2-\gamma, \qquad s\downarrow 0, \tag{3.8}$$

and if $\int_0^\delta 1/(tL(t))\,dt < \infty$ for some $\delta > 0$, then

$$\lim_{s\downarrow 0}\frac{1/L(s)}{\int_0^s 1/(tL(t))\,dt} = 0. \tag{3.9}$$

The results of the next theorem were obtained by M. Csörgő and Horváth (1985a).

THEOREM 3.6. *Let L be a slowly varying function at zero and $\{k_n\}$ as in* (1.1) *and* (1.2). *Then for all $\mu > 0$, as $n \to \infty$, we have*

$$k_n^{-\mu}\sup_{1/(n+1)\le s\le k_n/n} L(k_n/n)/L(s) = O(1), \tag{3.10}$$

$$\sup_{1/k_n\le s\le 1} s^{\mu}L(k_n/n)/L(sk_n/n) = O(1), \tag{3.11}$$

$$\max_{1\le i\le k_n} i^{-\mu}L(1/n)/L(i/n) = O(1), \tag{3.12}$$

$$(k_n/n)^{\mu}\sup_{k_n/n\le s\le 1/2} L(k_n/n)/(s^{\mu}L(s)) = O(1), \tag{3.13}$$

and

$$\sup_{1/(n+1)\le s\le k_n/n}(ns)^{-\mu}\left|\frac{L(1/n)}{L(s)}-1\right| = o(1). \tag{3.14}$$

4. Proofs.

PROOF OF THEOREM 2.1. Given any $0 < \varepsilon < 1/2$, by (3.1) we get

$$\int_\varepsilon^{1/2}\big||u_n(s)|^p - |B_n(s)|^p\big|/q(s)\,ds = o_p(1). \tag{4.1}$$

By Markov's inequality we obtain

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} P\left\{ \int_{1/(n+1)}^{\varepsilon} |B_n(s)|^p / q(s)\, ds > \delta \right\} = 0, \tag{4.2}$$

for all $\delta > 0$. Now using (3.1) with $\alpha = 0$, we have

$$\begin{aligned} &\int_{1/(n+1)}^{\varepsilon} |u_n(s)|^p / q(s)\, ds \\ &\quad \le 2^p \int_{1/(n+1)}^{\varepsilon} |u_n(s) - B_n(s)|^p / q(s)\, ds \\ &\qquad + 2^p \int_{1/(n+1)}^{\varepsilon} |B_n(s)|^p / q(s)\, ds \\ &\quad = O_P(1) \int_0^{\varepsilon} s^{p/2} / q(s)\, ds + 2^p \int_{1/(n+1)}^{\varepsilon} |B_n(s)|^p / q(s)\, ds. \end{aligned} \tag{4.3}$$

Therefore

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} P\left\{ \int_{1/(n+1)}^{\varepsilon} |u_n(s)|^p / q(s)\, ds > \delta \right\} = 0, \tag{4.4}$$

for all $\delta > 0$. Hence (2.2) and (2.3) are proven. The proofs of (2.4) and (2.5) are similar, and hence are omitted. □

PROOF OF THEOREM 2.2. For later use we note that if $p \ge 1$, then we have

$$\left| |a|^p - |b|^p \right| \le p 2^{p-1} |a - b|^p + p 2^{p-1} |b|^{p-1} |a - b|. \tag{4.5}$$

Using the Markov inequality, (3.8), and assuming $1 + q/2 - \tau > 0$, $\tau < 1 + q/2$, we obtain

$$\begin{aligned} &\limsup_{n \to \infty} P\left\{ \left(\frac{k_n}{n}\right)^{\tau - (q/2+1)} L\left(\frac{k_n}{n}\right) \int_0^{k_n/n} \frac{|W(t)|^q}{t^\tau L(t)}\, dt > K \right\} \\ &\quad \le \frac{m(q)}{K} \lim_{n \to \infty} \left(\frac{k_n}{n}\right)^{\tau - (q/2+1)} L\left(\frac{k_n}{n}\right) \int_0^{k_1/n} \frac{t^{q/2-\tau}}{L(t)\, dt} \\ &\quad = m(q) \frac{1 + q/2 - \tau}{K}, \end{aligned}$$

for all $K > 0$. Consequently,

$$\left(\frac{k_n}{n}\right)^{\tau - (q/2+1)} L\left(\frac{k_n}{n}\right) \int_0^{k_n/n} \frac{|W(t)|^q}{t^\tau L(t)}\, dt = O_p(1), \tag{4.6}$$

and a similar argument also yields

$$\left(\frac{k_n}{n}\right)^{\tau - (q/2+1)} L\left(\frac{k_n}{n}\right) \int_0^{k_n/n} \frac{|B(t)|^q}{t^\tau L(t)}\, dt = O_P(1). \tag{4.7}$$

Let $0 < \alpha < 1/2$ so that $\nu < p/2 + 1 - p\alpha$. First using (4.5) and then (3.1), we obtain

$$\begin{aligned}
&\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{1/(n+1)}^{k_n/n} \frac{\big||u_n(s)|^p - |B_n(s)|^p\big|}{s^\nu L(s)}\,ds \\
&\quad\le p2^{p-1}\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{1/(n+1)}^{k_n/n} \frac{|u_n(s)-B_n(s)|^p}{s^\nu L(s)}\,ds \\
&\qquad + p2^{p-1}\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{1/(n+1)}^{k_n/n} \frac{|B_n(s)|^{p-1}|u_n(s)-B_n(s)|}{s^\nu L(s)}\,ds \qquad (4.8)\\
&\quad= O_P(1)\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right) n^{-p\alpha}\int_{1/(n+1)}^{k_n/n} \frac{s^{p/2-p\alpha-\nu}}{L(s)}\,ds \\
&\qquad + O_P(1)\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right) n^{-\alpha}\int_{1/(n+1)}^{k_n/n} \frac{B_n(s)^{p-1}}{s^{\nu-1/2+\alpha}L(s)}\,ds \\
&\quad= o_P(1),
\end{aligned}$$

where we have used (3.10) with $\mu = p\alpha/2$, and (4.7). Thus, in order to prove (2.10) it is enough to verify

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{1/(n+1)}^{k_n/n} \frac{|B(s)|^p}{s^\nu L(s)}\,ds \to_{\mathscr{D}} \int_0^1 s^{-\nu}|W(s)|^p\,ds. \qquad (4.9)$$

First we observe that (4.5), (3.8) and (4.6) imply

$$\begin{aligned}
&\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{1/(n+1)}^{k_n/n} \frac{\big||W(s)|^p - |W(s)-sW(1)|^p\big|}{s^\nu L(s)}\,ds \\
&\quad\le p2^{p-1}\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)|W(1)|^p\int_{1/(n+1)}^{k_n/n} s^{p-\nu}/L(s)\,ds \qquad (4.10)\\
&\qquad + p2^{p-1}\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)|W(1)|\int_{1/(n+1)}^{k_n/n} \frac{|W(s)|^{p-1}}{s^{\nu-1}L(s)}\,ds \\
&\quad= o_P(1).
\end{aligned}$$

Integrating, we obtain

$$\begin{aligned}
&\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{1/(n+1)}^{k_n/n} \frac{|W(s)|^p}{s^\nu L(s)}\,ds \qquad (4.11)\\
&\quad= \left(\frac{k_n}{n}\right)^{-p/2} L\left(\frac{k_n}{n}\right)\int_{(1/k_n)(n/(n+1))}^{1} \frac{|W(tk_n/n)|^p}{t^\nu L(tk_n/n)}\,dt.
\end{aligned}$$

Let $0 < \delta < 1$. Then by (3.11) with $0 < \mu < p/2 + 1 - \nu$ we have

$$\begin{aligned}
&\left(\frac{k_n}{n}\right)^{-p/2} L\left(\frac{k_n}{n}\right) \int_{(1/k_n)(n/(n+1))}^{\delta} \left|W\left(\frac{sk_n}{n}\right)\right|^p \Big/ \left(s^\nu L\left(\frac{sk_n}{n}\right)\right) ds \\
&\qquad \le O(1)\left(\frac{k_n}{n}\right)^{-p/2} \int_0^\delta \frac{|W(sk_n/n)|^p}{s^{\nu+\mu}}\, ds \qquad (4.12)\\
&\qquad =_{\mathscr{D}} O(1) \int_0^\delta \frac{|W(s)|^p}{s^{\nu+\mu}}\, ds.
\end{aligned}$$

Consequently, by our choice of μ and ν, we have as $\delta \downarrow 0$,

$$\int_0^\delta |W(s)|^p / s^{\nu+\mu}\, ds \to_P 0. \qquad (4.13)$$

Utilizing (3.7), it follows that

$$\begin{aligned}
&\left(\frac{k_n}{n}\right)^{-p/2} L\left(\frac{k_n}{n}\right) \int_\delta^1 \frac{|W(sk_n/n)|^p}{(s^\nu L(sk_n/n))}\, ds \qquad (4.14)\\
&\qquad =_{\mathscr{D}} (1 + o(1)) \int_\delta^1 s^{-\nu} |W(s)|^p\, ds.
\end{aligned}$$

Hence, by

$$\{B(s); 0 \le s \le 1\} =_{\mathscr{D}} \{W(s) - sW(1); 0 \le s \le 1\} \qquad (4.15)$$

and (4.10)–(4.14), we have (4.9), which completes the proof of (2.10).

Using (3.2) and (3.3) it can be shown, similarly to (4.8), that

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right) \int_{U_n(1/n)}^{U_n(k_n/n)} \frac{\left||e_n(s)|^p - |B_n(s)|^p\right|}{s^\nu L(s)}\, ds = o_P(1). \qquad (4.16)$$

Thus (2.11) will be proved provided we can show

$$\begin{aligned}
&\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right) \int_{U_n(1/n)}^{U_n(k_n/n)} \frac{|B_n(s)|^p}{s^\nu L(s)}\, ds \qquad (4.17)\\
&\qquad \to_{\mathscr{D}} \int_0^1 s^{-\nu} |W(s)|^p\, ds.
\end{aligned}$$

On account of (4.9) we will have (4.17) if we show

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right) \int_{U_n(1/n)\wedge 1/(n+1)}^{U_n(1/n)\vee 1/(n+1)} \frac{|B_n(s)|^p}{s^\nu L(s)}\, ds \to_P 0 \qquad (4.18)$$

and

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} \int_{U_n(k_n/n)\wedge(k_n/n)}^{U_n(k_n/n)\vee(k_n/n)} \frac{|B_n(s)|^p}{s^\nu L(s)}\, ds \to_P 0. \qquad (4.19)$$

For every $\lambda > 1$ we have

$$\lim_{n\to\infty}\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right) E\int_0^{\lambda/n}\frac{|B(s)|^p}{s^\nu L(s)}\,ds = 0, \tag{4.20}$$

by (3.8) and (3.6). Now using Markov's inequality and Theorem 3.2, we obtain (4.18). Let $0 < \delta < 1$. Combining (4.15), (4.10) and (3.7), we get

$$\begin{aligned}&\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{(1-\delta)k_n/n}^{(1+\delta)k_n/n}\frac{|B(s)|^p}{s^\nu L(s)}\,ds\\ &\qquad \to_{\mathscr{D}} \int_{1-\delta}^{1+\delta} s^{-\nu}|W(s)|^p\,ds.\end{aligned} \tag{4.21}$$

As $\delta \downarrow 0$, then

$$\int_{1-\delta}^{1+\delta} s^{-\nu}|W(s)|^p\,ds \to_P 0. \tag{4.22}$$

Consequently, Theorem 3.3, (4.21) and (4.22) result in (4.19). This also completes the proof of (2.11). □

Proof of Remark 2.3. Using (3.8) and (3.10) we obtain

$$\begin{aligned}&\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_0^{1/(n+1)}\frac{|u_n(s)|^p}{s^\nu L(s)}\,ds\\ &\quad\le \left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)2^p n^{p/2}\int_0^{1/(n+1)}\frac{1}{s^\nu L(s)}\,ds\{(U_{1,n})^p + n^{-p}\}\\ &\quad= o_P(1).\end{aligned} \tag{4.23}$$

Hence we get (2.12) from (2.10). The statement of (2.14) follows by (2.8). □

Proof of Remark 2.4. We have by (3.8), (3.3) and (3.10) that

$$\begin{aligned}&\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)n^{p/2}\int_0^{U_n(1/n)}\frac{s^{p-\nu}}{L(s)}\,ds\\ &\quad= O_P(1)\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} n^{p/2}(U_{1,n})^{p-\nu+1}\frac{L(k_n/n)}{L(U_{1,n})}\\ &\quad= o_P(1),\end{aligned} \tag{4.24}$$

and now (2.11) implies (2.15). □

Proof of Theorem 2.3. First we note that it follows by Markov's inequality that

$$n^{-\alpha}\int_{1/(n+1)}^{n/(n+1)}|B_n(s)|^{p-1}/(s(1-s))^{p/2+1/2+\alpha}\,ds = O_P(1), \tag{4.25}$$

for $\alpha > 0$. Using (4.5) and (3.1) with $\alpha > 0$ we get immediately

$$\int_{1/(n+1)}^{n/(n+1)} \frac{\left| |u_n(s)|^p - |B_n(s)|^p \right|}{(s(1-s))^{p/2+1}}\, ds$$

$$\leq p2^{p-1} \int_{1/(n+1)}^{n/(n+1)} \frac{|u_n(s) - B_n(s)|^p}{(s(1-s))^{p/2+1}}\, ds \tag{4.26}$$

$$+p2^{p-1} \int_{1/(n+1)}^{n/(n+1)} \frac{|u_n(s) - B_n(s)| \cdot |B_n(s)|^{p-1}}{(s(1-s))^{p/2+1}}\, ds$$

$$= O_P(1).$$

Hence (2.16) is proven. In a similar way we get (2.17) from (3.2). Now (2.17) and Theorem 3.2 imply (2.18). □

PROOF OF COROLLARY 2.1. It is easy to check that

$$\int_a^b |B(s)|^p/(s(1-s))^{p/2+1}\, ds = 2\int_0^{\log((b/a)(1-a)/(1-b))/2} |V(t)|^p\, dt, \tag{4.27}$$

for all $0 < a, b < 1$. Hence Theorems 2.3 and 3.4 imply (2.20). □

PROOF OF COROLLARY 2.2. Let $1 \leq r_n \leq n - n/(n+1)$. Using Theorems 3.2 and 3.3, routine calculations yield

$$\int_{U_n(r_n/n) \wedge (r_n/n)}^{U_n(r_n/n) \vee (r_n/n)} \frac{|B_n(t)|^p}{(t(1-t))^{p/2+1}}\, dt = O_P(1),$$

$$\int_{U_{n,n}}^1 \frac{|e_n(t)|^p}{(t(1-t))^{p/2+1}}\, dt = O_P(1)$$

and

$$\int_0^{U_{1,n}} \frac{|e_n(t)|^p}{(t(1-t))^{p/2+1}}\, dt = O_P(1).$$

Then (2.21) follows immediately from (2.18) and Theorem 3.4. □

PROOF OF THEOREM 2.4. By (3.13) we get, with any $\alpha > 0$,

$$\lim_{n\to\infty} \left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right) n^{-\alpha} E \int_{k_n/n}^{1/2} \frac{|B_n(s)|^{p-1}}{s^{\nu-1/2+\alpha} L(s)}\, ds = 0. \tag{4.28}$$

Hence by Markov's inequality we have, as $n \to \infty$,

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right) n^{-\alpha} \int_{k_n/n}^{1/2} \frac{|B_n(s)|^{p-1}}{s^{\nu-1/2+\alpha} L(s)}\, ds = o_P(1). \tag{4.29}$$

Now applying (4.5), (3.1) with $\alpha > 0$ and (3.13) with $\mu = \alpha$, we obtain

$$\begin{aligned}
&\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{k_n/n}^{1/2}\frac{\left||u_n(s)|^p - |B_n(s)|^p\right|}{s^\nu L(s)}\,ds \\
&\quad \le \left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)p2^{p-1}\left\{\int_{k_n/n}^{1/2}\frac{|u_n(s)-B_n(s)|^p}{s^\nu L(s)}\,ds\right. \\
&\qquad\qquad \left. + \int_{k_n/n}^{1/2}\frac{|B_n(s)|^{p-1}|u_n(s)-B_n(s)|}{s^\nu L(s)}\,ds\right\} \qquad (4.30)\\
&\quad = O_P(1)\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)-\mu} n^{-p\alpha}\int_{k_n/n}^{1/2} s^{p/2-p\alpha-\nu+\mu}\,ds \\
&\qquad + O_P(1)\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)n^{-\alpha}\int_{k_n/n}^{1/2}\frac{|B_n(s)|^{p-1}}{s^{\nu-1/2+\alpha}L(s)}\,ds \\
&\quad = o_P(1).
\end{aligned}$$

Consequently, in order to verify (2.22), it suffices to show that

$$\begin{aligned}
&\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{k_n/n}^{1/2}\frac{|B(s)|}{s^\nu L(s)}\,ds \qquad (4.31)\\
&\qquad \to_{\mathscr{D}} \int_0^1 s^{\nu-(p+2)}|W(s)|^p\,ds.
\end{aligned}$$

Similarly to (4.29) it can be seen that

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{k_n/n}^{1/2}\frac{s|W(s)|^{p-1}}{s^\nu L(s)}\,ds = o_P(1). \qquad (4.32)$$

Hence by (4.5), (3.13) and (4.32) we obtain

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{k_n/n}^{1/2}\frac{\left||W(s)|^p - |W(s)-sW(1)|^p\right|}{s^\nu L(s)}\,ds = o_P(1). \qquad (4.33)$$

Letting $s = k_n/(nt)$ and using the fact that

$$\{tW(1/t);\ 0 < t \le 1\} =_{\mathscr{D}} \{W(t);\ 0 < t < \infty\},$$

we have

$$\begin{aligned}
&\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{k_n/n}^{1/2}\frac{|W(s)|^p}{s^\nu L(s)}\,ds \qquad (4.34)\\
&\qquad =_{\mathscr{D}} L\left(\frac{k_n}{n}\right)\int_{2k_n/n}^{1} t^{\nu-(p+2)}\frac{|W(t)|^p}{L(k_n/nt)}\,dt.
\end{aligned}$$

Let $0 < \delta < 1$, and $\mu > 0$ be such that $\nu > p/2 + 1 + \mu$. Then by (3.13) and

Markov's inequality we get

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} P\left\{L\left(\frac{k_n}{n}\right)\int_{2k_n/n}^{\delta} t^{\nu-(p/2)}\frac{|W(t)|^p}{L(k_n/nt)}\,dt>\varepsilon\right\}=0, \tag{4.35}$$

for all $\varepsilon > 0$. From (3.7) we have

$$L(k_n/n)\int_{\delta}^{1} t^{\nu-(p+2)}|W(t)|/L(k_n/(nt))\,dt \to_{\mathscr{D}} \int_{\delta}^{1} t^{\nu-(p+2)}|W(t)|\,dt. \tag{4.36}$$

Since δ in (4.36) can be taken arbitrarily small we obtain (4.31) by (4.33), (4.35) and (4.15). This completes the proof of (2.22).

Similarly to the proof of (4.30) it can be shown that

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{U_n(k_n/n)}^{U_n(1/2)}\frac{\big||e_n(s)|^p-|B_n(s)|^p\big|}{s^{\nu}L(s)}\,ds=o_P(1). \tag{4.37}$$

If we can show that

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{U_n(k_n/n)\wedge(k_n/n)}^{U_n(k_n/n)\vee(k_n/n)}\frac{|B_n(s)|^p}{s^{\nu}L(s)}\,ds=o_P(1) \tag{4.38}$$

and

$$\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{U_n(1/2)\wedge(1/2)}^{U_n(1/2)\vee(1/2)}\frac{|B_n(s)|^p}{s^{\nu}L(s)}\,ds=o_P(1), \tag{4.39}$$

then (2.23) will follow from (4.31). Using (3.13) and Markov's inequality we obtain

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} P\left\{\left(\frac{k_n}{n}\right)^{\nu-(p/2+1)} L\left(\frac{k_n}{n}\right)\int_{(1-\delta)k_n/n}^{(1+\delta)k_n/n}\frac{|B(s)|^p}{s^{\nu}L(s)}\,ds>\varepsilon\right\}=0, \tag{4.40}$$

for all $\varepsilon > 0$. Now Theorem 3.3 and (4.40) imply (4.38). The proof of (4.39) is similar and hence omitted. The proof of (2.23) is now complete. □

Proof of Theorem 2.5. By (3.14) we have

$$\begin{aligned} &n^{(p/2+1)-\nu}\int_{\lambda/n}^{k_n/n} s^{-\nu}|u_n(s)|^p|L(1/n)/L(s)-1|\,ds \\ &\quad= o(1)n^{(p/2+1)+\mu-\nu}\int_{\lambda/n}^{k_n/n} s^{\mu-\nu}|u_n(s)|^p\,ds, \end{aligned} \tag{4.41}$$

for all $0<\mu<\nu-(p/2+1)$. Consequently, it is enough to prove Theorem 2.5 with $L=1$.

First we note that as $n\to\infty$,

$$n^{1-\alpha}\int_{\lambda/n}^{k_n/n} t^{-\alpha}|S_{[\![nt]\!]+1}-tn|^q\,dt \to_P \int_{\lambda}^{\infty} t^{-\alpha}|S_{[\![t]\!]+1}-t|^q\,dt, \tag{4.42}$$

if $\alpha > q/2 + 1$, and by (3.5) we have

$$n^{(p/2+1)-\nu} \int_{\lambda/n}^{k_n/n} \frac{|u_n(s)|^p}{s^\nu}\, ds$$
(4.43)
$$=_{\mathscr{D}} n^{3/2-\nu-p/2}\left(\frac{n}{S_{n+1}}\right)^p \int_{\lambda/n}^{k_n/n} t^{-\nu}|S_{[\![nt]\!]+1} - tS_{n+1}|^p\, dt.$$

Now applying (4.5) and (4.42) we obtain

$$(4.44)\quad n^{3/2-\nu-p/2} \int_{\lambda/n}^{k_n/n} t^{-\nu}\big||S_{[\![nt]\!]+1} - tS_{n+1}|^p - |S_{[\![nt]\!]+1} - tn|^p\big|\, dt = o_P(1).$$

Now observing that $S_{n+1}/n \to_P 1$, (2.25) follows from (4.42)–(4.44). □

Proof of Theorem 2.6. First we note that by combining (3.3) and (3.7) we get

$$(4.45)\quad \sup_{U_n(1/n) \le s \le U_n(k_n/n)} (ns)^{-\mu}|L(1/n)/L(s) - 1| = o_P(1).$$

Hence, similarly to (4.41) we obtain

$$n^{(p/2+1)-\nu} \int_{U_n(1/n)}^{U_n(k_n/n)} s^{-\nu}|e_n(s)|^p|L(1/n)/L(s) - 1|\, ds$$
(4.46)
$$= o_P(1) n^{(p/2+1)+\mu-\nu} \int_{U_n(1/n)}^{U_n(k_n/n)} s^{\mu-\nu}|e_n(s)|^p\, ds,$$

whenever $0 < \mu < \nu - (p/2 + 1)$. Hence it suffices to prove (2.26) and (2.27) with $L = 1$. We have by (3.5)

$$n^{p+1-\nu} \int_{U_n(1/n)}^{U_n(k_n/n)} s^{-\nu}|E_n(s) - s|^p\, ds$$
(4.47)
$$= \sum_{i=1}^{k_n-1} n^{p+1-\nu} \int_{U_n(i/n)}^{U_n((i+1)/n)} s^{-\nu}|i/n - s|^p\, ds$$
$$=_{\mathscr{D}} \sum_{i=1}^{k_n-1} \int_{(n/S_{n+1})S_i}^{(n/S_{n+1})S_{i+1}} s^{-\nu}|i - s|^p\, ds.$$

Next we show that

$$(4.48)\quad \max_{1 \le i \le k_n - 1} \int_{(n/S_{n+1} \wedge 1)S_i}^{(n/S_{n+1} \vee 1)S_i} s^{-\nu}|i - s|^p\, ds = o_P(1)$$

and

$$(4.49)\quad \max_{1 \le i \le k_n - 1} \int_{(n/S_{n+1} \wedge 1)S_{i+1}}^{(n/S_{n+1} \vee 1)S_{i+1}} s^{-\nu}|i - s|^p\, ds = o_P(1).$$

Details of the proof are given only for (4.48). The proof of (4.49) is similar. We consider

$$\max_{1\le i\le k_n-1}\int_{(n/S_{n+1}\wedge)S_i}^{(n/S_{n+1}\vee 1)S_i} s^{-\nu}|i-s|^p\,ds$$

$$(4.50)\qquad \begin{aligned}&\le \max_{1\le i\le k_n-1} S_i\left|\frac{n}{S_{n+1}}-1\right|\left(\left(\frac{n}{S_{n+1}}\vee 1\right)S_i\right)^{-\nu}\left|\left(\frac{n}{S_{n+1}}\vee 1\right)S_i-i\right|^p\\ &\quad+\max_{1\le i\le k_n-1} S_i\left|\frac{n}{S_{n+1}}-1\right|\left(\left(\frac{n}{S_{n+1}}\wedge 1\right)S_i\right)^{-\nu}\left|\left(\frac{n}{S_{n+1}}\wedge 1\right)S_i-i\right|^p\\ &= A_n^{(1)}+A_n^{(2)}.\end{aligned}$$

We will only deal with $A_n^{(1)}$. The details for $A_n^{(2)}$ are similar. We have

$$A_n^{(1)}\le\left(\frac{n}{S_{n+1}}\vee 1\right)^{-\nu}\left|\frac{n}{S_{n+1}}-1\right|\max_{1\le i\le k_n-1}S_i^{1-\nu}\left|\left(\frac{n}{S_{n+1}}\vee 1\right)S_i-i\right|^p$$

$$= O_P(1)n^{-1/2}\max_{1\le i\le k_n-1}\left\{S_i\left|\frac{n}{S_{n+1}}-1\right|+|S_i-i|\right\}^p$$

$$(4.51)\qquad = O_P(1)n^{-1/2}\left|\frac{n}{S_{n+1}}-1\right|^p S_{[k_n]}^p + O_P(1)n^{-1/2}\max_{1\le i\le k_n}|S_i-i|^p$$

$$= O_P(1)\left(\frac{S_{[k_n]}}{k_n}\right)^p\left(\frac{k_n}{n}\right)^p n^{p/2-1/2}+O_P(1)$$

$$+O_P(1)\left(\frac{k_n}{n}\right)^{p/2}n^{p/2-1/2}\left\{k_n^{-1/2}\max_{1\le i\le k_n}|S_i-i|\right\}^p$$

$$= o_P(1).$$

Now using the law of iterated logarithm for the Poisson process $\{N(t);\ 0<t<\infty\}$ we obtain

$$(4.52)\qquad \begin{aligned}\sum_{i=1}^{k_n-1}\int_{S_i}^{S_{i+1}} s^{-\nu}|i-s|^p\,ds &= \int_{S_1}^{S_{[k_n]}} s^{-\nu}|N(s)-s|^p\,ds\\ &\to \int_{S_1}^{\infty} s^{-\nu}|N(s)-s|^p\,ds \quad \text{a.s.,}\end{aligned}$$

which also completes the proof of (2.26).

In order to prove (2.27), we first observe that for any $\varepsilon>0$ there exist an n_0 and j, $1\le j\le k_n$, such that for all $n\ge n_0$,

$$(4.53)\qquad P\{U_{j-1,n}<\lambda/n\le U_{j,n}\}\ge 1-\varepsilon,$$

by Theorem 3.2. Hence by (3.5) a proof of (2.27) reduces to show that

$$\begin{aligned} (4.54)\quad & n^{p+1-\nu}\left\{\int_{\lambda/n}^{S_j/S_{n+1}} s^{-\nu}\left|\frac{j-1}{n}-s\right|^p ds+\sum_{k=j}^{k_n-1}\int_{S_k/S_{n+1}}^{S_{k+1}/S_{n+1}} s^{-\nu}\left|\frac{k}{n}-s\right|^p ds\right\} \\ &= \int_\lambda^{S_{[k_n]}} t^{-\nu}|N(t)-t|^p\,dt+o_P(1)\to_{\mathscr{D}}\int_\lambda^\infty t^{-\nu}|N(t)-t|^p\,dt. \end{aligned}$$

The latter in turn goes along the lines of the proofs of (4.48), (4.49) and (4.52). □

PROOF OF REMARK 2.5. By (3.5) we have with $K(s)=1/K_{p-\nu}(s)$,

$$\begin{aligned} & n^{p+1-\nu}L(1/n)\int_0^{U_n(1/n)} s^{p-\nu}/L(s)\,ds \\ (4.55)\quad &=_{\mathscr{D}} n^{p+1-\nu}L(1/n)\int_0^{S_1/S_{n+1}} s^{p-\nu}/L(s)\,ds \\ &= K(S_1/S_{n+1})(S_1/S_{n+1})^{p+1-\nu}n^{p+1-\nu}L(1/n)/L(S_1/S_{n+1}), \end{aligned}$$

where by (3.8)

$$(4.56)\qquad K(S_1/S_{n+1})\to_P 1/(p+1-\nu),$$

and by (2.9) and (3.7)

$$(4.57)\qquad L(1/n)/L(S_1/S_{n+1})\to_P 1.$$

Consequently,

$$\begin{aligned} n^{p/2+1-\nu}L(1/n)\int_0^{U_n(1/n)}|e_n(s)|^p/L(s)\,ds &\to_{\mathscr{D}} S_1^{p+1-\nu}/(p+1-\nu) \\ &= \int_0^{S_1} t^{-\nu}|N(t)-t|^p\,dt, \end{aligned}$$

and therefore (2.28) follows from (2.26).

To prove (2.29), we notice that whenever

$$(4.58)\qquad \int_0^\delta s^{p-\nu}/L(s)\,ds=\infty,\quad \text{for some } \delta>0,$$

then

$$\int_0^{U_n(1/n)}|e_n(s)|^p/(s^\nu L(s))\,ds\to_P\infty.$$

If, on the other hand, the integral of (4.58) is finite for some $\delta>0$, then $\nu=p+1$, and

$$\int_0^\delta 1/(sL(s))\,ds<\infty.$$

Hence

$$L(1/n)\int_0^{U_n(1/n)}1/(sL(s))\,ds=K(U_n(1/n))L(1/n)/L(U_n(1/n))\to_P\infty,$$

by (4.57), and because $K(x)\to\infty$ as $x\to 0$ by (3.9). □

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