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MOMENTS OF RANDOM VECTORS WHICH BELONG TO SOME DOMAIN OF NORMAL ATTRACTION

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Let X be a random vector on $\mathbb{R}^k$ whose distribution μ belongs to the domain of normal attraction of some operator stable law ν. For a given ν it has been shown elsewhere that for certain ranges of α depending on ν, either $E|\langle X, \theta\rangle|^\alpha$ is finite for every $\theta \neq 0$ or is infinite for every $\theta \neq 0$. In this paper we show that the set of α for which $E|\langle X, \theta\rangle|^\alpha$ exists depends, in general, on both θ and ν, and we obtain a complete description of the cases in which $E|\langle X, \theta\rangle|^\alpha$ can be guaranteed either to exist or to diverge, just on the basis of θ and ν.

1. Introduction. Let μ, ν denote probability distributions on $\mathbb{R}^k$ and suppose that ν is full, i.e., it cannot be supported on any $(k-1)$ dimensional affine subspace of $\mathbb{R}^k$. If B is a linear operator on $\mathbb{R}^k$ we denote by t^B the operator $\exp(B \log t)$. We say that μ belongs to the domain of normal attraction of ν if for a sequence of independent random vectors $\{X_n\}$ with common distribution μ there exists a linear operator B and constants $b_n \in \mathbb{R}^k$ such that

$$n^{-B}(X_1 + \cdots + X_n) - b_n \Rightarrow Y, \tag{1.1}$$

where Y is a random vector with distribution ν and $\Rightarrow$ denotes convergence in distribution. In this case Y (or ν) is operator stable with exponent B. That is to say, equation (1.1) holds with equality replacing weak convergence and $\mu = \nu$.

Suppose now that X belongs to the domain of normal attraction of Y. In this paper we are concerned with the existence of absolute moments $E|\langle X, \theta\rangle|^\alpha$. Let B be an exponent of Y and define $m = \min\{\mathrm{Re}(\lambda)\}$, $M = \max\{\mathrm{Re}(\lambda)\}$, where λ ranges over the eigenvalues of B. Sharpe (1969) showed that $m \geq \frac{1}{2}$. The following result was obtained independently by Hudson, Veeh and Weiner (1988) and Meerschaert (1986).

THEOREM A. *For all nonzero $\theta \in \mathbb{R}^k$, $E|\langle X, \theta\rangle|^\alpha$ is finite for all $0 < \alpha < 1/M$ and, if $m > \frac{1}{2}$, then $E|\langle X, \theta\rangle|^\alpha = \infty$ for all $\alpha > 1/m$.*

This leaves open the existence question for α between $1/m$ and $1/M$. In this paper we will settle the existence question via an extension of the approach employed in Meerschaert (1986). We will define an index function $\alpha^*(\theta)$ based on the spectral properties of B such that the following holds:

THEOREM 1. *Suppose that X belongs to the domain of normal attraction of Y operator stable with exponent B. Define α^* as in* (2.5). *Then for all $\theta \neq 0$ in*

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$\mathbb{R}^k$ *we have*:

(a) *If* $\alpha^*(\theta) \neq 2$, *then* $E|\langle X, \theta\rangle|^\alpha$ *is finite for* $0 < \alpha < \alpha^*(\theta)$ *and infinite for* $\alpha \geq \alpha^*(\theta)$.

(b) *If* $\alpha^*(\theta) = 2$, *then* $E|\langle X, \theta\rangle|^\alpha < \infty$ *for* $0 < \alpha \leq \alpha^*(\theta)$.

If $\alpha^*(\theta) = 2$ and $\alpha > 2$, then $E|\langle X, \theta\rangle|^\alpha$ will be finite for some X attracted to Y and infinite for others. In connection with Theorem A, we have that $1/m$ and $1/M$ are, respectively, the maximum and minimum values of α^*. Since every eigenvalue of an operator stable exponent has a real part equal to or exceeding $\frac{1}{2}$, we will always have $0 < \alpha^*(\theta) \leq 2$.

2. Operator stable laws and exponents. Operator stable laws were characterized by Sharpe (1969). In view of (1.1), an operator stable law ν must be infinitely divisible with Lévy representation (a, Q, ϕ). If we let ν^t denote the t-fold convolution product of ν with itself and if B is an exponent of ν, then for all $t > 0$,

$$\nu^t = t^B \nu * \delta(b_t), \tag{2.1}$$

where $t^B\nu\{dx\} = \nu\{t^{-B}\, dx\}$ and $\delta(a)$ is the unit mass at $a \in \mathbb{R}^k$. It follows that $tQ(x) = Q(t^{B^*}x)$ and $t\phi = t^B\phi$.

The collection of linear operators $\{t^B\colon t > 0\}$ is simply a reparametrization of a one-parameter subgroup of $\mathrm{GL}(\mathbb{R}^k)$. The behavior of orbits $\{t^Bx\colon t > 0\}$ can be obtained easily by reference to standard results from the theory of linear differential equations in $\mathbb{R}^k$. The following characterization of t^B has been so adapted from Hirsch and Smale (1974).

There is a basis $b_1 \cdots b_k$ for $\mathbb{R}^k$ with respect to which the matrix representing B has a block diagonal form, each block corresponding to either a single real eigenvalue a or a single complex conjugate pair $a \pm ib$. For a real eigenvalue the corresponding block has a's along the diagonal, 1's along the subdiagonal and zero entries elsewhere. For complex eigenvalues the same is true if we consider B as a linear operator on a complex vector space and associate complex numbers with 2×2 matrices in the usual way.

Now it is simple to compute t^B. Because of the block diagonal form of B we obtain a direct sum decomposition of $\mathbb{R}^k$ into B-invariant subspaces $V_1 \cdots V_p$. Let V denote one of these subspaces. Without loss of generality we have $V = \mathrm{Span}\{b_1 \cdots b_n\}$. Using the coordinates associated with the basis $b_1 \cdots b_k$, for x and y in V associated with a real eigenvalue a we have $y = t^Bx$ where

$$y_i = t^a \sum_{q=0}^{i-1} (\log t)^q x_{i-q}/q!, \qquad i = 1, \dots, n. \tag{2.2}$$

For complex eigenvalues we associate V with a complex vector space $z_j = x_j + iy_j$, $w_j = u_j + iv_j$. In this case $\dim V$ must be even, so without loss of generality we can suppose that $V = \mathrm{Span}\{b_1 \cdots b_{2n}\}$. If $w = t^B z$ we again have

$$w_j = t^{a+ib} \sum_{q=0}^{j-1} (\log t)^q z_{j-q}/q!, \qquad j = 1, \dots, n, \tag{2.3}$$

or in real coordinates,

$$\begin{aligned} u_i &= t^a \sum_{q=0}^{i-1} (\log t)^q \left[x_{i-q} \cos(b \log t) - y_{i-q} \sin(b \log t) \right] / q!, \\ v_i &= t^a \sum_{q=0}^{i-1} (\log t)^q \left[y_{i-q} \cos(b \log t) + x_{i-q} \sin(b \log t) \right] / q!. \end{aligned} \tag{2.4}$$

We will also be interested in t^{B^*} where B^* is the transpose of B, i.e., $\langle x, By \rangle = \langle B^* x, y \rangle$ for all $x, y \in \mathbb{R}^k$. If we form the dual basis $f_1 \cdots f_k$ ($\langle b_i, f_j \rangle = 1$ if $i = j$ and 0 otherwise), then the matrix of B^* with respect to the dual basis is just the transpose (interchange rows and columns) of the matrix for B with respect to $b_1 \cdots b_k$. If $b_r \cdots b_s$ span V_j, then $W_j = \mathrm{Span}\{f_r \cdots f_s\}$ is the dual space for V_j. W_j is B^* invariant and $\mathbb{R}^k = W_1 \oplus \cdots \oplus W_p$. The restriction of B^* to W_j has the same eigenvalue (or complex conjugate pair) as the restriction of B to V_j. Using the dual coordinates (the ones associated with $f_1 \cdots f_k$) we obtain the same formulas (2.2) through (2.4) for t^{B^*} except that we sum from zero to $n - i$, where $n = \dim V$ (real eigenvalue) or $2n = \dim V$ (complex conjugate pair).

Let a_j denote the real part of the eigenvalue associated with V_j, W_j. For $x \neq 0$ write $x = x_1 + \cdots + x_p$ the unique direct sum decomposition with respect to $\{V_j\}$ and $x = x_1^* + \cdots + x_p^*$ the same with respect to $\{W_j\}$. Define

$$\begin{aligned} \alpha(x) &= \min\{1/a_j : x_j \neq 0\}, \\ \alpha^*(x) &= \min\{1/a_j : x_j^* \neq 0\}. \end{aligned} \tag{2.5}$$

Let U_1 denote the direct sum of all V_j with $a_j = \frac{1}{2}$ and U_2 the direct sum of the remaining V_j. Sharpe (1969) showed that the Lévy measure ϕ of ν is concentrated on U_2 and cannot be supported on any proper subspace of U_2. Notice that $x \neq 0$ is in U_1 if and only if $\alpha(x) = 2$.

Now form the dual spaces U_j^* to U_j as above. Sharpe also showed that the quadratic form Q in the Lévy representation for ν may be considered as an extension to $\mathbb{R}^k$ of a positive definite quadratic form on U_1^*. Notice that $x \neq 0$ is in U_1^* if and only if $\alpha^*(x) = 2$.

Now a consideration of the Lévy representation shows that the operator stable law ν may be written as a convolution product of a full normal law on U_1 and a full operator stable law on U_2 having no normal component. Now let π_i denote the natural projection map onto U_i. If we write the direct sum representation $x = x_1 + x_2$ with respect to U_1, U_2, then $\pi_i(x) = x_i$.

Since π_1 commutes with n^{-B} for all n we have that $\pi_1 X$ belongs to the domain of normal attraction of $\pi_1 Y$, which is nondegenerate normal on U_1.

Now for any $\theta \in U_1^*$ we have $\langle X, \theta \rangle = \langle \pi_1 X, \theta \rangle$. Thus the question of existence of moments $E|\langle X, \theta \rangle|^\alpha$ for $\theta \in U_1^*$ can be reduced to the case of a normal limit.

3. Regular variation. A real-valued function $R(t)$ defined for $t \geq A$ is regularly varying if it is positive, Borel measurable and if for all $\lambda > 0$,

$$\lim_{t \to \infty} \frac{R(\lambda t)}{R(t)} = \lambda^\alpha \tag{3.1}$$

for some real constant α called the index of R. Our references on regular variation are Feller (1971) and Seneta (1976).

In order to express the path behavior of t^B, it will be convenient to use the Euclidean norm $\|x\| = (x_1^2 + \cdots + x_k^2)^{1/2}$ where these are the coordinates associated with the basis $b_1 \cdots b_k$ chosen in Section 2. Define $R(t) = \|t^B x\|$, $x \neq 0$. It is not hard to check, using (2.2) through (2.4), that $R(t)$ varies regularly with index $1/\alpha(x)$. By a result in Seneta (1976), page 21, there exists an asymptotic inverse function t, a regularly varying function with index $\alpha(x)$ such that $R(t(r)) \sim t(R(r)) \sim r$ as $r \to \infty$. In fact we can take $t(r) = \inf\{t: R(t) \geq r\}$.

With regard to $t^{B^*} = \exp(B^* \log t)$ we will use the norm associated with the dual basis. For $x \neq 0$, we have once again that $R^*(t) = \|t^{B^*} x\|$ varies regularly with index $1/\alpha^*(x)$. The asymptotic inverse function t^* varies regularly with index $\alpha^*(x) > 0$ and so $t^*(r) \to \infty$ as $r \to \infty$.

Suppose now that μ, ν are as in the Introduction, i.e., μ is in the domain of normal attraction of ν operator stable. In order to study the tail behavior of μ we will define $f(x) = \mu(H_x)$, where

$$H_x = \{y \in \mathbb{R}^k: |\langle x, y \rangle| > 1\}. \tag{3.2}$$

We will also define $g(x) = \phi(H_x)$, where ϕ is the Lévy measure of ν. Notice that $f(\theta/r) = \Pr\{|\langle X, \theta \rangle| > r\}$.

Using the standard convergence criteria for triangular arrays of random vectors we obtain from (1.1) that $n\mu\{n^B\, dx\} \to \phi\{dx\}$ in the sense of Lévy measures. It follows easily that $t\mu\{t^B\, dx\} \to \phi\{dx\}$ and so for all $x \neq 0$ for which $\phi(\partial H_x) = 0$ we have

$$\lim_{t \to \infty} t f(t^{-B^*} x) = g(x). \tag{3.3}$$

A result due to Sharpe (1969) states that ϕ is a mixture of Lévy measures M concentrated on a single orbit of t^B, with $M\{t^B x_0: t > r\} = 1/r$. From this it is not hard to show that (3.3) holds for all nonzero $x \in \mathbb{R}^k$ and the convergence is uniform on relatively compact subsets of $\mathbb{R}^k - \{0\}$. [This same construction was used in Meerschaert (1986) in a more general context.]

Now, as above, let t^* denote the asymptotic inverse of $R^*(t) = \|t^{B^*} x\|$. If we define $\theta_r = t^*(r)^{B^*}(x/r)$, then $\{\theta_r\}$ is sequentially compact and, since $\|\theta_r\| = R^*(t^*(r))/r$, every limit point is a unit vector in $\mathbb{R}^k$. If $\alpha^*(x) < 2$, then every limit point lies in the set U_2^*. On the other hand, if $\alpha^*(x) = 2$ (i.e., if $x \in U_1^*$), then $\theta_r \in U_1^*$ and so $g(\theta_r) = 0$ for all $r > 0$.

Consider the expression

$$t^*(r)f(x/r) = t^*(r)f\left(t^*(r)^{-B^*}\theta_r\right). \tag{3.4}$$

For any fixed $\varepsilon > 0$, in view of the uniform convergence in (3.3) we have that $|t^*(r)f(x/r) - g(\theta_r)| < \varepsilon$ for all large r. If $\alpha^*(x) = 2$ this means that $t^*(r)f(x/r) \to 0$. Otherwise if $\alpha^*(x) < 2$, then the fact that ϕ is full on U_2 means that g is bounded away from zero and infinity on compact sets in $U_2^* - \{0\}$ and we have that

$$\{t^*(r)f(x/r): r \geq r_0\} \tag{3.5}$$

is bounded away from zero and infinity. Thus we have tied the tail behavior of μ to the growth rate of a regularly varying function.

Suppose that $\alpha^*(x) < 2$. Since t^* varies regularly with index $\alpha^*(x)$, we have for all $\delta > 0$ that

$$r^{\alpha^*(x)-\delta} < t^*(r) < r^{\alpha^*(x)+\delta} \tag{3.6}$$

for r sufficiently large. It follows that for all $\varepsilon > 0$ there exists $r_0 > 0$ such that for all $r \geq r_0$,

$$r^{-\alpha^*(x)-\varepsilon} < f(x/r) < r^{-\alpha^*(x)+\varepsilon}. \tag{3.7}$$

We can go further. Letting $a = 1/\alpha^*(x)$ we have that $R^*(t) > ct^a(\log t)^j$ for some $c > 0$ and some $j \in \{0, 1, 2, \dots, k-1\}$. Since $R^*(t) > (c - \varepsilon)t^a$ for t large we must have $t^*(r) \leq Kr^{1/a}$ for r large. Thus we have for some $m > 0$ that for some $r_0 > 0$, for all $r \geq r_0$,

$$f(x/r) \geq mr^{-\alpha^*(x)}. \tag{3.8}$$

We also note that for any $\lambda_0 > 1$ sufficiently large we can choose $r_0 > 0$ so that for all $r \geq r_0$ we have

$$\frac{f(x/\lambda_0 r)}{f(x/r)} \leq \frac{1}{2}. \tag{3.9}$$

Specifically we can choose any λ_0 for which $\lambda_0^{-\alpha^*(x)} < \frac{1}{2}(a/b)$, where a and b are, respectively, the lower and upper bounds of the quantity in (3.5).

4. The proof of Theorem 1. Suppose that X belongs to the domain of normal attraction of Y, an operator stable law with exponent B. The consideration of moments $E|\langle X, \theta\rangle|^\alpha$ divides naturally into the two cases $\alpha^*(\theta) = 2$ ($\theta \in U_1^*$) and $\alpha^*(\theta) \in (0, 2)$. We will consider each case separately.

Suppose $\theta \in U_1^*$. We showed in Section 2 that $\langle X, \theta\rangle$ is the marginal of a random vector in the domain of normal attraction of a full normal law. Now a result of Klosowksa (1980) states that $\langle X, \theta\rangle$ belongs to the domain of normal attraction of the standard normal law on $\mathbb{R}^1$. It is well known that any such law has a finite variance, i.e., $E|\langle X, \theta\rangle|^2 < \infty$. Then of course $E|\langle X, \theta\rangle|^\alpha < \infty$ for all $0 < \alpha \leq 2$. To see that the existence of $E|\langle X, \theta\rangle|^\alpha$ for $\alpha > 2$ depends on X, consider a one-dimensional random variable $X \geq 0$ with $\Pr\{X > r\} = r^{-\beta}$. Then EX^α is finite if and only if $\alpha < \beta$.

Suppose now that $\alpha^*(\theta) < 2$. We adapt the notation of Feller for the truncated moments of $\langle X, \theta \rangle$. Let

$$(4.1)\qquad \begin{aligned} U_\alpha(r,\theta) &= \int_0^r t^\alpha F_\theta\{dt\}, \\ V_\beta(r,\theta) &= \int_r^\infty t^\beta F_\theta\{dt\}, \end{aligned}$$

where $F_\theta(t) = \Pr\{|\langle X, \theta\rangle| \le t\}$. Note that $V_0(r,\theta) = f(\theta/r) = 1 - F_\theta(r)$. An integration by parts in (4.1) yields

$$(4.2)\qquad U_\alpha(r) = -r^\alpha V_0(r) + \int_0^r \alpha t^{\alpha-1} V_0(t)\, dt,$$

where for ease of notation we have suppressed θ. [This is a special case of equation (9.17) in Feller 8.] For each $\alpha \in (0, \alpha^*)$ we wish to show that each term on the right-hand side of (4.2) is bounded. But this follows immediately from the second inequality in (3.7).

If $\alpha \ge \alpha^*$, we wish to show that

$$(4.3)\qquad E|\langle X, \theta\rangle|^\alpha = U_\alpha(r_0) + \int_{r_0}^\infty t^\alpha F_\theta\{dt\}$$

is infinite, i.e., the integral diverges. Select $r_0 > 0$ large enough to ensure that both (3.8) and (3.9) hold. Subdivide the domain of integration into disjoint subintervals $J_n = [r_0\lambda_0^n, r_0\lambda_0^{n+1})$ and denote by I_n the integral over J_n. Since t^α is monotone we obtain

$$\begin{aligned} I_n &\ge (\lambda_0^n r_0)^\alpha \left[V_0(\lambda_0^n r_0) - V_0(\lambda_0^{n+1} r_0) \right] \\ &\ge \tfrac{1}{2} V_0(\lambda_0^n r_0)(\lambda_0^n r_0)^\alpha \ge \tfrac{1}{2} m (\lambda_0^n r_0)^{-\alpha^*} (\lambda_0^n r_0)^\alpha. \end{aligned}$$

Obviously $\Sigma I_n = \infty$ and the theorem is established.

5. Remarks. Although the index function α^* is defined in (2.5) in terms of the (not necessarily unique) exponent B of ν, it is clear from the statement of Theorem 1 that α^* is the same for every such exponent.

It may be though that the function $V_0(r,\theta) = f(\theta/r)$ is actually a regularly varying function of $r > 0$ for all θ. In fact this is true, for example, if B has k distinct eigenvalues. [In this case B is diagonalizable. See Meerschaert (1987).] The following example will show that, in general, V_0 cannot be expected to vary regularly.

We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. With this identification each linear operator on $\mathbb{C}$ (i.e., multiplication by a complex constant) corresponds to a linear operator on $\mathbb{R}^2$ whose matrix representation with respect to the standard basis is skew symmetric. Take $B = 1 + i$ so that $t^B = t^{1+i} = tR(\ln t)$ where $R(\theta)$ is a rotation in the plane through a counterclockwise angle θ. Let ϕ be a Lévy measure concentrated on $\{t^B e_1 : t > 0\}$ with $\phi\{t^B e_1 : t > r\} = r^{-1}$. Then the infinitely divisible distribution ν with Lévy representation $(0, 0, \phi)$ is operator stable with exponent B. By the arguments in Section 3, in order to show that V_0 does not vary regularly, it will suffice to show that $g(\theta)$ is not constant on $\|\theta\| = 1$. We compute that

$$g(\theta) = \int_0^\infty t^{-2} I\{|t\cos(\log t - \theta)| > 1\}\, dt. \tag{5.1}$$

For $\theta = e_1$, the indicator is positive on $(1, t_1)$, where t_1 is the smallest root of $t\cos(\log t) = 1$, $t > 1$. Therefore, since $t_1 \approx 3.64$, $g(e_1) > 0.7$. But for $\theta = e_2$ the indicator is zero on $(0, t_2)$, where t_2 is the smallest root of $t\sin(\log t) = 1$, $t > 1$. Since $t_2 \approx 1.8$, $g(e_2) < 0.6$, which concludes the example.

In the construction of $R^*(t)$ in Section 3 above we use a special Euclidean norm which depends on the choice of exponent B. We could have used any Euclidean norm, including the standard norm on $\mathbb{R}^k$, since any other set of basis vectors yields coordinates which can be expressed as linear combinations of our coordinates. Another norm which played a central role in Hudson, Jurek and Veeh (1986) is defined by

$$|||\, x \,||| = \int_0^1 \int_{S(\nu)} \|g t^B x\| t^{-1} H(dg)\, dt, \tag{5.2}$$

where H denotes Haar measure on the symmetry group $S(\nu)$ of the operator-stable law ν. It is possible to use this norm as well. The advantage is that R, R^* are monotone and continuous, so that we can take inverses instead of using asymptotic inverses. The disadvantage is that it becomes more difficult to establish the necessary growth conditions on R^*.

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