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A NECESSARY AND SUFFICIENT CONDITION FOR THE MARKOV PROPERTY OF THE LOCAL TIME PROCESS[1]

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Let X be a Markov process on an interval E of $\mathbb{R}$, with lifetime ζ, admitting a local time at each point and such that $P_x(X \text{ hits } y) > 0$ for all x, y in E. We prove here that the local times process $(L_\zeta^x,\ x \in E)$ is a Markov process if and only if X has fixed birth and death points and X has continuous paths.

The sufficiency of this condition has been established by Ray, Knight and Walsh. The necessity is proved using arguments based on excursion theory. This result has been proved before in Eisenbaum and Kaspi for symmetric processes using the existence of a zero mean Gaussian process with the Green function as covariance.

1. Introduction and preliminaries. Let X be a diffusion with values in $\mathbb{R}$ that starts and dies at fixed points. For each $x \in \mathbb{R}$, let L_ζ^x be the accumulated local time at x. Since X is assumed to be transient, L_ζ^x is almost surely finite for all $x \in E$. The Ray–Knight theorem asserts that $(L_\zeta^x)_{x \in \mathbb{R}}$ is a time inhomogeneous Markov process, which after a scale and a time change is composed of squares of various Bessel processes. This Markov property in the space variable of (L_ζ^x) stems from sample path properties of X, namely the continuity and the fixed birth and death points [see Walsh (1978)].

In this study, we are asking the opposite question. Given a transient Hunt process X with values in $\mathbb{R}$ that admits a local time at each point, and assuming that $(L_\zeta^x)_{x \in \mathbb{R}}$ is a Markov process under the measure governing X, does this imply that X is a diffusion with fixed birth and death points? The answer is generally, no. If X is a Poisson process starting from 0, then the local time at points not in $\mathbb{N}$ is 0, and at points in $\mathbb{N}$ the local times are i.i.d. random variables with an exponential distribution, so that $(L_\zeta^x)_{x \in \mathbb{R}}$ is a Markov process.

In Eisenbaum and Kaspi (1991), we assumed that X was a symmetric process, defined P_{ab} to be the law of X that starts at a and is killed at the last exit from b, and assumed further that the potential densities $g(x, y)$ were strictly positive and continuous in both variables. Using a connection, established by Dynkin and used by Atkinson, with a Gaussian field having g as covariance function, we have shown that the Markov property of (L_ζ^x) in $[a, b]$ is equivalent to the continuity of X restricted to $[a, b]$.

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In the present study, we relax the assumptions of symmetry and continuity of g. Here is our result. [We shall use here and in the sequel the notation of Blumenthal and Getoor (1968).]

THEOREM 1.1. *Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a transient Hunt process taking values in an interval $E \subset \mathbb{R}$, with cemetery Δ and death time ζ, such that*:

(i) *All points of E are regular for themselves.*
(ii) *For $y \in E$ let $T_y = \inf\{t > 0; X_t = y\}$. Then for every x, y, $P^x(T_y < \infty) > 0$ and $(x, y) \to E^y(e^{-T_x})$ is $\mathscr{E} \times \mathscr{E}$ measurable.*

Let L_t^x be a local time at x, normalized so that $E^y \int_0^\infty e^{-t}\, dL_t^x = E^y(e^{-T_x})$.
Suppose that there exists a probability measure μ on E, such that under P^μ, $(L_\zeta^x;\, x \in E)$ is a Markov process. Then, almost surely:

(a) *X has continuous paths.*
(b) *μ is concentrated at one point.*
(c) *X dies at a fixed point (which may be $+\infty$ or $-\infty$ when these are accumulation points of E).*

The proof of this result is composed of several steps and we defer it to the next section. We end this section with a preliminary result that we shall need for that proof.

Recall that a measure λ on $\mathscr{E}$ which is a countable sum of finite measures is a reference measure for X provided that a set $A \in \mathscr{E}$ is of potential 0 [$U(x, A) = 0$ for all $x \in E$, where $U(x, \cdot\,)$ is the potential] if and only if $\lambda(A) = 0$.

LEMMA 1.2. *Under the assumptions of* (1.1), *X has a reference measure.*

PROOF. Let $\lambda(A) = U(x_0, A)$ for some $x_0 \in E$. Since X is transient $U(x, \cdot\,)$ is a σ-finite measure [see Getoor (1979)]. By the strong Markov property of X, for every $x, y \in E$,

$$U(x, A) = E^x \int_0^{T_y} 1_A(X_t)\, dt + P^x(T_y < \infty) U(y, A). \tag{1.3}$$

Now since $P^{x_0}(T_y < \infty) > 0$ for all y, it follows from (1.3) that $U(x_0, A) = 0$ implies $U(y, A) = 0$ for all $y \in E$. The converse is trivial. □

COROLLARY 1.4. *Let A be a continuous additive functional of X, and let μ_A be its Revuz measure with respect to λ defined above. Assume condition* (ii) *of Theorem* (1.1) *and let $u^1(x, y)$ be the jointly measurable version of the Radon–Nykodim derivative of $U^1(x, \cdot\,)$ with respect to λ. Then up to an* *evanescent set* (P^x *evanescent for all* $x \in E$)

$$A_t = \int \mu_A(dx) u^1(x,x) L_t^x.$$

In particular, for $D \in \mathscr{E}$,

$$\int_0^t 1_D(X_s)\,ds = \int_D \lambda(dx) u^1(x,x) L_t^x.$$

PROOF. This is a direct consequence of (75.12) of Sharpe (1988) and the fact that (L_t^x) are normalized so that

$$E^y \int_0^\infty e^{-t}\,dL_t^x = E^y(e^{-T_x}). \qquad \square$$

2. Proof of Theorem 1.1. The proof of Theorem (1.1) is based on the following consequence of the Markov property of (L_ζ^x) under P^μ: For every x, y, z in E, such that $x < y < z$, $P^\mu(L_\zeta^y \in \cdot)$-a.e. $l \in \mathbb{R}_+$ and $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$,

$$(2.1)\qquad \begin{aligned} &P^\mu\Big(e^{-\lambda_1 L_\zeta^x} e^{-\lambda_2 L_\zeta^z} | L_\zeta^y = l\Big) \\ &\quad = P^\mu\Big(e^{-\lambda_1 L_\zeta^x} | L_\zeta^y = l\Big) P^\mu\Big(e^{-\lambda_2 L_\zeta^z} | L_\zeta^y = l\Big). \end{aligned}$$

STEP 1. We shall compute each member of (2.1) in terms of the excursion laws $({}^*P^x)_{x\in E}$ from the points of E.

By the strong Markov property at T_y,

$$(2.2)\qquad \begin{aligned} &P^\mu\Big(e^{-\lambda_1 L_\zeta^x - \lambda_2 L_\zeta^z - \lambda L_\zeta^y};\, L_\zeta^y > 0\Big) \\ &\quad = P^\mu\Big(e^{-\lambda_1 L_{T_y}^x - \lambda_2 L_{T_y}^z};\, T_y < \zeta\Big) P^y\big(e^{-\lambda_1 L_\zeta^x - \lambda_2 L_\zeta^z - \lambda L_\zeta^y}\big). \end{aligned}$$

For the second term on the right-hand side of (2.2), we use excursion theory from $M = \{t\colon X_t = y\}^-$ (A^- the closure of A):

$$(2.3)\qquad \begin{aligned} &P^y\big(e^{-\lambda_1 L_\zeta^x - \lambda_2 L_\zeta^z - \lambda L_\zeta^y}\big) \\ &\quad = P^y\Bigg(\sum_{s\in G} 1_{\{T_y\circ\theta_s = \infty\}}\, e^{-\lambda_1 L_s^x - \lambda_2 L_s^z - \lambda L_s^y} \times e^{-\lambda_1 L_\zeta^x\circ\theta_s - \lambda_2 L_\zeta^z\circ\theta_s}\Bigg), \end{aligned}$$

where G is the set of left endpoints of intervals contiguous to M. By the key formula of excursion theory [see Revuz and Yor (1991) or Maisonneuve (1975)], the right-hand side of (2.3) is equal to

$$P^y\bigg(\int_0^\infty e^{-\lambda L_s^y - \lambda_1 L_s^x - \lambda_2 L_s^z}\, {}^*P^y\Big(T_y = \infty;\, e^{-\lambda_1 L_\zeta^x - \lambda_2 L_\zeta^z}\Big)\,dL_s^y\bigg),$$

where ${}^*P^y$ denotes the excursion law from y. This, after the time change via $u = L_s^y$ and $\tau_s = \inf\{u > 0;\, L_u^y > s\}$, is equal to

$$(2.4)\qquad {}^*P^y\Big(T_y = \infty;\, e^{-\lambda_1 L_\zeta^x - \lambda_2 L_\zeta^z}\Big) P^y \int_0^{L_\zeta^y} e^{-\lambda s}\, e^{-\lambda_1 L_{\tau_s}^x - \lambda_2 L_{\tau_s}^z}\,ds.$$

Using now the exponential key formula [see Revuz and Yor (1991), page 434],

$$P^y\left(1_{\{L_\zeta^y > s\}}\, e^{-\lambda_1 L_{\tau_s}^x - \lambda_2 L_{\tau_s}^z}\right) = \exp\left\{-s\, {}^*P^y\left(1 - e^{-\lambda_1 L_{T_y}^x - \lambda_2 L_{T_y}^z} 1_{\{T_y < \infty\}}\right)\right\}. \tag{2.5}$$

Inserting now (2.5) in (2.4) we obtain

$$P^y\left(e^{-\lambda_1 L_\zeta^x - \lambda_2 L_\zeta^z - \lambda L_\zeta^y}\right) = \frac{{}^*P^y\left(T_y = \infty;\, e^{-\lambda_1 L_\zeta^x - \lambda_2 L_\zeta^z}\right)}{\lambda + {}^*P^y\left(1 - e^{-\lambda_1 L_{T_y}^x - \lambda_2 L_{T_y}^z} 1_{\{T_y < \infty\}}\right)}.$$

Finally we get

$$P^\mu\left(e^{-\lambda_1 L_\zeta^x - \lambda_2 L_\zeta^z - \lambda L_\zeta^y};\, L_\zeta^y > 0\right) = \frac{\alpha(\lambda_1, \lambda_2)}{\lambda + \beta(\lambda_1, \lambda_2)}, \tag{2.6}$$

where

$$\alpha(\lambda_1, \lambda_2) = P^\mu\left(e^{-\lambda_1 L_{T_y}^x - \lambda_2 L_{T_y}^z};\, T_y < \zeta\right) {}^*P^y\left(T_y = \infty;\, e^{-\lambda_1 L_\zeta^x - \lambda_2 L_\zeta^z}\right), \tag{2.7}$$

$$\beta(\lambda_1, \lambda_2) = {}^*P^y\left(1 - 1_{\{T_y < \infty\}}\, e^{-\lambda_1 L_{T_y}^x - \lambda_2 L_{T_y}^z}\right). \tag{2.8}$$

As on one hand we have

$$P^\mu\left(e^{-\lambda_1 L_\zeta^x - \lambda_2 L_\zeta^z - \lambda L_\zeta^y};\, L_\zeta^y > 0\right) = \int_{0_+}^\infty e^{-\lambda l} P^\mu\left(e^{-\lambda_1 L_\zeta^x - \lambda_2 L_\zeta^z} \middle| L_\zeta^y = l\right) P^\mu\left(L_\zeta^y \in dl\right),$$

and on the other, we have

$$P^\mu\left(e^{-\lambda_1 L_\zeta^x - \lambda_2 L_\zeta^z - \lambda L_\zeta^y};\, L_\zeta^y > 0\right) = \alpha(\lambda_1, \lambda_2) \int_{0_+}^\infty e^{-\lambda l}\, e^{-\beta(\lambda_1, \lambda_2) l}\, dl,$$

it follows that

$$P^\mu\left(L_\zeta^y \in dl\right) = \alpha(0,0)\, e^{-\beta(0,0) l}\, dl;\ l > 0, \tag{2.9}$$

$$P^\mu\left(e^{-\lambda_1 L_\zeta^x - \lambda_2 L_\zeta^z} \middle| L_\zeta^y = l\right) = \frac{\alpha(\lambda_1, \lambda_2)}{\alpha(0,0)} e^{-\{\beta(\lambda_1, \lambda_2) - \beta(0,0)\} l}. \tag{2.10}$$

Substituting (2.9) and (2.10) back in (2.1) we obtain

$$\alpha(\lambda_1, \lambda_2) = \frac{\alpha(\lambda_1, 0)\alpha(0, \lambda_2)}{\alpha(0,0)} \tag{2.11}$$

$$\beta(\lambda_1, \lambda_2) - \beta(0,0) = (\beta(\lambda_1, 0) - \beta(0,0)) + (\beta(0, \lambda_2) - \beta(0,0)). \tag{2.12}$$

STEP 2. We are now going to use (2.12) to prove the continuity of X. Toward this end we shall prove the following:

LEMMA 2.13. *Let $x < y < z$ be in E, then*

$$P^x(T_z < T_y) = 0, \qquad P^z(T_x < T_y) = 0.$$

PROOF. For $x < y < z$ in E, (2.12) is equivalent to

$$*P^y\Big(1_{\{T_y<\infty\}}(1 - e^{-\lambda_1 L^x_{T_y}})(1 - e^{-\lambda_2 L^z_{T_y}})\Big) = 0,$$

which is also equivalent to

$$*P^y(T_y < \infty; T_x < T_y; T_z < T_y) = 0. \tag{2.14}$$

Hence, by the strong Markov property under $*P^y$,

$$\begin{aligned} 0 &= *P^y(T_x < T_y \wedge T_z)P^x(T_z < T_y)P^z(T_y < \infty) \\ &\quad + *P^y(T_z < T_y \wedge T_x)P^z(T_x < T_y)P^x(T_y < \infty). \end{aligned} \tag{2.15}$$

By our assumptions,

$$0 < P^y(T_x < \infty) \le P^y\Big(\sum_{s\in G} 1_{\{T_x\circ\theta_s < (T_y\wedge T_z)\circ\theta_s\}} + \sum_{s\in G} 1_{\{T_z\circ\theta_s < T_x\circ\theta_s < T_y\circ\theta_s\}}\Big)$$

$$= P^y\int_0^\infty *P^y(T_x < T_y \wedge T_z)\,dL^y_t + P^y\int_0^\infty *P^y(T_z < T_x < T_y)\,dL^y_t.$$

Suppose now that $P^x(T_z < T_y) > 0$. Then by (2.15),

$$0 = *P^y(T_x < T_y \wedge T_z),$$

$$0 = *P^y(T_z < T_y \wedge T_x)P^z(T_x < T_y) = *P^y(T_z < T_x < T_y),$$

and consequently $P^y(T_x < \infty) = 0$, which is absurd.

Repeating the same argument now with $0 < P^y(T_z < \infty)$, we can show that $P^z(T_x < T_y) = 0$, which completes the proof of our lemma. □

Let $\mathscr{O}$ be the set of intervals in E with say, rational end points. Since X is assumed right continuous with left limits, $\Omega_d = \{\omega \in \Omega\colon \exists\ t \in (0,\zeta)X_{t^-}(\omega) \ne X_t(\omega)\}$ is the set on which X doesn't have continuous paths. Further, if we define, for measurable B, $T_B = \inf\{t > 0\colon X_t \in B\}$, then

$$\Omega_d = \bigcup_{A\in O}\ \bigcup_{s\in Q_+} \big\{\omega \in \Omega\colon 0 < T_{A^c}(\theta_s\omega) < \infty,\ X_{T_{A^c}}\circ\theta_s\omega \in E - \partial A\big\}.$$

Since each $A \in \mathscr{O}$ is an open interval, all we need to show is that for any $x \in A$, $0 = P^x(X_{T_{A^c}} \in E \setminus \partial A)$. Let $A = (a, b)$ and $x \in A$. Since X is right continuous, it follows that on $X_{T_{A^c}} \notin \{a, b, \Delta\}$, X will spend some time in

$E\setminus[a,b]$ before hitting a or b or dying. Consequently, on $\{T_{A^c} < \zeta,\ X_{T_{A^c}} \notin \{a,b\}\}$,

$$\int_0^{T_{\{a,b\}}} 1_{\bar{A}^c}(X_t)\,dt > 0,$$

but

$$\int_0^{T_{\{a,b\}}} 1_{\bar{A}^c}(X_t)\,dt = \int_{\bar{A}^c} L^z_{T_{\{a,b\}}} u^1(z,z)\lambda(dz),$$

where λ, as was defined in Lemma (1.2), is the reference measure we work with. But we have seen that for any $x < b < z$ $(z < a < x)$,

$$P^x(T_z < T_b) = 0, \qquad (P^x(T_z < T_a) = 0),$$

and so

$$P^x\int_0^{T_{\{a,b\}}} 1_{\bar{A}^c}(X_t)\,dt = \int_{\bar{A}^c} P^x\big(L^z_{T_{\{a,b\}}}\big)u^1(z,z)\lambda(dz) = 0.$$

Hence, P^x a.s. on $\{T_{A^c} < \zeta\}$, $X_{T_{A^c}} \in \{a,b\}$, and our result follows.

STEP 3. We shall use (2.11) to show that:

(i) μ is concentrated at one point.

(ii) X dies at a fixed point (which may be $+\infty$ or $-\infty$ if these are accumulation points of E).

Equation (2.11) is equivalent to

$$\begin{aligned} &P^\mu\big(e^{-\lambda_1 L^x_{T_y} - \lambda_2 L^z_{T_y}} | T_y < \zeta\big)\, {}^*P^y\big(e^{-\lambda_1 L^x_\zeta - \lambda_2 L^z_\zeta} | T_y = \infty\big)\\ &\quad = P^\mu\big(e^{-\lambda_1 L^x_{T_y}} | T_y < \zeta\big) P^\mu\big(e^{-\lambda_2 L^z_{T_y}} | T_y < \zeta\big)\\ &\qquad \times {}^*P^y\big(e^{-\lambda_1 L^x_\zeta} | T_y = \infty\big)\, {}^*P^y\big(e^{-\lambda_2 L^z_\zeta} | T_y = \infty\big), \end{aligned} \tag{2.16}$$

where for $A \in \mathscr{F}$,

$${}^*P^y\big(A | T_y = \infty\big) = {}^*P^y(A;\, T_y = \infty)/\, {}^*P^y(T_y = \infty).$$

Let $P^\mu_y = P^\mu(\cdot | T_y < \zeta)$ and τ_y be the last exit from y, which is finite since the process is transient. A simple use of the key formula of excursion theory [Revuz and Yor (1991)] yields

$$P^\mu_y(e^{-\lambda L^x_\zeta \circ \theta_{\tau_y}}) = P^y(L^y_\zeta)\, {}^*P^y\big(e^{-\lambda L^x_\zeta} 1_{T_y = \infty}\big), \tag{2.17}$$

$$P^y(L^y_\zeta)\, {}^*P^y(T_y = \infty) = P^y(\tau_y < \infty) = 1. \tag{2.18}$$

For $x \in E$, write A_x (resp., B_x) for $L^x_{T_y}$ (resp., $L^x_\zeta \circ \theta_{\tau_y}$). Then (2.16) [via (2.17) and (2.18)] amounts to the statement that $A_x + B_x$ and $A_z + B_z$ are P^μ_y-independent for $x < y < z$. Also, it is clear that $A_x A_z = 0 = B_x B_z$, a.s. P^μ_y, and that A_x and B_x are P^μ_y-independent, as are A_x and B_z, A_z and B_x, A_z and B_z. Since all these random variables have finite moments (of all orders), we can use the facts cited above to compute $P^\mu_y((A_x + B_x)(A_z + B_z))$ in two different ways. What one obtains is the identity

$$P_y^\mu(A_x)P_y^\mu(A_z) + P_y^\mu(B_x)P_y^\mu(B_z) = 0,$$

whence

$$A_x = 0 \quad \text{a.s. } P_y^\mu \quad \text{or} \quad A_z = 0 \quad \text{a.s. } P_y^\mu \tag{2.19}$$

and

$$B_x = 0 \quad \text{a.s. } P_y^\mu \quad \text{or} \quad B_z = 0 \quad \text{a.s. } P_y^\mu. \tag{2.20}$$

By varying x, y and z (2.19) implies that μ is a one point mass, while (2.20) implies that $X_{\zeta-}$ has a degenerate law. □

3. Remark. If instead of assuming that E is an interval contained in $\mathbb{R}$, one assumes that E is any ordered set, then Theorem (1.1) remains true, except that the continuity has to be interpreted as the property described in Lemma (2.13), and the birth and death occur either at a fixed point or at two neigbouring points. Note that the Poisson process becomes continuous with this interpretation. The following example illustrates, however, that property (ii) in Theorem (1.1), namely that points communicate, is essential for it to hold.

Consider a discrete state Markov process with state space the integers and infinitesimal generator satisfying

$$q_{i,i+1} = 1, \qquad q_{i,i+2} = 1, \qquad q_{i,i} = -2.$$

This process is clearly not continuous even with the interpretation of Lemma (2.13), and for $i < j$, $P_j(T_i < \infty) = 0$. Nevertheless, as can be easily checked, its local time process $(L^i\colon i \in \text{integers})$ is a discrete time Markov process.

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