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THE ESCAPE RATE OF FAVORITE SITES OF SIMPLE RANDOM WALK AND BROWNIAN MOTION

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Consider a simple symmetric random walk on the integer lattice $\mathbb{Z}$. For each n, let $V(n)$ denote a favorite site (or most visited site) of the random walk in the first n steps. A somewhat surprising theorem of Bass and Griffin [*Z. Wahrsch. Verw. Gebiete* **70** (1985) 417–436] says that V is almost surely transient, thus disproving a previous conjecture of Erdős and Révész [*Mathematical Structures–Computational Mathematics–Mathematical Modeling* **2** (1984) 152–157]. More precisely, Bass and Griffin proved that almost surely, $\liminf_{n\to\infty} \frac{|V(n)|}{n^{1/2}(\log n)^{-\gamma}}$ equals 0 if $\gamma < 1$, and is infinity if $\gamma > 11$ (eleven). The present paper studies the rate of escape of $V(n)$. We show that almost surely, the "lim inf" expression in question is 0 if $\gamma \le 1$, and is infinity otherwise. The corresponding problem for Brownian motion is also studied.

1. Introduction. Consider the movement $(S_n, n \ge 0)$ of a simple symmetric random walk on the integer lattice $\mathbb{Z}$ starting from $S_0 := 0$. That is, $S_n = \sum_{i=1}^n X_i$, where $(X_i, i \ge 1)$ is a sequence of independent and identically distributed random variables with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$. We write, for $n \ge 0$ and $x \in \mathbb{Z}$,

$$N(n, x) := \#\{k \in [0, n] \cap \mathbb{Z} : S_k = x\}, \tag{1.1}$$

which records the number of visits of the random walk during the first n steps. We define

$$\mathbb{V}(n) := \left\{x \in \mathbb{Z} : N(n, x) = \max_{y\in\mathbb{Z}} N(n, y)\right\}. \tag{1.2}$$

In words, $\mathbb{V}(n)$ denotes the set of sites which are the "most visited" by the random walk at step n. Following Erdős and Révész [11], an element of $\mathbb{V}(n)$ is called "favorite site."

Erdős and Révész [11] initiated the study of favorite sites. One of the questions they asked is the following: what is the probability that $0 \in \mathbb{V}(n)$ for infinitely many n? In view of the recurrence of the random walk (Pólya's theorem), it seems natural to expect this probability to be positive. However, things do not go like this; a theorem of Bass and Griffin [3] implies that

$$\inf_{x\in\mathbb{V}(n)} |x| \to \infty \qquad \text{a.s.} \tag{1.3}$$

In particular, (1.3) disproves a conjecture of Erdős and Révész [11].

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We now give the precise statement of the Bass–Griffin theorem. Throughout the paper, $V(n)$ denotes an arbitrary element of $\mathbb{V}(n)$; when we state a limit result for $V(n)$ [such as (1.4)], it is to be understood that the result holds uniformly in all $V(n) \in \mathbb{V}(n)$.

THEOREM A ([3]). *Almost surely,*

$$\liminf_{n\to\infty} \frac{|V(n)|}{n^{1/2}(\log n)^{-\gamma}} = \begin{cases} 0, & \text{if } \gamma < 1, \\ \infty, & \text{if } \gamma > 11. \end{cases} \tag{1.4}$$

The transience of the process V stated in (1.3) is clearly a consequence of (1.4). What about the rate of escape of V?

THEOREM 1.1. *We have, almost surely,*

$$\liminf_{n\to\infty} \frac{|V(n)|}{n^{1/2}(\log n)^{-\gamma}} = \begin{cases} 0, & \text{if } \gamma \le 1, \\ \infty, & \text{otherwise.} \end{cases} \tag{1.5}$$

Theorem 1.1 gives accurate information of the escape rate of V. A few comments about its proof. The upper bound in Theorem 1.1 (the "if" part) slightly improves the corresponding Bass–Griffin result by bringing in the critical case $\gamma = 1$. In some situations, the critical case can be very hard to deal with. It is not the case here. The main contribution of the present paper is to deal with the very delicate lower bound (the "otherwise" part in Theorem 1.1). Our method of proving this part is interesting, not only because it solves a hard problem, but, perhaps more importantly, because it can be adapted to handle other rate of escape problems. For example, by means of a variation of this method, we have recently obtained in [17] some interesting results for the lower class (in the sense of P. Lévy) of the empirical process.

There were many open questions for favorite sites of random walk, raised in the pioneer work of Erdős and Révész [11]. These questions were later summarized in the book of Révész ([21], pages 130 and 131). A great number of them remain unanswered so far. Let us just mention an innocent-looking conjecture here.

It is clear that almost surely there are infinitely many n for which $\#\mathbb{V}(n) = 1$; and it is equally clear that almost surely there are infinitely many n such that $\#\mathbb{V}(n) = 2$. What is not known is whether these are the only possibilities; in fact, Erdős and Révész [11] conjectured that

$$\mathbb{P}\{\#\mathbb{V}(n) \ge 3; \text{ for infinitely many } n\} = 0. \tag{1.6}$$

The best result so far has been obtained by Tóth [24], who proved that the probability in (1.6) is 0 if we replace "$\#\mathbb{V}(n) \ge 3$" by "$\#\mathbb{V}(n) \ge 4$." See also Section 5.4 for a few comments about this conjecture.

We now make a (short) list of some related references. Theorem A has an obvious analogue for Brownian motion—also proved in [3]—which implies that the process of favorite sites of Brownian motion is transient. This has been extended to symmetric stable processes by Bass, Eisenbaum and Shi [2], and recently by Marcus [18] to a class of Lévy processes. (See Section 5.3 for more details.) Two of the Erdős–Révész problems for random walk were solved in [6] concerning, respectively, the large jumps of favorite sites, and the joint asymptotic behavior of favorite sites and local times. Other interesting questions and results can be found in [16] for Brownian motion, in [8] for symmetric stable processes, and in [10] for more general symmetric Markov processes. We refer to the survey paper [23] for an updated overview of various problems for favorite sites.

The main idea of our approach in the proof of Theorem 1.1 can be described as follows. It is more convenient to study the case of Brownian motion in view of the powerful Ray–Knight theorem (Fact 2.3). Therefore, instead of studying the number of visits $N(\cdot,\cdot)$ of our random walk, we will be studying a similar object for Brownian motion, namely the local time $L(\cdot,\cdot)$ [defined in (2.1)]. That we are allowed to make this passage from N (associated with a random walk) to L (associated with a Brownian motion) is guaranteed by a strong invariance principle of Révész, recalled as Fact 3.1. To get the lower bound in Theorem 1.1, we bound a probability of type [for some appropriate (q_k), (a_k) and (b_k)]

$$\mathbb{P}\left\{\sup_{|x|\le q_k} L(t,x) > \sup_{|x|>q_k} L(t,x) - \Delta_k,\ \text{for some } t\in[a_k,b_k]\right\},$$

where Δ_k is "very small," and its presence is only to compensate the small error term we get in the passage from N to L. If this probability is summable for k, then we will be able to apply the Borel–Cantelli lemma to see that almost surely for all large k, $\sup_{|x|\le q_k} L(t,x) \le \sup_{|x|>q_k} L(t,x) - \Delta_k$ for all $t\in[a_k,b_k]$, and this will tell us that the maximum of $x\mapsto N(t,x)$ is realized at some site $|x|>q_k$, which in turn will give us the desired lower bound in Theorem 1.1. Unfortunately, since our basic tool (the Ray–Knight theorem) works more smoothly with some random times instead of deterministic times, we will have to work with *random* times a_k and b_k, and moreover our b_k can be much greater than a_k (indeed, b_k/a_k is almost surely unbounded). This will cause us some additional difficulties. The way in which we deal with this is to make a special partition of the random interval $[a_k,b_k]$, and make a very careful analysis within each element of the partition. The exact statement of the probability estimate is in Lemma 2.1, and its proof is the heart of the paper.

To obtain the upper bound in Theorem 1.1, we choose some random (q_k), (t_k) and prove that

$$\sum_k \mathbb{P}\left\{\sup_{|x|\le q_k} L(t_k,x) > \sup_{|x|>q_k} L(t_k,x) - \Delta_k\right\} = \infty, \tag{1.7}$$

where again the presence of Δ_k is to compensate the error term originating from the use of invariance principle. The estimate (1.7) is proved directly by means of the Ray–Knight theorem, with some technical arrangements to get rid of the dependence structure in order to apply the Borel–Cantelli lemma. From there, the upper bound in Theorem 1.1 follows by means of a routine argument. The statement of the probability estimate is in Lemma 2.2.

Here is how the rest of the paper is organized. Section 2 is devoted to the statement and the proof of the two probability estimates for the local time of Brownian motion which are briefly described as above. We make use of these estimates to prove Theorem 1.1: the "otherwise" part in Section 3, and the "if" part in Section 4. Finally, some further discussions and a few related questions are provided in Section 5.

2. Key estimates. In this section, we study the local time of Brownian motion, and prove the two main probability estimates described in the Introduction. Throughout, $W := (W(t), t \geq 0)$ denotes a standard Brownian motion. Let $L := (L(t,x), t \geq 0, x \in \mathbb{R})$ be the process of local time of W. That is, for all Borel function $f \geq 0$,

$$\int_0^t f(W(s))\,ds = \int_{-\infty}^{\infty} f(x)L(t,x)\,dx. \tag{2.1}$$

In the sense of (2.1), the process L relates to W in the same way $N(n,x)$ [defined in (1.1)] does to the random walk (S_n). Moreover, according to Trotter [25], L is jointly continuous in t and x (except on a null set).

We introduce the (right-continuous) inverse local time at 0:

$$\tau_r := \inf\{t \geq 0 : L(t,0) > r\}, \qquad r > 0. \tag{2.2}$$

Here are the main probability estimates of the section, which will play important roles in the proofs of, respectively, the lower and upper bounds of Theorem 1.1. As we have already mentioned before, Lemma 2.1 is the key result of the present paper; the estimate in Lemma 2.2, on the other hand, despite its resemblance somewhat to Lemma 2.1, is not as deep by far.

LEMMA 2.1. *Fix $0 < \rho < 1$. For all $R \geq 3$, $0 < a \leq (\log R)^{-1-5\rho}$ and $[1-(\log R)^{-\rho}]R \leq r < R$, we have*

$$\begin{aligned} &\mathbb{P}\left\{ \sup_{|x|\leq a\sqrt{t}} L(t,x) > \sup_{|y|>a\sqrt{t}} L(t,y) - \frac{R}{(\log R)^{(1+3\rho)/2}}, \text{ for some } t \in [\tau_r, \tau_R] \right\} \\ &\qquad \leq \frac{c_1}{(\log R)^{1+2\rho}}, \end{aligned} \tag{2.3}$$

where $c_1 = c_1(\rho) \in \mathbb{R}_+^*$ is a constant whose value depends only on ρ.

LEMMA 2.2. *Let $R > 0$ and $0 < a \leq 1$. Then*

$$\mathbb{P}\left\{ \sup_{|x|\leq a\sqrt{\tau_R}} L(\tau_R,x) > \sup_{|y|>a\sqrt{\tau_R}} L(\tau_R,y) + \sqrt{a}R \right\} \geq c_2 a, \tag{2.4}$$

where $c_2 \in \mathbb{R}_+^*$ is an absolute constant.

Before we proceed to prove the lemmas, let us recall our basic tool. Indeed, the main reason for which τ is of interest is the following Ray–Knight theorem, which describes the law of the process $(L(\tau_1, x), x \in \mathbb{R})$. The independence part in the theorem is a consequence of excursion theory.

FACT 2.3 ([15, 19]). The processes $(L(\tau_1, x), x \geq 0)$ and $(L(\tau_1, -x), x \geq 0)$ are independent squared Bessel processes of dimension 0, starting from 1.

For an account of definition and general properties of (squared) Bessel processes, we refer to the books of Borodin and Salminen [4] and Revuz and Yor [22].

The proof of Lemma 2.1 also relies on two useful results. The first one (Fact 2.4), which can be proved for example by means of the Ray–Knight theorem, is borrowed from [5] and is stated here in a weakened form; see also [3] for a slightly different version. The second (Fact 2.5), which we have learnt from [26], concerns level crossings of the Ornstein–Uhlenbeck process.

FACT 2.4 ([5]). For all $a \in (0, 1)$ and $\lambda \in (0, 1)$,

$$\mathbb{P}\left\{\sup_{|x| \leq a} L(\tau_1, x) > 1 + \lambda\right\} \leq 8 \exp\left(-\frac{\lambda^2}{16a}\right), \tag{2.5}$$

$$\mathbb{P}\left\{\sup_{|x| \leq a} L(\tau_1, x) < 1 - \lambda\right\} \leq 8 \exp\left(-\frac{\lambda^2}{16a}\right). \tag{2.6}$$

FACT 2.5 ([26]). There exist absolute constants $c_3 \in \mathbb{R}_+^*$ and $c_4 \in \mathbb{R}_+^*$ such that for all $a \in (0, 1)$ and $\lambda \in (0, 1)$,

$$\mathbb{P}\{W(t) < \lambda\sqrt{t}, \ \forall t \in [a, 1]\} \leq c_3 a^{1/2} \exp[c_4 \lambda \log(1/a)]. \tag{2.7}$$

We now start to prove Lemma 2.1. In the rest of this section, the letter c with subscripts denotes unimportant constants which are finite and positive.

PROOF OF LEMMA 2.1. It is clear that the probability expression on the left-hand side of (2.3) is nondecreasing in a and nonincreasing in r; therefore, we only have to prove the lemma for

$$a := (\log R)^{-1-5\rho}, \tag{2.8}$$

$$r := [1 - (\log R)^{-\rho}]R. \tag{2.9}$$

There is nothing to prove if R is bounded [in this case, we only have to take a large value of c_1 so that (2.3) holds trivially]. We will thus concentrate ourselves on the case $R \geq R_0$, where $R_0 = R_0(\rho)$ is a constant whose value will be given in (2.25).

For notational convenience, we write

$$\Delta = \Delta(R, \rho) := \frac{R}{(\log R)^{(1+3\rho)/2}}.$$

We define $T = T(\omega, R, \rho)$ by

$$T := \inf\left\{t \geq \tau_r : \sup_{|x| \leq a\sqrt{t}} L(t, x) > \sup_{|y| > a\sqrt{t}} L(t, y) - \Delta\right\},$$

with the usual convention $\inf \varnothing := \infty$. Then

$$\begin{aligned} &\left\{\sup_{|x| \leq a\sqrt{t}} L(t, x) > \sup_{|y| > a\sqrt{t}} L(t, y) - \Delta, \text{ for some } t \in [\tau_r, \tau_R]\right\} \\ &\quad = \{T < \tau_R\}, \end{aligned} \tag{2.10}$$

so that the probability term on the left-hand side of (2.3) is nothing else but $\mathbb{P}\{T < \tau_R\}$.

When $T < \infty$, we clearly have

$$\sup_{|x| \leq a\sqrt{T}} L(T, x) \geq \sup_{|y| > a\sqrt{T}} L(T, y) - \Delta, \tag{2.11}$$

$$|W(T)| \leq a\sqrt{T}. \tag{2.12}$$

In fact both (2.11) and (2.12) are identities in case $T > \tau_r$; in the other case $T = \tau_r$, we simply have $|W(T)| = 0$ while (2.11) is a strict inequality.

There are two main difficulties in dealing with $\mathbb{P}\{T < \tau_R\}$:

(i) In the event on the left-hand side of (2.10), we are taking maxima with respect to x and y over some *random* and *moving* intervals.

(ii) The random time T can lie anywhere between τ_r and τ_R (in the event which is of interest to us), and τ_r and τ_R can fall quite apart from each other (indeed, nothing prevents W from making a long excursion away from 0 after τ_r). If we do not know where T lies, then it will be hard to estimate the probability of an event involving the local time at T.

We overcome difficulty (i) by making a (deterministic) partition of the (random) interval $[\tau_r, \tau_R]$. Let $1 < b_0 < b_1 < \cdots < b_M < R/2$, whose values will be given in (2.23) and (2.24). For each fixed j, if $a\sqrt{T} \in [b_{j-1}, b_j]$, then (2.11) and (2.12) imply, respectively,

$$\sup_{|x| \leq b_j} L(T, x) \geq \sup_{|y| > b_j} L(T, y) - \Delta, \tag{2.13}$$

$$|W(T)| \leq b_j. \tag{2.14}$$

To overcome difficulty (ii), we introduce a "nice" event which, together with condition (2.13), leads us to something concerning the local time at τ_R, the probability of which we know how to estimate by means of the Ray–Knight theorem [the precise formulation of this idea is in (2.15)]. In order to introduce this "nice" event, we define, for $\omega \in \{T < \infty\}$,

$$D(T) := \inf\{t \ge T : W(t) = 0\},$$

which is the first zero of the Brownian motion W after T. Accordingly, there exists a (random) $\Theta \ge r$ such that $D(T) = \tau_\Theta$, and since $W(\tau_R) = 0$, it is clear that $\Theta \in [r, R]$ on $\{T < \tau_R\}$.

Let $c_5 := c_5(\rho) \in \mathbb{R}_+^*$ be a constant (depending only on ρ) whose value is determined in (2.20). For notational simplification, we write

$$f(u) = f_\rho(u) := (c_5)^2[\log(|u| \vee e)]^{\rho/2}, \qquad u \in \mathbb{R},$$

with the usual notation $p \vee q := \max\{p, q\}$. Note that f is nondecreasing on $\mathbb{R}_+$.

The "nice" event we have in mind is the intersection of the following two events, which are well defined for $\omega \in \{T < \tau_R\}$:

$$E_{j-} := \left\{\sup_{y\in\mathbb{R}}[L(\tau_\Theta, y) - L(T, y)] \le b_j\right\} \cap \left\{\sup_{T\le t\le\tau_\Theta} |W(t)| \le 2b_j\right\},$$

$$\begin{aligned} E_{j+} := \Big\{&|L(\tau_R, x) - L(\tau_\Theta, x) - (R - \Theta)| \\ &\le \sqrt{|x|(R-\Theta) f\big((R-\Theta)/|x|\big)} \text{ for all } |x| \le R - \Theta\Big\} \\ &\cap \left\{\sup_{\tau_\Theta\le t\le\tau_R} |W(t)| \le R - \Theta\right\}. \end{aligned}$$

On $E_{j+} \cap \{T < \tau_R\}$, we clearly have $L(\tau_R, x) = L(\tau_\Theta, x)$ for all $|x| > R$ and, moreover, for $|x| \le R$,

$$|L(\tau_R, x) - L(\tau_\Theta, x) - (R - \Theta)| \le \sqrt{|x|(R - r) f(R/|x|)}.$$

If $\omega \in \{a\sqrt{T} \in [b_{j-1}, b_j]\} \cap \{T < \tau_R\} \cap E_{j-} \cap E_{j+}$ and $b_j < |y| \le R$, then, in view of (2.13),

$$\begin{aligned} \sup_{|x|\le b_j} L(\tau_R, x) &\ge L(\tau_R, y) - \Delta - b_j - \sqrt{|y|(R-r) f(R/|y|)} \\ &\quad - \sup_{|x|\le b_j} \sqrt{|x|(R-r) f(R/|x|)}. \end{aligned}$$

Since

$$\sup_{|x|\le b_j} \sqrt{|x|(R-r) f(R/|x|)} \le c_6\sqrt{b_j(R-r) f(R)}$$

for some $c_6 = c_6(\rho)$, we have, for $\omega \in \{a\sqrt{T} \in [b_{j-1}, b_j]\} \cap \{T < \tau_R\} \cap E_{j-} \cap E_{j+}$ (writing $c_7 := 1 + c_6$)

$$\sup_{|x| \le b_j} L(\tau_R, x) \ge \sup_{b_j < |y| \le R} \bigl[L(\tau_R, y) - \Delta - b_j - c_7\sqrt{|y|(R-r)f(R)}\,\bigr].$$

If $\omega \in \{a\sqrt{T} \in [b_{j-1}, b_j]\} \cap \{T < \tau_R\} \cap E_{j-} \cap E_{j+}$ and $|y| > R$, then

$$\begin{aligned}\sup_{|x| \le b_j} L(\tau_R, x) &\ge L(\tau_R, y) - \Delta - b_j - \sup_{|x| \le b_j} \sqrt{|x|(R-r)f(R/|x|)} \\ &\ge L(\tau_R, y) - \Delta - b_j - c_7\sqrt{b_j(R-r)f(R)}.\end{aligned}$$

(Of course, in the last line, c_6 would have done the job in lieu of c_7.) Accordingly, by introducing the event

$$\begin{aligned}E_j := &\left\{\sup_{|x| \le b_j} L(\tau_R, x) \ge \sup_{b_j < |y| \le R} \bigl[L(\tau_R, y) - \Delta - b_j - c_7\sqrt{|y|(R-r)f(R)}\,\bigr]\right\} \\ &\cap \left\{\sup_{|x| \le b_j} L(\tau_R, x) \ge \sup_{|y| > R} L(\tau_R, y) - \Delta - b_j - c_7\sqrt{b_j(R-r)f(R)}\right\} \\ &\cap \left\{\tau_R \ge \frac{b_{j-1}^2}{a^2}\right\} \cap \left\{\sup_{0 \le t \le \tau_R} |W(t)| \le \sup_{0 \le t \le (b_j/a)^2} |W(t)| \vee R\right\},\end{aligned}$$

we have, for all $1 \le j \le M$,

$$\bigl(\{a\sqrt{T} \in [b_{j-1}, b_j]\} \cap \{T < \tau_R\} \cap E_{j-} \cap E_{j+}\bigr) \subset E_j. \tag{2.15}$$

This is our solution to overcome difficulty (ii).

Let us outline the rest of the proof of Lemma 2.1. Both E_{j-} and E_{j+} are "typical" events in the sense that $\mathbb{P}(E_{j-})$ and $\mathbb{P}(E_{j+})$ are greater than some positive constants. This will lead us, by exploiting the strong Markov property, to the estimate that $\mathbb{P}\{a\sqrt{T} \in [b_{j-1}, b_j], T < \tau_R\}$ is bounded by, say, $c_8\mathbb{P}\{E_j\}$. Summing over j, we will arrive at: $\mathbb{P}\{T < \tau_R\} \le c_8 \sum_j \mathbb{P}\{E_j\}$. Despite the somewhat complicated form of the event E_j, we will be able to estimate $\mathbb{P}\{E_j\}$ by means of the Ray–Knight theorem, which will give us an upper bound for $\mathbb{P}\{T < \tau_R\}$, and will thus yield Lemma 2.1. Now let us make things rigorous.

Let $(\mathcal{F}_t)_{t \ge 0}$ denote the natural filtration of W, then by (2.15),

$$\begin{aligned}\mathbb{P}(E_j) &\ge \mathbb{P}\bigl(b_{j-1} \le a\sqrt{T} \le b_j, T < \tau_R, E_{j-}, E_{j+}\bigr) \\ &= \mathbb{E}\Bigl[\mathbf{1}_{\{b_{j-1} \le a\sqrt{T} \le b_j, T < \tau_R, E_{j-}\}}\mathbb{P}\bigl(E_{j+} | \mathcal{F}_{D(T)}\bigr)\Bigr],\end{aligned} \tag{2.16}$$

the equality following from the fact that $\{a\sqrt{T} \in [b_{j-1}, b_j]\}$, $\{T < \tau_R\}$ and E_{j-} are all measurable with respect to $\mathcal{F}_{D(T)} = \mathcal{F}_{\tau_\Theta}$. By the strong Markov and scaling properties, on $\{T < \tau_R\}$,

$$\begin{aligned}
&\mathbb{P}\{E_{j+}|\mathcal{F}_{D(T)}\} \\
&\quad = \mathbb{P}\Big\{|L(\tau_1, x) - 1| \le \sqrt{|x| f(1/|x|)},\ \forall\, 0 < |x| \le 1,\ \sup_{0\le s\le\tau_1} |W(s)| \le 1\Big\} \qquad (2.17)\\
&\quad \ge \mathbb{P}\big\{|L(\tau_1, x) - 1| \le \sqrt{|x| f(1/|x|)},\ \forall\, 0 < |x| \le 1\big\} - c_9,
\end{aligned}$$

where

$$c_9 := \mathbb{P}\Big\{\sup_{0\le s\le\tau_1} |W(s)| > 1\Big\} < 1. \qquad (2.18)$$

(It is easy to see, e.g., by means of excursion theory, that $[\sup_{0\le s\le\tau_1} |W(s)|]^{-1}$ has the exponential distribution with mean 1, so that $c_9 = 1 - e^{-1}$, but we are not going to use the exact value of c_9.) According to the Ray–Knight theorem recalled in Fact 2.3, if $(Z(t), t \ge 0)$ denotes a *squared* Bessel process of dimension 0 starting from $Z(0) = 1$, then

$$\mathbb{P}\big\{|L(\tau_1, x) - 1| \le \sqrt{|x| f(1/|x|)},\ \forall\, 0 < |x| \le 1\big\} = [\mathbb{P}(A)]^2, \qquad (2.19)$$

where

$$A := \big\{|Z(t) - 1| \le \sqrt{t f(1/t)},\ \forall\, 0 < t \le 1\big\}.$$

We now estimate $\mathbb{P}(A)$. Since Z satisfies

$$Z(t) = 1 + B\bigg(4\int_0^t Z(s)\, ds\bigg),$$

where B is a standard Brownian motion, we can apply the usual iterated logarithm law to B to see that

$$\limsup_{t\to 0} \frac{|Z(t) - 1|}{\sqrt{2t \log\log(1/t)}} = 2 \qquad \text{a.s.},$$

a fortiori,

$$\sup_{0<t\le 1} \frac{|Z(t) - 1|}{\sqrt{t (\log \max\{e, 1/t\})^{\rho/2}}} < \infty \qquad \text{a.s.}$$

We can therefore choose a constant $c_5 := c_5(\rho)$ sufficiently large (how large depending only on ρ), such that

$$\mathbb{P}\bigg(\sup_{0<t\le 1} \frac{|Z(t) - 1|}{\sqrt{t (\log \max\{e, 1/t\})^{\rho/2}}} < c_5\bigg) > \sqrt{c_9}, \qquad (2.20)$$

where $c_9 < 1$ is the absolute constant defined in (2.18). Let $c_{10} = c_{10}(\rho)$ denote the probability on the left-hand side of (2.20). We have proved that $\mathbb{P}(A) = c_{10} > \sqrt{c_9}$, which, considered jointly with (2.19) and (2.18), yields that on $\{T < \tau_R\}$,

$$\mathbb{P}\big\{E_{j+}|\mathcal{F}_{D(T)}\big\} \ge (c_{10})^2 - c_9 > 0.$$

Plugging this into (2.16) we obtain [writing $c_{11} := (c_{10})^2 - c_9$]

$$\begin{aligned}\mathbb{P}(E_j) &\geq c_{11}\mathbb{P}\big(b_{j-1} \leq a\sqrt{T} \leq b_j, T < \tau_R, E_{j-}\big)\\ &= c_{11}\mathbb{E}\Big[\mathbf{1}_{\{b_{j-1}\leq a\sqrt{T}\leq b_j, T<\tau_R\}}\mathbb{P}(E_{j-}|\mathcal{F}_T)\Big].\end{aligned}$$

By the strong Markov property, if we write $\mathbb{P}^x$ for the probability under which W starts from $W(0) = x$ (one can work in the canonical space), and $H_0 := \inf\{t \geq 0 : W(t) = 0\}$, then on $\{a\sqrt{T} \in [b_{j-1}, b_j]\}$,

$$\begin{aligned}\mathbb{P}\{E_{j-}|\mathcal{F}_T\} &\geq \inf_{|x|\leq b_j} \mathbb{P}^x\Big\{\sup_{y\in\mathbb{R}} L(H_0, y) \leq b_j, \sup_{0\leq t\leq H_0} |W(t)| \leq 2b_j\Big\}\\ &= \inf_{|x|\leq 1} \mathbb{P}^x\Big\{\sup_{y\in\mathbb{R}} L(H_0, y) \leq 1, \sup_{0\leq t\leq H_0} |W(t)| \leq 2\Big\}\\ &:= c_{12} > 0.\end{aligned}$$

Accordingly, by writing $c_{13} := 1/(c_{11}c_{12})$, we arrive at

$$\mathbb{P}\big(b_{j-1} \leq a\sqrt{T} \leq b_j, T < \tau_R\big) \leq c_{13}\mathbb{P}(E_j), \qquad 1 \leq j \leq M.$$

Since $T \geq \tau_r$ by definition, we obtain

$$\mathbb{P}(T < \tau_R) \leq \mathbb{P}\Big(\sqrt{\tau_R} > \frac{b_M}{a}\Big) + \mathbb{P}\Big(\sqrt{\tau_r} < \frac{b_0}{a}\Big) + c_{13}\sum_{j=1}^{M}\mathbb{P}(E_j).$$

By Lévy's identity, for any fixed $u > 0$, $u/\sqrt{\tau_u}$ is distributed as the modulus of a standard Gaussian random variable, so that

$$\begin{aligned}\mathbb{P}(T < \tau_R) &\leq \frac{aR}{b_M} + \exp\Big(-\frac{a^2r^2}{2b_0^2}\Big) + c_{13}\sum_{j=1}^{M}\mathbb{P}(E_j)\\ &\leq \frac{aR}{b_M} + \exp\Big(-c_{14}\frac{a^2R^2}{b_0^2}\Big) + c_{13}\sum_{j=1}^{M}\mathbb{P}(E_j),\end{aligned} \tag{2.21}$$

the last inequality being a consequence of (2.9).

We now bound $\mathbb{P}(E_j)$. For notational simplicity, let us write

$$\begin{aligned}\varepsilon_j &:= \frac{b_j}{R},\\ \delta &:= \Big(1 - \frac{r}{R}\Big)f(R) = \frac{(c_5)^2}{(\log R)^{\rho/2}},\\ \Delta_j &:= \frac{\Delta + b_j}{R} = \frac{1}{(\log R)^{(1+3\rho)/2}} + \varepsilon_j,\end{aligned}$$

[the second identity for δ following from (2.9) and from the definition of f].

By scaling,

$$\begin{aligned}\mathbb{P}(E_j) = \mathbb{P}\biggl\{ & \sup_{|x|\le\varepsilon_j} L(\tau_1, x) \ge \sup_{\varepsilon_j<|y|\le 1} \bigl(L(\tau_1, y) - \Delta_j - c_7\sqrt{\delta|y|}\bigr),\\
& \sup_{|x|\le\varepsilon_j} L(\tau_1, x) \ge \sup_{|y|>1} L(\tau_1, y) - \Delta_j - c_7\sqrt{\varepsilon_j\delta},\\
& \tau_1 \ge \frac{b_{j-1}^2}{a^2R^2}, \sup_{0\le t\le\tau_1} |W(t)| \le \sup_{0\le t\le b_j^2/(aR)^2} |W(t)| \vee 1\biggr\}.\end{aligned} \tag{2.22}$$

From here, the estimate of $\mathbb{P}(E_j)$ will depend on whether $j = 1$ or $j \ge 2$. For the case $j \ge 2$, we note that

$$\begin{aligned}\mathbb{P}(E_j) \le \mathbb{P}\biggl\{ & \sup_{|x|\le\varepsilon_j} L(\tau_1, x) \ge \sup_{|y|>\varepsilon_j} L(\tau_1, y) - \Delta_j - c_7\sqrt{\delta},\\
& \tau_1 \ge \frac{b_{j-1}^2}{a^2R^2}, \sup_{0\le t\le\tau_1} |W(t)| \le \sup_{0\le t\le b_j^2/(aR)^2} |W(t)| \vee 1\biggr\}.\end{aligned}$$

By the usual Brownian tail estimate,

$$\mathbb{P}\biggl\{\sup_{0\le t\le b_j^2/(aR)^2} |W(t)| > \frac{b_j\sqrt{2f(R)}}{aR}\biggr\} \le 2e^{-f(R)}.$$

Accordingly,

$$\begin{aligned}\mathbb{P}(E_j) \le 2e^{-f(R)} + \mathbb{P}\biggl\{ & \sup_{|x|\le\varepsilon_j} L(\tau_1, x) \ge \sup_{|y|>\varepsilon_j} L(\tau_1, y) - \Delta_j - c_7\sqrt{\delta},\\
& \tau_1 \ge \frac{b_{j-1}^2}{a^2R^2}, \sup_{0\le t\le\tau_1} |W(t)| \le \frac{b_j\sqrt{2f(R)}}{aR}\biggr\},\end{aligned}$$

where we used the fact that we can choose $R_0 = R_0(\rho)$ sufficiently large so that $b_2\sqrt{2f(R)} \ge aR$ for all $R \ge R_0$, the choice of b_2 being fixed in (2.24). When the event $\{\cdots\}$ in the probability expression on the right-hand side is realized, we have $\sup_{x\in\mathbb{R}} L(\tau_1, x) \le \sup_{|x|\le\varepsilon_j} L(\tau_1, x) + \Delta_j + c_7\sqrt{\delta}$, so that

$$\begin{aligned}\frac{b_{j-1}^2}{a^2R^2} \le \tau_1 &= \int_{|x|\le b_j\sqrt{2f(R)}/(aR)} L(\tau_1, x)\, dx\\
&\le \frac{2b_j\sqrt{2f(R)}}{aR}\Bigl(\sup_{|x|\le\varepsilon_j} L(\tau_1, x) + \Delta_j + c_7\sqrt{\delta}\Bigr).\end{aligned}$$

As a result,

$$\mathbb{P}(E_j) \le 2e^{-f(R)} + \mathbb{P}\left\{ \sup_{|x|\le \varepsilon_j} L(\tau_1, x) \ge \frac{b_{j-1}^2}{ab_j R\sqrt{8f(R)}} - \Delta_j - c_7\sqrt{\delta} \right\}.$$

We now choose the values for b_j, $0 \le j \le M$:

$$b_0 := aR(\log R)^{-\rho}, \tag{2.23}$$

$$b_j := aR(\log R)^{(j+1)\rho/2}, \qquad 1 \le j \le M := \lfloor 2(1+2\rho)/\rho \rfloor. \tag{2.24}$$

Recall that we are only interested in $R \ge R_0$. Our choice of $R_0 = R_0(\rho)$ is as follows. For $2 \le j \le M$,

$$\begin{aligned}
&\frac{b_{j-1}^2}{ab_j R\sqrt{8f(R)}} - \Delta_j - c_7\sqrt{\delta} \\
&\quad = \frac{(\log R)^{j\rho/2 - 3\rho/4}}{\sqrt{8}c_5} - \frac{1}{(\log R)^{(1+3\rho)/2}} \\
&\qquad - (\log R)^{j\rho/2 - 1 - 9\rho/2} - \frac{c_7 c_5}{(\log R)^{\rho/4}} \\
&\quad \ge 2,
\end{aligned} \tag{2.25}$$

and R_0 is such that the last inequality in (2.25) holds for all $R \ge R_0$. If need be, we can enlarge the value of R_0 in order to ensure that conditions $b_2\sqrt{2f(R)} \ge aR$ and $b_M < R/2$ are fulfilled for all $R \ge R_0$.

Accordingly, for $2 \le j \le M$,

$$\mathbb{P}(E_j) \le 2e^{-f(R)} + \mathbb{P}\left\{ \sup_{|x|\le \varepsilon_j} L(\tau_1, x) \ge 2 \right\} \le 2e^{-f(R)} + 8\exp\left(-\frac{1}{16\varepsilon_j}\right),$$

the last inequality being a consequence of (2.5). Since for $j \le M$, $\varepsilon_j = a(\log R)^{(j+1)\rho/2} \le a(\log R)^{1+5\rho/2} = (\log R)^{-5\rho/2}$, we obtain

$$\mathbb{P}(E_j) \le 2e^{-f(R)} + 8\exp\left(-\frac{(\log R)^{5\rho/2}}{16}\right), \qquad j \ge 2.$$

Since M, defined in (2.24), is a constant depending only on ρ, we have

$$\begin{aligned}
\sum_{j=2}^{M} \mathbb{P}(E_j) &\le 2Me^{-f(R)} + 8M\exp\left(-\frac{(\log R)^{5\rho/2}}{16}\right) \\
&= 2M\exp\left(-(c_5)^2(\log R)^{\rho/2}\right) + 8M\exp\left(-\frac{(\log R)^{5\rho/2}}{16}\right) \\
&\le \frac{c_{15}}{(\log R)^{1+2\rho}}.
\end{aligned} \tag{2.26}$$

We now estimate $\mathbb{P}(E_1)$. By (2.22),

$$\mathbb{P}(E_1) \le \mathbb{P}\biggl\{ \sup_{|x|\le\varepsilon_1} L(\tau_1, x) \ge \sup_{\varepsilon_1<|y|\le 1} \bigl(L(\tau_1, y) - \Delta_1 - c_7\sqrt{\delta|y|}\bigr)\biggr\}.$$

If the event $\{\cdots\}$ in the probability term on the right-hand side is realized, and if furthermore $\sup_{|x|\le\varepsilon_1} L(\tau_1, x) \le 1 + 4\sqrt{\varepsilon_1 \log(1/\varepsilon_1)}$, then

$$\begin{aligned}\sup_{\varepsilon_1<|y|\le\delta} \bigl(L(\tau_1, y) - c_7\sqrt{\delta|y|}\bigr) &\le \sup_{\varepsilon_1<|y|\le 1} \bigl(L(\tau_1, y) - c_7\sqrt{\delta|y|}\bigr)\\ &\le \sup_{|x|\le\varepsilon_1} L(\tau_1, x) + \Delta_1\\ &\le 1 + 4\sqrt{\varepsilon_1 \log(1/\varepsilon_1)} + \Delta_1.\end{aligned}$$

Therefore, by introducing

$$\widehat{\Delta} := 4\sqrt{\varepsilon_1 \log(1/\varepsilon_1)} + \Delta_1 = 4\sqrt{\varepsilon_1 \log(1/\varepsilon_1)} + \frac{1}{(\log R)^{(1+3\rho)/2}} + \varepsilon_1,$$

$$\widehat{E} := \biggl\{ \sup_{\varepsilon_1<|y|\le\delta} \bigl(L(\tau_1, y) - c_7\sqrt{\delta|y|}\bigr) < 1 + \widehat{\Delta}, \inf_{|y|\le\delta} L(\tau_1, y) > \tfrac{1}{4}\biggr\},$$

we have

$$\begin{aligned}\mathbb{P}(E_1) \le \mathbb{P}(\widehat{E}) &+ \mathbb{P}\biggl\{ \sup_{|x|\le\varepsilon_1} L(\tau_1, x) > 1 + 4\sqrt{\varepsilon_1 \log(1/\varepsilon_1)}\biggr\}\\ &+ \mathbb{P}\biggl\{ \inf_{|y|\le\delta} L(\tau_1, y) \le \tfrac{1}{4}\biggr\}.\end{aligned}$$

According to (2.5),

$$\mathbb{P}\biggl\{ \sup_{|x|\le\varepsilon_1} L(\tau_1, x) > 1 + 4\sqrt{\varepsilon_1 \log(1/\varepsilon_1)}\biggr\} \le 8\varepsilon_1,$$

whereas by means of (2.6),

$$\mathbb{P}\biggl\{ \inf_{|y|\le\delta} L(\tau_1, y) \le \frac{1}{4}\biggr\} \le 8\exp\biggl(-\frac{9}{256\delta}\biggr) = 8\exp[-c_{16}(\log R)^{\rho/2}].$$

Therefore,

$$\mathbb{P}(E_1) \le \mathbb{P}(\widehat{E}) + 8\varepsilon_1 + 8\exp[-c_{16}(\log R)^{\rho/2}]. \tag{2.27}$$

It remains to estimate $\mathbb{P}(\widehat{E})$. According to the Ray–Knight theorem (Fact 2.3), if Y denotes a zero-dimensional Bessel process with $Y(0) = 1$, then

$$\begin{aligned}\mathbb{P}(\widehat{E}) &= \biggl[\mathbb{P}\biggl(\sup_{\varepsilon_1<t\le\delta} \bigl(Y^2(t) - c_7\sqrt{\delta t}\bigr) < 1 + \widehat{\Delta}, \inf_{0\le t\le\delta} Y^2(t) > \frac{1}{4}\biggr)\biggr]^2\\ &\le \biggl[\mathbb{P}\biggl(\sup_{\varepsilon_1<t\le\delta} \bigl(Y(t) - c_7\sqrt{\delta t}\bigr) < 1 + \frac{\widehat{\Delta}}{2}, \inf_{0\le t\le\delta} Y(t) > \frac{1}{2}\biggr)\biggr]^2.\end{aligned}$$

Recall that Y satisfies the stochastic differential equation (B denoting a standard Brownian motion)

$$Y(t) = 1 + B(t) - \frac{1}{2}\int_0^t \frac{ds}{Y(s)},$$

for all $0 \le t < \zeta_Y := \inf\{s > 0 : Y(s) = 0\}$. Therefore, if $\inf_{0\le t\le\delta} Y(t) > 1/2$, then $\zeta_Y > \delta$, and for all $t \le \delta$, $Y(t) \ge 1 + B(t) - t \ge 1 + B(t) - \sqrt{\delta t}$. Consequently,

$$\begin{aligned}\mathbb{P}(\widehat{E}) &\le \left[\mathbb{P}\left(B(t) < (1+c_7)\sqrt{\delta t} + \frac{\widehat{\Delta}}{2}, \forall\, \varepsilon_1 < t \le \delta\right)\right]^2 \\ &\le \left[\mathbb{P}\big(B(t) < (2+c_7)\sqrt{\delta t}, \forall\, \nu \le t \le \delta\big)\right]^2,\end{aligned}$$

where ν is such that $\sqrt{\delta\nu} := \widehat{\Delta}/2$, or equivalently, $\nu := (4\sqrt{\varepsilon_1 \log(1/\varepsilon_1)} + \Delta_1)^2/(4\delta)$, which is greater than ε_1. By scaling,

$$\mathbb{P}(\widehat{E}) \le \left[\mathbb{P}\big(B(s) < (2+c_7)\sqrt{\delta s}, \forall\, s \in [\delta^{-1}\nu, 1]\big)\right]^2,$$

which, in light of (2.7), yields

$$\begin{aligned}\mathbb{P}(\widehat{E}) &\le \left[c_3\frac{\sqrt{\nu}}{\sqrt{\delta}}\exp\left(c_4(2+c_7)\sqrt{\delta}\log\frac{\delta}{\nu}\right)\right]^2 \\ &\le c_{17}\frac{\nu}{\delta} \\ &\le c_{18}\frac{\varepsilon_1\log(1/\varepsilon_1) + \Delta_1^2}{\delta^2} \\ &\le c_{19}\frac{\varepsilon_1\log(1/\varepsilon_1)}{\delta^2} + \frac{c_{20}}{\delta^2(\log R)^{1+3\rho}}.\end{aligned}$$

Since $\varepsilon_1 = a(\log R)^\rho = (\log R)^{-1-4\rho}$ [cf. (2.8)], we have

$$\frac{\varepsilon_1\log(1/\varepsilon_1)}{\delta^2} = \frac{1+4\rho}{(c_5)^4}\frac{\log\log R}{(\log R)^{1+3\rho}} \le \frac{c_{21}}{(\log R)^{1+2\rho}},$$

whereas $\delta^2(\log R)^{1+3\rho} = (c_5)^4(\log R)^{1+2\rho}$. Therefore,

$$\mathbb{P}(\widehat{E}) \le \frac{c_{19}c_{21}}{(\log R)^{1+2\rho}} + \frac{c_{20}/(c_5)^4}{(\log R)^{1+2\rho}} := \frac{c_{22}}{(\log R)^{1+2\rho}}.$$

Plugging this into (2.27) gives

$$\mathbb{P}(E_1) \le \frac{c_{23}}{(\log R)^{1+2\rho}}.$$

This estimate, together with (2.27) and (2.21), implies that

$$\mathbb{P}(T < \tau_R) \le \frac{aR}{b_M} + \exp\left(-c_{14}\frac{a^2R^2}{b_0^2}\right) + \frac{c_{24}}{(\log R)^{1+2\rho}}.$$

Since $aR/b_M = (\log R)^{-(M+1)\rho/2} \le (\log R)^{-(1+2\rho)}$ and $a^2R^2/b_0^2 = (\log R)^{2\rho}$, we obtain

$$\mathbb{P}(T < \tau_R) \le \frac{1}{(\log R)^{1+2\rho}} + \exp\bigl[-c_{14}(\log R)^{2\rho}\bigr] + \frac{c_{24}}{(\log R)^{1+2\rho}} \le \frac{c_{25}}{(\log R)^{1+2\rho}}.$$

In view of (2.10), this completes the proof of Lemma 2.1. □

PROOF OF LEMMA 2.2. By scaling,

$$p(a) := \mathbb{P}\Bigl\{\sup_{|x|\le a\sqrt{\tau_R}} L(\tau_R, x) > \sup_{|y|>a\sqrt{\tau_R}} L(\tau_R, y) + \sqrt{a}R\Bigr\} = \mathbb{P}\Bigl\{\sup_{|x|\le a\sqrt{\tau_1}} L(\tau_1, x) > \sup_{|y|>a\sqrt{\tau_1}} L(\tau_1, y) + \sqrt{a}\Bigr\}.$$

There is nothing to prove in case $a \in [1/17, 1]$ (it suffices then to take a small value of the constant c_2). So we assume $a \in (0, 1/17)$ from now on. Note that

$$p(a) \ge \mathbb{P}\Bigl\{\sup_{|x|\le a} L(\tau_1, x) > \sup_{|y|>a} L(\tau_1, y) + \sqrt{a}, \tau_1 \ge 1\Bigr\}.$$

By the occupation time formula, $\tau_1 = \int_{\mathbb{R}} L(\tau_1, y)\,dy$, so that

$$p(a) \ge \mathbb{P}\Bigl\{\sup_{|x|\le a} L(\tau_1, x) > \sup_{|y|>a} L(\tau_1, y) + \sqrt{a}, \inf_{1\le y\le 3} L(\tau_1, y) \ge \tfrac{1}{2}\Bigr\}.$$

According to the Ray–Knight theorem (Fact 2.3), if Z and $\widetilde{Z}$ are two independent zero-dimensional *squared* Bessel processes starting from 1, then

$$\begin{aligned}
p(a) &\ge \mathbb{P}\Bigl\{\sup_{0\le t\le a}\bigl(Z(t)\vee\widetilde{Z}(t)\bigr) > \sup_{t>a}\bigl(Z(t)\vee\widetilde{Z}(t)\bigr) + \sqrt{a}, \inf_{1\le t\le 3} Z(t) \ge \tfrac{1}{2}\Bigr\}\\
&\ge \mathbb{P}\Bigl\{\sup_{0\le t\le a} Z(t) \ge 1 + 2\sqrt{a}, 1 - \sqrt{a} \le Z(a) \le 1\Bigr\}\\
&\quad \times \mathbb{P}\bigl\{1 - \sqrt{a} \le \widetilde{Z}(a) \le 1\bigr\}\\
&\quad \times \inf_{1-\sqrt{a}\le x\le 1} \mathbb{P}^x\Bigl\{\sup_{t\ge 0} Z(t) < 1 + \sqrt{a}, \inf_{1-a\le t\le 3-a} Z(t) \ge \tfrac{1}{2}\Bigr\}\\
&\quad \times \inf_{1-\sqrt{a}\le x\le 1} \mathbb{P}^x\Bigl\{\sup_{t\ge 0} \widetilde{Z}(t) < 1 + \sqrt{a}\Bigr\},
\end{aligned}$$

where $\mathbb{P}^x$ is the probability under which the squared Bessel process Z starts from x (thus $\mathbb{P} = \mathbb{P}^1$).

If we could show that

$$\mathbb{P}\Big\{\sup_{0\le t\le a} Z(t) \ge 1+2\sqrt{a},\, 1-\sqrt{a}\le Z(a)\le 1\Big\} \ge c_{26}, \tag{2.28}$$

$$\inf_{1-\sqrt{a}\le x\le 1} \mathbb{P}^x\Big\{\sup_{t\ge 0} Z(t) < 1+\sqrt{a},\, \inf_{1-a\le t\le 3-a} Z(t) \ge \tfrac12\Big\} \ge c_{27}\sqrt{a}, \tag{2.29}$$

then we would have $p(a) \ge (c_{26}c_{27}\sqrt{a}\,)^2$, which would complete the proof of Lemma 2.2, with $c_2 := (c_{26}c_{27})^2$.

It remains to verify (2.28) and (2.29). We start with the proof of (2.29). Write

$$\sigma_r := \inf\{t\ge 0 : Z_t = r\}, \qquad r\ge 0.$$

Recall that $a < 1/17$. For all $1-\sqrt{a}\le x\le 1$, we have

$$\begin{aligned}
&\mathbb{P}^x\Big\{\sup_{t\ge 0} Z(t) < 1+\sqrt{a},\, \inf_{1-a\le t\le 3-a} Z(t) \ge \tfrac12\Big\}\\
&\quad\ge \mathbb{P}^x\Big\{\sigma_{3/4} < \sigma_{1+\sqrt{a}},\, \inf_{\sigma_{3/4}\le t\le 3+\sigma_{3/4}} Z(t)\ge \tfrac12,\, \sup_{t\ge\sigma_{3/4}} Z(t)\le 1\Big\} \qquad (2.30)\\
&\quad= \mathbb{P}^x\{\sigma_{3/4} < \sigma_{1+\sqrt{a}}\}\mathbb{P}^{3/4}\Big\{\inf_{0\le t\le 3} Z(t)\ge\tfrac12,\, \sup_{t\ge 0} Z(t)\le 1\Big\},
\end{aligned}$$

the last identity being a consequence of the strong Markov property. The second probability expression on the right-hand side is a (strictly) positive constant. Since Z is a diffusion process in its natural scale ([22], Chapter XI), we have, uniformly in $x\in[1-\sqrt{a},1]$,

$$\mathbb{P}^x\{\sigma_{3/4} < \sigma_{1+\sqrt{a}}\} = \frac{1+\sqrt{a}-x}{1+\sqrt{a}-3/4} \ge \frac{1+\sqrt{a}-1}{1+1-3/4} = \frac45\sqrt{a},$$

which, with the aid of (2.30), yields (2.29).

To check (2.28), we note that by the Markov property, the probability on the left-hand side of (2.28) is greater than or equal to

$$\mathbb{P}\{1+2\sqrt{a}\le Z(a/2)\le 1+3\sqrt{a}\,\} \inf_{x\in[1+2\sqrt{a},1+3\sqrt{a}\,]} \mathbb{P}^x\{1-\sqrt{a}\le Z(a/2)\le 1\}.$$

Recall the semigroup of the squared Bessel process Z ([22], Chapter XI): for $x>0$ and $y>0$,

$$\mathbb{P}^x\{Z(t)\in dy\} = \frac{1}{2t}\frac{x^{1/2}}{y^{1/2}}\exp\Big(-\frac{x+y}{2t}\Big) I_1\Big(\frac{(xy)^{1/2}}{t}\Big)\,dy$$

(and $\mathbb{P}^x\{Z(t)=0\} = e^{-x/(2t)}$), where $I_1(\cdot)$ is the modified Bessel function of index 1. It is known (see, e.g., [1], page 377, Formula 9.7.1) that

$$I_1(z) = \frac{1+o(1)}{(2\pi z)^{1/2}}e^z, \qquad z\to\infty,$$

from which it follows that uniformly in $a \in (0, 1/17)$,

$$\mathbb{P}\big\{1 + 2\sqrt{a} \le Z(a/2) \le 1 + 3\sqrt{a}\big\} \ge c_{28},$$
$$\inf_{x\in[1+2\sqrt{a},1+3\sqrt{a}\,]} \mathbb{P}^x\big\{1 - \sqrt{a} \le Z(a/2) \le 1\big\} \ge c_{29}.$$

This yields (2.28), and completes the proof of Lemma 2.2. □

3. Proof of Theorem 1.1: the "otherwise" part. This section is devoted to the proof of the lower bound in Theorem 1.1: for any $\gamma > 1$,

$$\lim_{n\to\infty} \frac{|V(n)|}{n^{1/2}(\log n)^{-\gamma}} = \infty \qquad \text{a.s.} \tag{3.1}$$

Before starting the proof, we recall two useful results. The first (Fact 3.1) is a strong invariance principle for local time, and the second (Fact 3.2) concerns the increments of the local time of Brownian motion.

FACT 3.1 ([20]). Possibly in an enlarged probability space, there exists a coupling for simple random walk (S_n) and standard Brownian motion W, such that for any $\varepsilon > 0$, when $n \to \infty$,

$$\sup_{x\in\mathbb{Z}} |L(n, x) - N(n, x)| = o\big(n^{(1/4)+\varepsilon}\big) \qquad \text{a.s.,} \tag{3.2}$$

where L is the local time of W, and N is the number of visits of (S_n) as in (1.1).

FACT 3.2 ([3]). Let L be the local time of Brownian motion. For any $\varepsilon > 0$, when $t \to \infty$,

$$\sup_{(x,y)\in\mathbb{R}^2,|x-y|\le 1} |L(t, x) - L(t, y)| = o\big(t^{(1/4)+\varepsilon}\big) \qquad \text{a.s.} \tag{3.3}$$

Without loss of generality, we shall be working with the coupled processes (S_n) and W defined in Fact 3.1. Fix $\gamma > 1$. It is possible to choose $\beta \in (1/2, 1)$ such that

$$\gamma > 1 + \frac{5(1-\beta)}{\beta}. \tag{3.4}$$

Let $r_k := \exp(k^\beta)$. It is easily checked that

$$1 - \frac{r_{k-1}}{r_k} = \frac{\beta + o(1)}{(\log r_k)^{(1-\beta)/\beta}}, \qquad k \to \infty.$$

Thus, for all large k, $r_{k-1} \ge [1 - (\log r_k)^{-(1-\beta)/\beta}]r_k$. By applying Lemma 2.1 to $\rho = (1-\beta)/\beta$, $R = r_k$, $r = r_{k-1}$ and $a = (\log r_k)^{-\gamma}$ [thus $a < (\log R)^{-1-5\rho}$ in view of (3.4)], we obtain: for all large k,

$$\mathbb{P}\Big\{ \sup_{|x|\le\sqrt{t}(\log r_k)^{-\gamma}} L(t,x) > \sup_{|y|>\sqrt{t}(\log r_k)^{-\gamma}} L(t,y) - \Delta_k, \text{ for some } t\in[\tau_{r_{k-1}}, \tau_{r_k}]\Big\}$$
$$\le \frac{c_1}{(\log r_k)^{1+2(1-\beta)/\beta}},$$

where $\Delta_k := r_k/(\log r_k)^{(3-2\beta)/(2\beta)}$. Since $\sum_k 1/(\log r_k)^{1+2(1-\beta)/\beta} < \infty$, it follows from the Borel–Cantelli lemma that with probability one, for all sufficiently large k and all $t\in[\tau_{r_{k-1}}, \tau_{r_k}]$,

$$\sup_{|x|\le\sqrt{t}(\log r_k)^{-\gamma}} L(t,x) \le \sup_{|y|>\sqrt{t}(\log r_k)^{-\gamma}} L(t,y) - \Delta_k.$$

Fix $\varepsilon>0$. In view of Facts 3.1 and 3.2, this yields that almost surely for all large k and all $n\in[\tau_{r_{k-1}}, \tau_{r_k}]$,

$$\begin{aligned}&\max_{|x|\le\sqrt{n}(\log r_k)^{-\gamma}} N(n,x)\\ &\qquad \le \max_{|y|>\sqrt{n}(\log r_k)^{-\gamma}} N(n,y) - \Delta_k + \left(\frac{\sqrt{\tau_{r_k}}}{(\log r_k)^{\gamma}}\right)^{(1/4)+\varepsilon}.\end{aligned} \tag{3.5}$$

At this stage, it is convenient to recall that

$$\limsup_{t\to\infty} \frac{\sup_{x\in\mathbb{R}} L(t,x)}{(2t\log\log t)^{1/2}} = 1 \qquad \text{a.s.,} \tag{3.6}$$

$$\lim_{t\to\infty} \frac{L(t,0)}{t^{1/2}(\log t)^{-2}} = \infty \qquad \text{a.s.} \tag{3.7}$$

Indeed, (3.6) is Kesten's iterated logarithm law for the local time [14], while (3.7) is a special case of Hirsch's law [13]. Whereas Hirsch's law was formulated for the supremum of partial sums (or equivalently for that of Brownian motion), it holds, by virtue of Lévy's identity, for the local time as well.

From (3.7) it follows that almost surely for all large r,

$$\tau_r \le r^2(\log r)^4. \tag{3.8}$$

Thus, $\varepsilon>0$ being a (small) fixed constant, almost surely for all large k,

$$\left(\frac{\sqrt{\tau_{r_k}}}{(\log r_k)^{\gamma}}\right)^{(1/4)+\varepsilon} \le \big(r_k(\log r_k)^{2-\gamma}\big)^{(1/4)+\varepsilon} < \Delta_k.$$

Going back to (3.5), we obtain: almost surely for all large k and all $n\in[\tau_{r_{k-1}}, \tau_{r_k}]$,

$$\max_{x\in\mathbb{Z}:\,|x|\le\sqrt{n}\,(\log r_k)^{-\gamma}} N(n,x) < \max_{y\in\mathbb{Z}:\,|y|>\sqrt{n}\,(\log r_k)^{-\gamma}} N(n,y),$$

which, by definition of $V(n)$, means

$$|V(n)| > \sqrt{n}\,(\log r_k)^{-\gamma}.$$

When $k \to \infty$ and $n \in [\tau_{r_{k-1}}, \tau_{r_k}]$, we have, almost surely,

$$(\log r_k)^{-\gamma} = \frac{1+o(1)}{(\log r_{k-1})^\gamma} \geq \frac{1+o(1)}{[\log L(n,0)]^\gamma} \geq \frac{1+o(1)}{[(1/2)\log n]^\gamma},$$

the last inequality being a consequence of (3.6). Therefore,

$$\liminf_{n\to\infty} \frac{|V(n)|}{n^{1/2}(\log n)^{-\gamma}} \geq 2^\gamma \qquad \text{a.s.}$$

Since $\gamma > 1$ is arbitrary, this yields (3.1), and therefore completes the proof of the "otherwise" part of Theorem 1.1.

4. Proof of Theorem 1.1: the "if" part. This section is devoted to the proof of the upper bound in Theorem 1.1. We note that only the case $\gamma = 1$ needs treated, namely,

$$\liminf_{n\to\infty} \frac{|V(n)|}{n^{1/2}(\log n)^{-1}} = 0 \qquad \text{a.s.} \tag{4.1}$$

As for the proof of the lower bound (cf. Section 3), we shall again be working with the random walk (S_n) and Brownian motion W defined in Fact 3.1. We consider the sequence $r_k := k^{5k}$. Let τ be as before the inverse local time at 0 of W [cf. (2.2)], and let

$$W^{(k)}(t) := W(t + \tau_{r_{k-1}}), \qquad t \geq 0.$$

The strong Markov property tells us that for each $k \geq 2$, $W^{(k)}$ is a standard Brownian motion independent of $\mathcal{F}_{\tau_{r_{k-1}}}$ [recalling that $(\mathcal{F}_t)$ is the natural filtration of W]. We can of course define for this new Brownian motion its local time [denoted by $L^{(k)}(t,x)$] and its inverse local time at 0 [denoted by $\tau_r^{(k)}$]. Clearly,

$$L^{(k)}(t,x) = L(t+\tau_{r_{k-1}}, x) - L(\tau_{r_{k-1}}, x), \qquad t \geq 0, x \in \mathbb{R}, \tag{4.2}$$

$$\tau_r^{(k)} = \tau_{r+r_{k-1}} - \tau_{r_{k-1}}, \qquad r > 0. \tag{4.3}$$

We fix $\lambda > 0$, and define

$$a_k := \frac{\lambda}{\log r_k},$$

$$s_k := r_k - r_{k-1},$$

$$A_k := \left\{ \sup_{|x| \leq a_k \sqrt{\tau_{s_k}^{(k)}}} L^{(k)}(\tau_{s_k}^{(k)}, x) > \sup_{|y| > a_k \sqrt{\tau_{s_k}^{(k)}}} L^{(k)}(\tau_{s_k}^{(k)}, y) + \sqrt{a_k} s_k \right\}.$$

By (4.2) and (4.3), each A_k is measurable with respect to $\mathcal{F}_{\tau_{s_k + r_{k-1}}} = \mathcal{F}_{\tau_{r_k}}$. Therefore the events $\{A_k : k \geq 2\}$ are independent. Moreover,

$$\mathbb{P}(A_k) = \mathbb{P}\left\{ \sup_{|x| \leq a_k \sqrt{\tau_{s_k}}} L(\tau_{s_k}, x) > \sup_{|y| > a_k \sqrt{\tau_{s_k}}} L(\tau_{s_k}, y) + \sqrt{a_k} s_k \right\},$$

which, according to Lemma 2.2, is greater than or equal to $c_2 a_k$. Since $\sum_k a_k = \infty$, the Borel–Cantelli lemma confirms that almost surely there exist infinitely many k such that

$$\sup_{|x|\le a_k\sqrt{\tau_{s_k}^{(k)}}} L^{(k)}\big(\tau_{s_k}^{(k)}, x\big) > \sup_{|y|> a_k\sqrt{\tau_{s_k}^{(k)}}} L^{(k)}\big(\tau_{s_k}^{(k)}, y\big) + \sqrt{a_k}\, s_k,$$

which, in light of (4.2) and (4.3), in turn implies

$$\sup_{|x|\le a_k\sqrt{\tau_{s_k}^{(k)}}} L\big(\tau_{r_k}, x\big) > \sup_{|y|> a_k\sqrt{\tau_{s_k}^{(k)}}} L\big(\tau_{r_k}, y\big) + \sqrt{a_k}\, s_k - \sup_{y\in\mathbb{R}} L\big(\tau_{r_{k-1}}, y\big).$$

Applying Facts 3.1 and 3.2, we get the corresponding result for random walk: almost surely there exist infinitely many k such that

$$\begin{aligned}(4.4)\qquad &\max_{|x|\le a_k\sqrt{\tau_{s_k}^{(k)}}} N\big(\lfloor \tau_{r_k}\rfloor, x\big)\\ &\quad > \max_{|y|> a_k\sqrt{\tau_{s_k}^{(k)}}} N\big(\lfloor \tau_{r_k}\rfloor, y\big) + \sqrt{a_k}\, s_k - \sup_{y\in\mathbb{R}} L\big(\tau_{r_{k-1}}, y\big) - \tau_{r_k}^{(1/4)+\varepsilon}.\end{aligned}$$

In view of (3.6) and (3.8), we have, almost surely for $k \to \infty$,

$$\begin{aligned}\sup_{y\in\mathbb{R}} L\big(\tau_{r_{k-1}}, y\big) + \tau_{r_k}^{1/4+\varepsilon} &\le \tau_{r_{k-1}}^{1/2} \log \tau_{r_{k-1}} + \tau_{r_k}^{1/4+\varepsilon}\\ &\le r_{k-1}(\log r_{k-1})^4 + r_k^{1/2+2\varepsilon}(\log r_k)^{1+4\varepsilon}\\ &= o\big(\sqrt{a_k}\, s_k\big).\end{aligned}$$

Plugging this into (4.4) yields that almost surely, we have, for infinitely many k,

$$\max_{|x|\le a_k\sqrt{\tau_{s_k}^{(k)}}} N\big(\lfloor \tau_{r_k}\rfloor, x\big) > \max_{|y|> a_k\sqrt{\tau_{s_k}^{(k)}}} N\big(\lfloor \tau_{r_k}\rfloor, y\big).$$

By definition, we have found infinitely many k such that

$$\big|V\big(\lfloor \tau_{r_k}\rfloor\big)\big| \le a_k\sqrt{\tau_{s_k}^{(k)}} \le a_k\sqrt{\tau_{r_k}}.$$

According to (3.8), we have $a_k \le (2\lambda + o(1))/\log \tau_{r_k}$, a.s. (for $k \to \infty$), which leads us to the following estimate:

$$\liminf_{n\to\infty} \frac{|V(n)|}{n^{1/2}(\log n)^{-1}} \le 2\lambda \qquad \text{a.s.}$$

Sending λ to 0 readily yields (4.1), and thus completes the proof of the "if" part in Theorem 1.1.

5. Remarks and questions. In this final section, we first mention some related results concerning, respectively, favorite sites in $\mathbb{Z}_+$ of random walk, and favorite sites of Brownian motion. We finish the paper with some unanswered questions.

5.1. *Nonnegative favorite sites.* Bass and Griffin [3] also considered *nonnegative* favorite sites: let $V_+(n) \in \mathbb{Z}_+$ be such that

$$N\big(n, V_+(n)\big) = \max_{x \in \mathbb{Z}_+} N(n, x). \tag{5.1}$$

They proved that, almost surely,

$$\liminf_{n\to\infty} \frac{V_+(n)}{n^{1/2}(\log n)^{-\gamma}} = \begin{cases} 0, & \text{if } \gamma < 2, \\ \infty, & \text{if } \gamma > 11. \end{cases}$$

An inspection of the proof of Theorem 1.1 readily yields the following result concerning the escape rate of $V_+(n)$:

THEOREM 5.1. *Let $V_+(n)$ be as in* (5.1). *Then*

$$\liminf_{n\to\infty} \frac{V_+(n)}{n^{1/2}(\log n)^{-\gamma}} = \begin{cases} 0, & \textit{if } \gamma \le 2, \\ \infty, & \textit{otherwise}. \end{cases}$$

5.2. *Favorite sites of Brownian motion.* The problem of favorite sites is also naturally posed for Brownian motion. Let W be a standard Brownian motion and let L denote the local time of W. For each t, we call

$$\mathbb{U}(t) := \Big\{x \in \mathbb{R} : L(t, x) = \sup_{y \in \mathbb{R}} L(t, y)\Big\}$$

the set of favorite sites at time t. In contrast to the case of random walk [cf. the Erdős–Révész conjecture stated in (1.6)], it is not hard to determine the cardinality of $\mathbb{U}(t)$. For each fixed t, $\mathbb{U}(t)$ is almost surely a singleton; and almost surely for all t, $\#\mathbb{U}(t)$ is either 1 or 2. See [8] and [16] for more details. This property actually holds for symmetric stable processes [8], and was recently extended to more general symmetric Markov processes by Eisenbaum and Khoshnevisan [10].

Similarly, one can consider $\mathbb{U}_+(t) := \{x \in \mathbb{R}_+ : L(t, x) = \sup_{y \ge 0} L(t, y)\}$, the set of nonnegative favorite sites at time t. It is easy to adapt our argument to show the analogues of Theorems 1.1 and 5.1 for Brownian motion.

THEOREM 5.2. *Let $U(t) \in \mathbb{U}(t)$ and $U_+(t) \in \mathbb{U}_+(t)$. With probability* 1,

$$\liminf_{t\to\infty} \frac{|U(t)|}{t^{1/2}(\log t)^{-\gamma}} = \begin{cases} 0, & \textit{if } \gamma \le 1, \\ \infty, & \textit{otherwise}, \end{cases}$$

$$\liminf_{t\to\infty} \frac{U_+(t)}{t^{1/2}(\log t)^{-\gamma}} = \begin{cases} 0, & \textit{if } \gamma \le 2, \\ \infty, & \textit{otherwise}. \end{cases}$$

Again, as for random walk, Theorem 5.2 holds uniformly in $U(t) \in \mathbb{U}(t)$ and $U_+(t) \in \mathbb{U}_+(t)$.

5.3. *Favorite sites of symmetric stable processes.* The Bass–Griffin theorem tells us that the process of favorite sites of Brownian motion (or simple random walk) is transient with probability 1. This property is enjoyed by symmetric stable processes. Let $(X_\alpha(t), t \ge 0)$ be a symmetric stable process of index $\alpha \in (1, 2]$. The condition $\alpha > 1$ is here to ensure the existence of the local time L_α of X_α. Let $V_\alpha(t)$ be a favorite site, in the sense that $L_\alpha(t, V_\alpha(t)) = \sup_{x\in\mathbb{R}} L_\alpha(t, x)$. It was proved by Bass, Eisenbaum and Shi [2] that for sufficiently large γ,

$$\lim_{t\to\infty} \frac{|V_\alpha(t)|}{t^{1/\alpha}(\log t)^{-\gamma}} = \infty \qquad \text{a.s.}$$

In particular, this implies the transience of V_α. The result was recently extended by Marcus [18] to a class of symmetric Lévy processes.

It seems interesting to know what the escape rate of V_α is. In order to answer this question, one needs to know some fine properties of fractional Brownian motion. As such, this can also be viewed as a problem in Gaussian theory.

5.4. *About the Erdős–Révész conjecture.* Let us say a few words about the Erdős–Révész conjecture stated in (1.6), concerning the probability that $\#\mathbb{V}(n) \ge 3$ for infinitely many n. One of the reasons it is hard to study the asymptotic behavior of $\#\mathbb{V}(n)$ is that little is known about the "site local time" process $x \mapsto N(n, x)$ [for the definition of N, see (1.1)], whether n is random or deterministic. In the literature, there is a description by Knight [15] for the "edge local time" process (i.e., instead of counting the number of visits at each site, one counts the visits at each edge) of random walk stopped at some carefully chosen random times, and this is the tool with which Tóth [24] managed to prove that almost surely for all large n, $\#\mathbb{V}(n) \le 3$. Tóth's theorem is quite close to conjecture (1.6), but the ultimate small gap might be very hard to fill.

Recently Eisenbaum [9] obtained an explicit and very interesting description of the site local time process N stopped at some random times. One would like to hope that this description might shed some new light upon the Erdős–Révész conjecture.

5.5. *Two-dimensional random walk.* Dimension two is critical for simple symmetric random walk, and some local time questions may be very delicate (note that local time does not exist for two-dimensional Brownian motion). Let $(\mathbf{S}_n, n \ge 0)$ be a simple symmetric random walk on $\mathbb{Z}^2$, starting from $\mathbf{S}_0 = \mathbf{0}$. Let $N(n, \mathbf{x}) := \#\{k \in [0, n] \cap \mathbb{Z} : \mathbf{S}_k = \mathbf{x}\}$, for any $\mathbf{x} \in \mathbb{Z}^2$. It is very hard to study the precise asymptotic behavior of $\sup_{\mathbf{x}\in\mathbb{Z}^2} N(n, \mathbf{x})$. Indeed, in 1960 Erdős and Taylor [12] raised the conjecture that

$$\lim_{n\to\infty} \frac{1}{(\log n)^2} \sup_{\mathbf{x}\in\mathbb{Z}^2} N(n, \mathbf{x}) = \frac{1}{\pi} \qquad \text{a.s.} \tag{5.2}$$

Some 40 years later (5.2) was confirmed by Dembo, Peres, Rosen and Zeitouni [7]. As an interesting consequence of (5.2), it was established in [7] that if $\mathbf{V}(n)$ is a favorite site at time n (with obvious definition), then

$$\lim_{n\to\infty} \frac{\log \|\mathbf{V}(n)\|}{\log n} = \frac{1}{2} \qquad \text{a.s.,} \tag{5.3}$$

where $\|\mathbf{x}\|$ denotes the Euclidean modulus of $\mathbf{x} \in \mathbb{Z}^2$.

It is a challenging problem to study the rate of escape of $\|\mathbf{V}(n)\|$, not only in the logarithmic scale, as in (5.3). For example, is it possible to prove something in dimension two which is similar to (1.4)?

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