The Annals of Probability
2003, Vol. 31, No. 4, 1862–1899

PROBABILISTIC MODELS FOR VORTEX FILAMENTS BASED ON FRACTIONAL BROWNIAN MOTION

BY DAVID NUALART,[1] CARLES ROVIRA[2] AND SAMY TINDEL

Universitat de Barcelona and Institut Galilée—Université Paris 13

We consider a vortex structure based on a three-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. We show that the energy $\mathbb{H}$ has moments of any order under suitable conditions. When $H \in (\frac{1}{2}, \frac{1}{3})$ we prove that the intersection energy $\mathbb{H}_{xy}$ can be decomposed into four terms, one of them being a weighted self-intersection local time of the fractional Brownian motion in $\mathbb{R}^3$.

1. Introduction. Observations of three-dimensional turbulent fluids in a number of experiments indicate that the vorticity field of the fluid is concentrated along thin structures called vortex filaments. In his book, Chorin (1994) suggested probabilistic descriptions of vortex filaments by trajectories of self-avoiding walks on a lattice. In Flandoli (2002), a model of vortex filaments based on a three-dimensional Brownian motion was introduced. In this model the Gibbs measure is given by $Z^{-1}e^{-\beta\mathbb{H}}\,d\mu_W$, where Z is a normalizing constant, β is a positive parameter, μ_W is the Wiener measure and $\mathbb{H}$ is the kinetic energy of a given configuration. The computation and integrability properties of the kinetic energy are basic problems in these models.

Denote by $u(x)$ the velocity field of the fluid at point $x \in \mathbb{R}^3$ and let $\xi = \operatorname{curl} u$ be the associated vorticity field. The kinetic energy of the fluid will be

$$\mathbb{H} = \frac{1}{2}\int_{\mathbb{R}^3} |u(x)|^2\,dx = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\xi(x)\cdot\xi(y)}{|x-y|}\,dx\,dy. \tag{1}$$

If the vorticity field is concentrated along a curve $\gamma = \{\gamma(t),\ 0 \le t \le T\}$ the kinetic energy has the formal expression

$$\mathbb{H} = \frac{\Gamma^2}{8\pi}\int_0^T\int_0^T \frac{\langle\dot\gamma_s, \dot\gamma_t\rangle}{|\gamma_t - \gamma_s|}\,ds\,dt, \tag{2}$$

where Γ is a parameter called the circuitation, which is divergent even if the curve γ is smooth. For this reason, we assume that the vorticity field is concentrated along a thin tube centered in a curve γ. Moreover, we choose a random model and consider this curve as the trajectory of a stochastic process. In Flandoli (2002), γ is the path of a three-dimensional Brownian motion. We

Received January 2002; revised September 2002.
[1]Supported in part by DGES Grant BFM2000-0598.
[2]Supported in part by DGES Grants BFM2000-0607 and HF2000-0002.
AMS 2000 subject classifications. 76M35, 60H05.
Key words and phrases. Filaments, energy, fractional Brownian motion.

assume that γ is the trajectory of a three-dimensional fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. In this way our filaments may have any kind of Hölder continuity between $\frac{1}{2}$ and 1.

Let us describe in detail our random model for vorticity filaments. Suppose that $B = \{B_t,\ t \in [0, T]\}$ is a three-dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$. This means that B is a zero-mean Gaussian stochastic process defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the covariance function

$$\mathbb{E}(B_t^i B_s^j) = \tfrac{1}{2}\delta_{i,j}(s^{2H} + t^{2H} - |t-s|^{2H}). \tag{3}$$

We assume that the vorticity field is concentrated along a trajectory of B. This can be formally expressed as

$$\xi(x) = \Gamma \int_{\mathbb{R}^3} \left(\int_0^T \delta(x - y - B_s)\dot{B}_s \, ds \right) \rho(dy), \tag{4}$$

where ρ is a probability measure on $\mathbb{R}^3$ with compact support. Substituting (4) into (2) we derive the formal expression for the kinetic energy,

$$\mathbb{H} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathbb{H}_{xy} \rho(dx)\rho(dy), \tag{5}$$

where the so-called interaction energy $\mathbb{H}_{xy}$ is given by the double integral

$$\mathbb{H}_{xy} = \frac{\Gamma^2}{8\pi} \sum_{i=1}^{3} \int_0^T \int_0^T \frac{1}{|x + B_t - y - B_s|} \, dB_s^i \, dB_t^i. \tag{6}$$

The purpose of this article is to give a rigorous meaning to expressions (5) and (6) and to show that $\mathbb{H}$ is a well-defined nonnegative random variable with moments of all orders.

Notice first that (3) implies that $\mathbb{E}|B_t - B_s|^2 = 3|t-s|^{2H}$. As a consequence, the trajectories $t \to B_t(\omega)$ of the fBm are Hölder continuous of order $H - \varepsilon$ for all $\varepsilon > 0$. Taking into account the results by Young (1936), for any Hölder continuous function f of order greater than $1 - H$, the Stieltjes integral $\int_0^T f(t)\, dB_t(\omega)$ exists. The double integral appearing in (6) is defined by an approximation argument, smoothing the function $|x|^{-1}$ by means of the convolution with an approximation of the identity.

The main results of this article are the following:

1. The energy $\mathbb{H}$ has moments of any order, provided the measure ρ satisfies the condition

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |x - y|^{1 - 1/H} \rho(dx)\rho(dy) < \infty. \tag{7}$$

2. If $H < \frac{2}{3}$ and (7) holds, the interaction energy $\mathbb{H}_{xy}$ can be decomposed into four terms, one of them being the weighted self-interaction local time of the fractional Brownian motion in $\mathbb{R}^3$. We also show that this decomposition does not hold true whenever $H \geq \frac{2}{3}$.

If $H = \frac{1}{2}$, the fBm B is a classical three-dimensional Brownian motion. In this case, condition (7) would be

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |x-y|^{-1} \rho(dx) \rho(dy) < \infty, \tag{8}$$

which is the assumption made by Flandoli (2002) and Flandoli and Gubinelli (2002). In this last article, using Fourier approach and Itô's stochastic calculus, the authors showed that $Ee^{-\beta \mathbb{H}} < \infty$ for sufficiently small negative β. Similar models were considered in Flandoli and Minelli (2001) for some stochastic processes with a prescribed Hölder continuity, under the assumption (8) on the measure ρ.

Condition (7) on ρ is sharp and provides an interesting relationship between the regularity of the filament described by the Hurst parameter H and the thickness of the cross section described by the measure ρ. More precisely, the more regular the filament is, the more singular its cross section can be.

The proofs of results 1 and 2 are based on the stochastic calculus of variations (Malliavin calculus) with respect to the fBm. A basic idea used in both proofs is the following one. Pathwise integrals with respect to the fractional Brownian motion can be decomposed into the sum of a divergence term plus a term involving the trace of the derivative operator. Both terms can be estimated by the techniques of the Malliavin calculus. In this way we have made use of recent progress in the stochastic calculus with respect to the fBm, for which we refer, among others, to Alòs, Mazet and Nualart (2001), Alòs and Nualart (2003), Carmona and Coutin (2000), Dai and Heyde (1996), Decreusefond and Üstünel (1998), Lin (1995) and Hu and Øksendal (2003).

At this stage let us also point out some of the technical difficulties that we met on our way to these results:

1. The definition of $\mathbb{H}$ through a Fourier analysis procedure reduces our problem to estimation of the moments of the divergence integrals

$$Y_\xi = p_\xi\left(\int_0^T e^{i\langle \xi, B_t\rangle} \, \delta B_t\right),$$

   where $\xi \in \mathbb{R}^3$ and p_ξ denotes the orthogonal projection on ξ. The moments of Y_ξ cannot be estimated by means of classical Burkholder inequalities. We have computed and estimated these moments using the duality relationship of the Malliavin calculus and taking advantage of some surprising cancellation relationships due to the presence of the projection operator p_ξ.

2. The self-intersection local time of the fBm has been studied by Rosen (1987) in dimension 2 and by Hu (2001). For our purposes, we need a weighted version of the self-intersection local time, which exists without any normalization. The proof of result 2 is based again on the Malliavin calculus for the fBm and some estimates which go beyond those of Hu (2001) and are based on the local nondeterministic property.

This article is organized as follows. Section 2 contains some preliminaries on the fractional Brownian motion and the stochastic calculus of variations with respect to it. Section 3 contains the definition of $\mathbb{H}$ and the study of its moments. Finally, Section 4 is devoted to the decomposition of the interaction energy $\mathbb{H}_{xy}$ when $H < \frac{2}{3}$.

2. Fractional Brownian motion. In this section we present some basic facts about the fractional Brownian motion and the stochastic calculus with respect to this process.

Fix a parameter $\frac{1}{2} < H < 1$. The fractional Brownian motion of Hurst parameter H is a centered Gaussian process $B = \{B_t, t \in [0, T]\}$ with the covariance function

$$R(t,s) = \tfrac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}). \tag{9}$$

We assume that B is defined in a complete probability space $(\Omega, \mathcal{F}, P)$. One can show [see, e.g., Alòs and Nualart (2003)] that

$$R(t,s) = \int_0^{t\wedge s} K(t,r)K(s,r)\,dr, \tag{10}$$

where $K(t,s)$ is the kernel defined by

$$K(t,s) = c_H s^{1/2-H} \int_s^t (r-s)^{H-3/2} r^{H-1/2}\,dr$$

for $s < t$, where $c_H = [H(2H-1)/B(2-2H, H-1/2)]^{1/2}$ and $B(\alpha,\beta)$ denotes the Beta function. We assume that $K(t,s) = 0$ if $s > t$. Notice that formula (10) implies that R is nonnegative definite and, therefore, there exists a Gaussian process with this covariance.

We denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R(t,s).$$

It is easy to show that

$$R(t,s) = \alpha_H \int_0^t \int_0^s |r-u|^{2H-2}\,du\,dr,$$

where $\alpha_H = H(2H-1)$. This implies

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u \, du \, dr \tag{11}$$

for all φ and ψ in $\mathcal{E}$. The mapping $\mathbf{1}_{[0,t]} \to B_t$ can be extended to an isometry between $\mathcal{H}$ and the first chaos H_1 associated with B. We denote this isometry by $\varphi \to B(\varphi)$. The elements of the Hilbert space $\mathcal{H}$ may not be functions, but distributions of negative order. For this reason, it is convenient to introduce the Banach space $|\mathcal{H}|$ of measurable functions φ on $[0, T]$ satisfying

$$\|\varphi\|^2_{|\mathcal{H}|} := \alpha_H \int_0^T \int_0^T |\varphi_r| |\varphi_u| |r-u|^{2H-2} \, dr \, du < \infty. \tag{12}$$

It was shown in Pipiras and Taqqu (2000) that the space $|\mathcal{H}|$ equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{H}}$ is not complete and it is isometric to a subspace of $\mathcal{H}$. We identify $|\mathcal{H}|$ with this subspace. The continuous embedding $L^{1/H}([0, T]) \subset |\mathcal{H}|$ was proved in Mémin, Mishura and Valkeila (2001).

The operator K^* from $\mathcal{E}$ to $L^2([0, T])$, defined by

$$(K^* \varphi)(s) = \int_s^T \varphi_r \, \partial_r K(r, s) \, dr,$$

can be extended to an isometry between the Hilbert spaces $\mathcal{H}$ and $L^2([0, T])$. Hence, the process $W = \{W_t, \, t \in [0, T]\}$, defined by

$$W_t = B\big((K^*)^{-1}(\mathbf{1}_{[0,t]})\big),$$

is a Wiener process, and the process B has an integral representation of the form

$$B_t = \int_0^t K(t, s) \, dW_s. \tag{13}$$

The fractional Brownian motion B possesses the property of local nondeterminism stated in the following lemma. This property was introduced in Berman (1973) and applied to the existence and smoothness of the local time of Gaussian processes.

LEMMA 1. *There exists a constant k_H such that for all $0 \le t_1 < \cdots < t_n$ and for any vector $(u_1, \ldots, u_n) \in \mathbb{R}^n$, we have*

$$\mathrm{Var}\left(\sum_{i=2}^n u_i (B_{t_i} - B_{t_{i-1}}) \right) \ge k_H \sum_{i=2}^n u_i^2 |t_i - t_{i-1}|^{2H}.$$

2.1. *Malliavin calculus and stochastic integrals for the fBm.* We can construct a stochastic calculus of variations with respect to the Gaussian process B following the general approach introduced, for instance, in Nualart (1995). Let us recall the definition of the derivative and divergence operators and some basic facts of this stochastic calculus of variations, taken mainly from Alòs and Nualart (2003).

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f\big(B(\phi_1), \dots, B(\phi_n)\big), \tag{14}$$

where $n \geq 1$, $f \in C_{\mathrm{b}}^{\infty}(\mathbb{R}^n)$ (f and all its partial derivatives are bounded) and $\phi_i \in \mathcal{H}$. The derivative operator D of a smooth and cylindrical random variable F of the form (14) is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\big(B(\phi_1), \dots, B(\phi_n)\big)\phi_j.$$

The derivative operator D is then a closable operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p \geq 1$. For any $p \geq 1$, the Sobolev space $\mathbb{D}^{1,p}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p}^{p} = E|F|^p + E\|DF\|_{\mathcal{H}}^{p}.$$

In a similar way, given a Hilbert space V, we denote by $\mathbb{D}^{1,p}(V)$ the corresponding Sobolev space of V-valued random variables.

The divergence operator δ is the adjoint of the derivative operator, defined by means of the duality relationship

$$E(F\delta(u)) = E\langle DF, u\rangle_{\mathcal{H}}, \tag{15}$$

where u is a random variable in $L^2(\Omega; \mathcal{H})$. We say that u belongs to the domain of the operator δ, denoted by $\operatorname{Dom}\delta$, if the mapping $F \mapsto E\langle DF, u\rangle_{\mathcal{H}}$ is continuous in $L^2(\Omega)$. A basic result says that the space $\mathbb{D}^{1,2}(\mathcal{H})$ is included in $\operatorname{Dom}\delta$.

Two basic properties of the divergence operator are the following:

P1. For any $u \in \mathbb{D}^{1,2}(\mathcal{H})$,

$$E\delta(u)^2 = E\|u\|_{\mathcal{H}}^2 + E\langle Du, (Du)^*\rangle_{\mathcal{H}\otimes\mathcal{H}}, \tag{16}$$

where $(Du)^*$ is the adjoint of (Du) in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

P2. For any $u \in \mathbb{D}^{2,2}(\mathcal{H})$, $\delta(u)$ belongs to $\mathbb{D}^{1,2}$ and for any h in $\mathcal{H}$,

$$\langle D\delta(u), h\rangle_{\mathcal{H}} = \delta(\langle Du, h\rangle_{\mathcal{H}}) + \langle u, h\rangle_{\mathcal{H}}. \tag{17}$$

Consider the space $|\mathcal{H}| \otimes |\mathcal{H}| \subset \mathcal{H} \otimes \mathcal{H}$ of measurable functions φ on $[0, T]^2$ such that

$$\|\varphi\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2 := \alpha_H^2 \int_{[0,T]^4} |\varphi_{r,s}||\varphi_{r',s'}||r - r'|^{2H-2}|s - s'|^{2H-2}\, dr\, ds\, dr'\, ds' < \infty.$$

Let $\mathbb{D}^{1,2}(|\mathcal{H}|)$ be the space of processes u such that

$$E\|u\|^2_{|\mathcal{H}|} + E\|Du\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} < \infty. \tag{18}$$

Then $\mathbb{D}^{1,2}(|\mathcal{H}|)$ is included in $\mathbb{D}^{1,2}(\mathcal{H})$ and for a process u in $\mathbb{D}^{1,2}(|\mathcal{H}|)$, we can write

$$E\|u\|^2_{\mathcal{H}} = \alpha_H \int_{[0,T]^2} u_s u_r |r-s|^{2H-2}\, dr\, ds \tag{19}$$

and

$$\begin{aligned} &E\langle Du, (Du)^*\rangle_{\mathcal{H}\otimes\mathcal{H}} \\ &\qquad = \alpha_H^2 \int_{[0,T]^4} D_r u_s\, D_{r'} u_{s'} |r-s'|^{2H-2} |r'-s|^{2H-2}\, dr\, dr'\, ds\, ds'. \end{aligned} \tag{20}$$

The elements of $\mathbb{D}^{1,2}(|\mathcal{H}|)$ are stochastic processes and we make use of the integral notation $\delta(u) = \int_0^T u_t\, \delta B_t$; we call this integral the Skorohod integral with respect to the fBm. Moreover, if $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$, one can also define an indefinite integral process given by $X_t = \int_0^t u_s\, \delta B_s$.

Let us define now a Stratonovich-type integral with respect to B. By convention we put $B_t = 0$ if $t \notin [0, T]$. Following the approach by Russo and Vallois (1993), we can give the following definition:

DEFINITION 2. Let $u = \{u_t, t \in [0, T]\}$ be a stochastic process with integrable trajectories. The Stratonovich integral of u with respect to B is defined as the limit in probability as ε tends to zero of

$$(2\varepsilon)^{-1} \int_0^T u_s (B_{s+\varepsilon} - B_{s-\varepsilon})\, ds,$$

provided this limit exists, and it is denoted by $\int_0^T u_t\, dB_t$.

It was shown in Alòs and Nualart (2003) that a process $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$, such that

$$\int_0^T \int_0^T |D_s u_t| |t-s|^{2H-2}\, ds\, dt < \infty$$

a.s. is Stratonovich integrable and

$$\int_0^T u_s\, dB_s = \int_0^T u_s\, \delta B_s + \alpha_H \int_0^T \int_0^T D_s u_t |t-s|^{2H-2}\, ds\, dt. \tag{21}$$

On the other hand, if the process u has a.s. λ-Hölder continuous trajectories with $\lambda > 1 - H$, then the Stratonovich integral $\int_0^T u_s\, dB_s$ exists and coincides with the pathwise Riemann–Stieltjes integral.

Our computations heavily rely on a slight extension of the Itô formula for the fBm shown in Theorem 2 in Alòs, Mazet and Nualart (2001):

PROPOSITION 3. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a bounded function in the class $C^{1,2}$, with bounded partial derivatives. Then the process $\{\partial_x f(t, B_t); t \in [0, T]\}$ is an element of $\mathbb{D}^{1,2}(|\mathcal{H}|)$ and for any $0 \le r < t \le T$, we have*

$$
\begin{aligned}
f(t, B_t) = f(r, B_r) &+ \int_r^t \partial_s f(s, B_s)\, ds + \int_r^t \partial_x f(s, B_s)\, \delta B_s \\
&+ H \int_r^t \partial_{xx}^2 f(s, B_s) s^{2H-1}\, ds
\end{aligned}
\tag{22}
$$

$$
= f(r, B_r) + \int_r^t \partial_s f(s, B_s)\, ds + \int_r^t \partial_x f(s, B_s)\, dB_s. \tag{23}
$$

We also use a substitution formula, the proof of which is similar to the one in Theorem 3.2.4 in Nualart (1995):

PROPOSITION 4. *Let $\zeta = \{\zeta_t(x); t \in [0, T], x \in \mathbb{R}\}$ be a random field that satisfies:*

1. *For each $x \in \mathbb{R}$, $\zeta(x) \in \mathbb{D}^{1,2}(|\mathcal{H}|)$.*
2. *For each $(t, \omega) \in [0, T] \times \Omega$, the mapping $x \mapsto \zeta_t(x)$ is in $C_{\mathrm{b}}^1(\mathbb{R})$.*

Let F be a random variable in $\mathbb{D}^{1,2}$. Then

$$
\begin{aligned}
\int_0^T \zeta_t(F)\, \delta B_t &= \int_0^T \zeta_t(x)\, \delta B_t \big|_{x=F} \\
&\quad - \alpha_H \int_0^T \int_0^T \partial_x \zeta_r(F) D_s F |r - s|^{2H-2}\, dr\, ds.
\end{aligned}
$$

3. Vortex filaments based on fractional Brownian motion. Let $B = \{B_t, t \in [0, T]\}$ be a three-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ defined on the complete probability space $(\Omega, \mathcal{F}, P)$. The derivative operator with respect to B^i is denoted by D^i, and the space of processes satisfying (18), using the derivative operator D^i, is denoted by $\mathbb{D}_i^{1,2}(|\mathcal{H}|)$. In a similar way, the divergence and Stratonovich differentials with respect to the process B^i are denoted by δB^i and dB^i, respectively.

The following lemma is a three-dimensional version of the isometry formula (16) for the divergence operator with respect to the fBm that will be useful later.

LEMMA 5. *Let u and v be two three-dimensional processes, the components of which are in $\mathbb{D}^{1,2}(|\mathcal{H}|)$. Then*

$$
\begin{aligned}
&E\left[\left(\int_0^T u_t\, \delta B_t\right)\left(\int_0^T v_t\, \delta B_t\right)\right] \\
&\qquad = E \sum_{i=1}^3 \langle u^i, v^i \rangle_{\mathcal{H}} + E \sum_{i,j=1}^3 \langle D_t^j u_s^i, D_s^i u_t^j \rangle_{\mathcal{H} \otimes \mathcal{H}}.
\end{aligned}
\tag{24}
$$

Using expressions (19) and (20), we can transform formula (24) into

$$
\begin{aligned}
&E\left[\left(\int_0^T u_t\,\delta B_t\right)\left(\int_0^T v_t\,\delta B_t\right)\right]\\
&\qquad=\alpha_H\sum_{i=1}^3\int_0^T\int_0^T E(u_r^i v_{r'}^i)|r-r'|^{2H-2}\,dr\,dr'\\
&\qquad\quad+\alpha_H^2\sum_{i,j=1}^3\int_{[0,T]^4}E(D_\theta^j u_r^i D_{r'}^i v_{\theta'}^j)\\
&\qquad\qquad\times|r-r'|^{2H-2}|\theta-\theta'|^{2H-2}\,dr\,dr'\,d\theta\,d\theta'.
\end{aligned}\tag{25}
$$

The trajectories of B constitute the core of the filament we want to introduce and the full filament is a collection of translates of $\{B_t\}$ of the form $\{x+B_t\}$, where x varies on a fractal set. This fractal set is described by means of a finite probability measure ρ supported by a compact set K. Formally we consider a vorticity field of the form (4), which leads to the formal expressions (5) and (6) for the kinetic energy $\mathbb{H}$ associated with the vorticity field $\xi(x)$ and for the interaction energy $\mathbb{H}_{xy}$.

To provide a precise meaning to expressions (5) and (6) we could use a definition of the double Stratonovich integral in the sense of Russo and Vallois (1993), as a double limit or as an iterated integral. Another approach is as follows.

We smooth the function $|x|^{-1}$ by a convolution with a Gaussian kernel,

$$
\sigma_n(x)=\int_{\mathbb{R}^3}\frac{1}{|x-y|}p_{1/n}(y)\,dy,\tag{26}
$$

where $p_{1/n}(y)=(2\pi/n)^{-3/2}e^{-n|y|^2/2}$, and we define

$$
\mathbb{H}_{xy}^n=\frac{\Gamma^2}{8\pi}\sum_{i=1}^3\int_0^T\left(\int_0^T\sigma_n(x+B_t-y-B_s)\,dB_s^i\right)dB_t^i,\tag{27}
$$

where the integrals are Stratonovich integrals in the sense of Definition 2. The proof of the existence of this iterated integral and a decomposition in terms of Skorohod integrals is given in Proposition 12 below. One can show that the integrals in (27) are also pathwise Riemann–Stieltjes integrals.

Then we have the following result which provides a definition of the energy $\mathbb{H}$.

THEOREM 6. *Suppose that the measure ρ satisfies condition* (7). *Let $\mathbb{H}_{xy}^n$ be the smoothed interaction energy defined by* (27). *Then*

$$
\mathbb{H}^n=\int_K\int_K H_{xy}^n\rho(dx)\rho(dy)\tag{28}
$$

converges, for all $k\ge 1$, in $L^k(\Omega)$ to a random variable $\mathbb{H}\ge 0$ that we call the energy associated with the vorticity field (4).

PROOF. For any $\alpha > 0$ we can compute the Fourier transform

$$\frac{1}{|z|^{\alpha}} = \int_{\mathbb{R}^3} \frac{C_\alpha}{|\xi|^{3-\alpha}} e^{-i\langle \xi, z\rangle}\, d\xi, \tag{29}$$

with $C_\alpha := 2^2\pi^3 \int_0^\infty u^{1-\alpha}\sin(u)\, du$ [see, e.g., Bochner and Chandrasekharan (1949)]. In particular, for $\alpha = 1$ and $C_1 = 1$, we obtain

$$\begin{aligned}\sigma_n(x) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{-i\langle \xi, x-y\rangle}}{|\xi|^2}\, d\xi\; p_{1/n}(y)\, dy \\ &= \int_{\mathbb{R}^3} e^{i\langle \xi, x\rangle} \frac{e^{-|\xi|^2/2n}}{|\xi|^2}\, d\xi.\end{aligned}$$

Substituting this expression in (27) and using a stochastic Fubini theorem, we get

$$\begin{aligned}\mathbb{H}^n_{xy} &= \frac{\Gamma^2}{8\pi}\sum_{j=1}^{3}\int_0^T\left(\int_0^T\left(\int_{\mathbb{R}^3} e^{i\langle \xi, x-y+B_t-B_s\rangle}\frac{e^{-|\xi|^2/2n}}{|\xi|^2}\, d\xi\right) dB^j_s\right) dB^j_t \\ &= \frac{\Gamma^2}{8\pi}\int_{\mathbb{R}^3} \|Y_\xi\|^2_{\mathbb{C}} \frac{e^{i\langle \xi, x-y\rangle - |\xi|^2/2n}}{|\xi|^2}\, d\xi,\end{aligned}$$

where

$$Y_\xi = \int_0^T e^{i\langle \xi, B_t\rangle}\, dB_t$$

and $\|Y_\xi\|^2_{\mathbb{C}} = \sum_{i=1}^3 Y^i_\xi \overline{Y^i_\xi}$. Now let us compute

$$\begin{aligned}\mathbb{H}^n &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathbb{H}^n_{xy}\rho(dx)\rho(dy) \\ &= \frac{\Gamma^2}{8\pi}\int_{\mathbb{R}^3} \|Y_\xi\|^2_{\mathbb{C}} \frac{|\widehat{\rho}(\xi)|^2}{|\xi|^2} e^{-|\xi|^2/2n}\, d\xi,\end{aligned}$$

where $\widehat{\rho}(\xi) = \int_{\mathbb{R}^3} e^{i\langle \xi, x\rangle}\rho(dx)$ is the Fourier transform of the measure ρ. This expression already implies that $\mathbb{H}^n \geq 0$ and, furthermore, $\mathbb{H}^n$ is increasing with n. Therefore, to show the convergence in $L^k(\Omega)$ to a random variable $\mathbb{H} \geq 0$ it suffices to show that

$$E\left(\left|\int_{\mathbb{R}^3} \|Y_\xi\|^2_{\mathbb{C}} \frac{|\widehat{\rho}(\xi)|^2}{|\xi|^2}\, d\xi\right|^k\right) < \infty. \tag{30}$$

Using formula (29) with $\alpha = \frac{1}{H} - 1$, we can write

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x-y|^{1-1/H}\rho(dx)\rho(dy) \\ &\quad = C_{1/H-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} |\xi|^{1/H-4} e^{-i\langle \xi, x-y\rangle}\, d\xi\right)\rho(dx)\rho(dy) \\ &\quad = C_{1/H-1}\int_{\mathbb{R}^3} |\widehat{\rho}(\xi)|^2 |\xi|^{1/H-4}\, d\xi < \infty\end{aligned}$$

by condition (7).

Notice that since the measure $d\mu = |\widehat{\rho}(\xi)|^2/|\xi|^{4-1/H}\,d\xi$ is finite, using Hölder's inequality, to prove (30) it suffices to check that

$$\int_{\mathbb{R}^3} \frac{E\|Y_\xi\|_{\mathbb{C}}^{2k}}{|\xi|^{k/H-2k}} \frac{|\widehat{\rho}(\xi)|^2}{|\xi|^{4-1/H}}\,d\xi < \infty, \tag{31}$$

and (31) will be a consequence of the estimate

$$E\|Y_\xi\|_{\mathbb{C}}^{2k} \le C_k|\xi|^{k/H-2k}$$

proved in Lemma 10. □

To complete the proof of Theorem 6 we need the following technical lemmas and propositions. We begin by introducing some notation.

Given $\xi, v \in \mathbb{R}^3$, let $p_\xi(v) = v - (\xi/|\xi|^2)\langle \xi, v\rangle$ be the orthogonal projection of v on $\langle \xi\rangle^\perp$. Observe that $p_\xi(v) = M_\xi v$, where M_ξ is the symmetric idempotent matrix given by

$$M_\xi^{j,l} = \delta_{j,l} - \frac{\xi_j\xi_l}{|\xi|^2}.$$

LEMMA 7. *The random field $Y_\xi = \int_0^T e^{i\langle \xi, B_t\rangle}\,dB_t$ satisfies*

$$Y_\xi = p_\xi\left(\int_0^T e^{i\langle \xi, B_t\rangle}\,\delta B_t\right) - \frac{i\xi}{|\xi|^2}\left(e^{i\langle \xi, B_T\rangle} - 1\right). \tag{32}$$

PROOF. Using relationship (21) we get

$$\begin{aligned} Y_\xi &= \int_0^T e^{i\langle \xi, B_t\rangle}\,\delta B_t + \alpha_H \int_0^T \int_0^t \left(D_s e^{i\langle \xi, B_t\rangle}\right)(t-s)^{2H-2}\,ds\,dt \\ &= \int_0^T e^{i\langle \xi, B_t\rangle}\,\delta B_t + H\int_0^T i\xi e^{i\langle \xi, B_t\rangle} t^{2H-1}\,dt. \end{aligned} \tag{33}$$

On the other hand, by a multidimensional version of the Itô type formula (22) we obtain

$$\begin{aligned} e^{i\langle \xi, B_T\rangle} = 1 &+ \sum_{j=1}^3 \int_0^T i\xi_j e^{i\langle \xi, B_t\rangle}\,\delta B_t^j \\ &- H\int_0^T t^{2H-1}|\xi|^2 e^{i\langle \xi, B_t\rangle}\,dt. \end{aligned}$$

Dividing by $|\xi|^2$ and multiplying by $i\xi$ we can write

$$\begin{aligned} H\int_0^T t^{2H-1} i\xi e^{i\langle \xi, B_t\rangle}\,dt \\ = -\frac{i\xi}{|\xi|^2}\big(e^{i\langle \xi, B_T\rangle}-1\big) - \frac{\xi}{|\xi|^2}\Big\langle \xi, \int_0^T e^{i\langle \xi, B_t\rangle}\,\delta B_t\Big\rangle. \end{aligned} \tag{34}$$

Then putting together (33) and (34) we obtain (32) easily. □

LEMMA 8. *Given $\xi \in \mathbb{R}^3$ and a smooth function ϕ from $\mathbb{R}$ to $\mathbb{C}$ bounded with bounded derivatives, we have*

$$p_\xi D[\phi(\langle \xi, B_t\rangle)] = 0.$$

PROOF. Observe that

$$p_\xi D[\phi(\langle \xi, B_t\rangle)] = \phi'(\langle \xi, B_t\rangle) p_\xi(\xi) = 0. \qquad \square$$

LEMMA 9. *Given $\xi \in \mathbb{R}^3$, $i, j \in \{1,2,3\}$ and ϕ a smooth function from $\mathbb{R}$ to $\mathbb{C}$ bounded with bounded derivatives, then*

$$p_\xi^i D_s p_\xi^j\left(\int_0^T \phi(\langle \xi, B_t\rangle)\,\delta B_t\right) = M_\xi^{i,j}\phi(\langle \xi, B_t\rangle).$$

PROOF. Using the definition of p_ξ^j we have

$$p_\xi^j\left(\int_0^T \phi(\langle \xi, B_t\rangle)\,\delta B_t\right) = \sum_{l=1}^3 M_\xi^{j,l}\int_0^T \phi(\langle \xi, B_t\rangle)\,\delta B_t^l.$$

For $k \in \{1,2,3\}$, using (17) we have

$$\begin{aligned} D_s^k p_\xi^j\left(\int_0^T \phi(\langle \xi, B_t\rangle)\,\delta B_t\right) \\ = M_\xi^{j,k}\phi(\langle \xi, B_s\rangle) + \sum_{l=1}^3 M_\xi^{j,l}\int_0^T D_s^k[\phi(\langle \xi, B_t\rangle)]\,\delta B_t^l. \end{aligned}$$

Applying now the projection p_ξ^i and using Lemma 8 and the idempotent character of the matrix M_ξ, we obtain

$$p_\xi^i D_s p_\xi^j\left(\int_0^T \phi(\langle \xi, B_t\rangle)\,\delta B_t\right) = M_\xi^{j,i}\phi(\langle \xi, B_s\rangle). \qquad \square$$

LEMMA 10. *There exists a positive constant C_k that depends only on k, T and H such that for all $\xi \in \mathbb{R}^3$ and for all $k \geq 1$,*

$$E\|Y_\xi\|_{\mathbb{C}}^{2k} \leq C_k|\xi|^{k/H-2k}. \tag{35}$$

Moreover, there exists a constant δ that depends on T and H such that for all $\xi \in \mathbb{R}^3$ and for all $k \geq 1$,

$$E\|Y_\xi\|_{\mathbb{C}}^{2k} \leq \frac{(2k)!}{k!}\delta^k. \tag{36}$$

PROOF. From (32) we get

$$E\|Y_k\|_{\mathbb{C}}^{2k} \leq 2^{k-1}\left(E\left\|\frac{i\xi}{|\xi|^2}(e^{i\langle \xi, B_T\rangle} - 1)\right\|_{\mathbb{C}}^{2k} + A_k\right), \tag{37}$$

where

$$A_k := E\left\|p_\xi\left(\int_0^T e^{i\langle \xi, B_t\rangle}\,\delta B_t\right)\right\|_{\mathbb{C}}^{2k}. \tag{38}$$

Using that $1 = e^{i\langle \xi, B_0\rangle}$ we have

$$E\left\|\frac{i\xi}{|\xi|^2}(e^{i\langle \xi, B_T\rangle} - 1)\right\|_{\mathbb{C}}^{2k} \leq E|B_T|^{2k},$$

which implies (36) for the first summand of (37). On the other hand, for $|\xi| \geq 1$ we have

$$E\left\|\frac{i\xi}{|\xi|^2}(e^{i\langle \xi, B_T\rangle} - 1)\right\|_{\mathbb{C}}^{2k} \leq \frac{2^{2k}}{|\xi|^{2k}} \leq 2^{2k}|\xi|^{k/H-2k}.$$

Thus the estimate (35) holds for the first summand on the right-hand side of (37).

Then, it suffices to study the term A_k. Observe first that A_k can be written as

$$A_k = E\sum_{(i_1,\dots,i_k)\in\{1,2,3\}^k}\prod_{l=1}^k p_\xi^{i_l}\left(\int_0^T e^{i\langle \xi, B_t\rangle}\,\delta B_t\right)p_\xi^{i_l}\left(\int_0^T e^{-i\langle \xi, B_t\rangle}\,\delta B_t\right). \tag{39}$$

Let us fix $(i_1,\dots,i_k) \in \{1,2,3\}^k$ and let us study the corresponding term of the above sum. Set $j_{2n-1} = j_{2n} = i_n$ and $\varepsilon_{2n-1} = -1$, $\varepsilon_{2n} = 1$ for $1 \leq n \leq k$. Then we can write

$$\begin{aligned} &E\prod_{l=1}^k p_\xi^{i_l}\left(\int_0^T e^{i\langle \xi, B_t\rangle}\,\delta B_t\right)p_\xi^{i_l}\left(\int_0^T e^{-i\langle \xi, B_t\rangle}\,\delta B_t\right)\\ &\qquad = E\prod_{l=1}^{2k} p_\xi^{j_l}\left(\int_0^T e^{i\varepsilon_l\langle \xi, B_t\rangle}\,\delta B_t\right). \end{aligned} \tag{40}$$

By the duality relationship between the derivative and divergence operators and by applying Lemma 9 we get that (40) is equal to

$$
\begin{aligned}
&E\left(p_\xi^{j_1}\left(\int_0^T e^{i\varepsilon_1\langle\xi,B_t\rangle}\,\delta B_t\right)\prod_{l=2}^{2k} p_\xi^{j_l}\left(\int_0^T e^{i\varepsilon_l\langle\xi,B_t\rangle}\,\delta B_t\right)\right)\\
&\quad= E\left(\left\langle e^{i\varepsilon_1\langle\xi,B.\rangle},\, p_\xi^{j_1} D.\left(\prod_{l=2}^{2k} p_\xi^{j_l}\left(\int_0^T e^{i\varepsilon_l\langle\xi,B_t\rangle}\,\delta B_t\right)\right)\right\rangle_{\mathcal{H}}\right)\\
&\quad= \sum_{l=2}^{2k} E\left(\left\langle e^{i\varepsilon_1\langle\xi,B.\rangle},\, p_\xi^{j_1} D.p_\xi^{j_l}\left(\int_0^T e^{i\varepsilon_l\langle\xi,B_t\rangle}\,\delta B_t\right)\right\rangle_{\mathcal{H}}\right.\\
&\qquad\qquad \left.\times \prod_{n=2,n\neq l}^{2k} p_\xi^{j_n}\left(\int_0^T e^{i\varepsilon_n\langle\xi,B_t\rangle}\,\delta B_t\right)\right) \qquad (41)\\
&\quad= \sum_{l=2}^{2k} M_\xi^{j_1,j_l} E\left(\langle e^{i\varepsilon_1\langle\xi,B.\rangle}, e^{i\varepsilon_l\langle\xi,B.\rangle}\rangle_{\mathcal{H}} \prod_{n=2,n\neq l}^{2k} p_\xi^{j_n}\left(\int_0^T e^{i\varepsilon_n\langle\xi,B_t\rangle}\,\delta B_t\right)\right).
\end{aligned}
$$

Set now $n(l)=3$ if $l=2$ and $n(l)=2$ otherwise. Following the same arguments we used before and applying now Lemma 8 we obtain that (41) is equal to

$$
\begin{aligned}
&\sum_{l=2}^{2k} M_\xi^{j_1,j_l} E\left(p_\xi^{j_{n(l)}}\left(\int_0^T e^{i\varepsilon_{n(l)}\langle\xi,B_t\rangle}\,\delta B_t\right)\langle e^{i\varepsilon_1\langle\xi,B.\rangle}, e^{i\varepsilon_l\langle\xi,B.\rangle}\rangle_{\mathcal{H}}\right.\\
&\qquad\qquad\left.\times \prod_{n=2,n\neq l,n(l)}^{2k} p_\xi^{j_n}\left(\int_0^T e^{i\varepsilon_n\langle\xi,B_t\rangle}\,\delta B_t\right)\right)\\
&\quad= \sum_{l=2}^{2k}\sum_{n=2,n\neq l,n(l)}^{2k} M_\xi^{j_1,j_l} M_\xi^{j_n,j_{n(l)}}\\
&\qquad\qquad \times E\left(\langle e^{i\varepsilon_1\langle\xi,B.\rangle}, e^{i\varepsilon_l\langle\xi,B.\rangle}\rangle_{\mathcal{H}}\langle e^{i\varepsilon_n\langle\xi,B.\rangle}, e^{i\varepsilon_{n(l)}\langle\xi,B.\rangle}\rangle_{\mathcal{H}}\right.\\
&\qquad\qquad\qquad \left.\times \prod_{m=2,m\neq l,n(l),n}^{2k} p_\xi^{j_m}\left(\int_0^T e^{i\varepsilon_m\langle\xi,B_t\rangle}\,\delta B_t\right)\right).
\end{aligned}
$$

Set now G_{2k} the set of all the permutations of $\{1,2,\ldots,2k\}$. Iterating these computations it is easy to check that (40) can be expressed as

$$
\frac{1}{2^k k!}\sum_{\sigma\in G_{2k}}\prod_{l=1}^{k} M_\xi^{j_{\sigma(2l)},j_{\sigma(2l-1)}} R_\xi^\sigma, \qquad (42)
$$

where

$$R_\xi^\sigma := E\left(\prod_{l=1}^k \langle e^{i\varepsilon_{\sigma(2l)}\langle \xi, B.\rangle}, e^{i\varepsilon_{\sigma(2l-1)}\langle \xi, B.\rangle}\rangle_{\mathcal{H}}\right).$$

Note that

$$\begin{aligned} R_\xi^\sigma = \alpha_H^k \int_{[0,T]^{2k}} & E\left(\exp\left(i\left\langle \xi, \sum_{j=1}^{2k} \varepsilon_{\sigma(j)} B_{r_j}\right\rangle\right)\right) \\ & \times \prod_{l=1}^k |r_{2l} - r_{2l-1}|^{2H-2}\, dr_1 \cdots dr_{2k} \end{aligned} \tag{43}$$

and

$$E\left(\exp\left(i\left\langle \xi, \sum_{j=1}^{2k} \varepsilon_{\sigma(j)} B_{r_j}\right\rangle\right)\right) = \exp\left(-\frac{3|\xi|^2}{2} \operatorname{Var}\left(\sum_{j=1}^{2k} \varepsilon_{\sigma(j)} B_{r_j}^1\right)\right).$$

As a consequence, $R_\xi^\sigma \geq 0$, and using the estimate $|M_\xi^{j_{\sigma(2l)}, j_{\sigma(2l-1)}}| \leq 2$ we deduce the upper bound

$$A_k \leq \frac{3^k}{k!} \sum_{\sigma \in G_{2k}} R_\xi^\sigma.$$

The obvious inequality

$$\begin{aligned} R_\xi^\sigma &\leq \alpha_H^k \int_{[0,T]^{2k}} \prod_{l=1}^k |r_{2l} - r_{2l-1}|^{2H-2}\, dr_1 \cdots dr_{2k} \\ &\leq \alpha_H^k \left(\int_{[0,T]^2} |s-t|^{2H-2}\, ds\, dt\right)^k \end{aligned}$$

yields (36) for the term A_k.

For each permutation $\beta \in G_{2k}$, consider the set

$$D_\beta = \{(r_1, \ldots, r_{2k}); r_{\beta(1)} < r_{\beta(2)} < \cdots < r_{\beta(2k)}\}.$$

It is clear that the integral in (43) can be decomposed into a sum of integrals over the sets D_β, when $\beta \in G_{2k}$. On the set D_β we have

$$\sum_{j=1}^{2k} \varepsilon_{\sigma(j)} B_{r_j}^1 = \sum_{j=2}^{2k} a_j (B_{r_{\beta(j)}}^1 - B_{r_{\beta(j-1)}}^1),$$

where $a_j = \sum_{j : r_j \geq r_{\beta(j+1)}} \varepsilon_{\sigma(j)}$. In the particular case, $\sigma^{-1} = \beta$, we obtain

$$\sum_{j=1}^{2k} \varepsilon_{\sigma(j)} B_{r_j}^1 = \sum_{l=1}^{k} (B_{r_{\beta(2l)}}^1 - B_{r_{\beta(2l-1)}}^1)$$

and the coefficients a_j are given by

$$a_j = \begin{cases} 0, & \text{if } j = 2l-1, \\ 1, & \text{if } j = 2l. \end{cases}$$

If we compose a given permutation σ with the permutation that interchanges the elements j and j', the coefficients a_j are transformed into a_j+2, a_j-2 or a_j. Hence, $a_j \neq 0$ for all $j = 2l$.

The local nondeterministic property of the fractional Brownian motion (see Lemma 1) implies that

$$\begin{aligned} &\operatorname{Var}\left(\sum_{j=2}^{2k} a_j \left(B^1_{r_{\beta(j)}} - B^1_{r_{\beta(j-1)}}\right)\right) \\ &\qquad \geq k_H \sum_{j=2}^{2k} a_j^2 |r_{\beta(j)} - r_{\beta(j-1)}|^{2H} \geq k_H \sum_{l=1}^{k} |r_{\beta(2l)} - r_{\beta(2l-1)}|^{2H}. \end{aligned}$$

As a consequence, we deduce the estimate

$$\begin{aligned} A_k \leq \frac{(2k)!(3\alpha_H)^k}{k!} \sum_{\beta \in G_{2k}} \int_{D_\beta} \exp\left(-\frac{3k_H|\xi|^2}{2} \sum_{l=1}^{k} |r_{\beta(2l)} - r_{\beta(2l-1)}|^{2H}\right) \\ \times \prod_{l=1}^{k} |r_{2l} - r_{2l-1}|^{2H-2}\, dr_1 \cdots dr_{2k}. \end{aligned} \tag{44}$$

For each $l = 1, \dots, k$ we estimate below the distance $|r_{2l} - r_{2l-1}|$ by $|r_{\beta(j)} - r_{\beta(j-1)}|$, where $(r_{\beta(j-1)}, r_{\beta(j)})$ is the first interval contained in the interval determined by the points r_{2l-1} and r_{2l}. In this way we obtain the estimate on the set D_β:

$$\prod_{l=1}^{k} |r_{2l} - r_{2l-1}|^{2H-2} \leq T^{(k-1)(2-2H)} \prod_{j=2}^{2k} |r_{\beta(j)} - r_{\beta(j-1)}|^{2H-2}.$$

Hence

$$\begin{aligned} A_k \leq \frac{(2k)!^2 d^k}{k!} \int_{\{r_1 < \cdots < r_{2k} < T\}} \exp\left(-\frac{3k_H|\xi|^2}{2} \sum_{l=1}^{k} |r_{2l} - r_{2l-1}|^{2H}\right) \\ \times \prod_{j=2}^{2k} |r_j - r_{j-1}|^{2H-2}\, dr_1 \cdots dr_{2k}, \end{aligned}$$

where $d = 3\alpha_H T^{2-2H}$.

Performing the change of variables $y_j = r_j - r_{j-1}$, $j = 1, \dots, 2k$ (with the convention $r_0 = 0$), we can bound the last integral by

$$\int_{\{\sum_{j=1}^{2k} y_j \le T\} \cap \mathbb{R}_+^{2k}} \exp\left(-\frac{3k_H|\xi|^2}{2} \sum_{l=1}^{k} y_{2l}^{2H}\right) \prod_{j=2}^{2k} y_j^{2H-2}\, dy_1 \cdots dy_{2k}$$
$$\le c_k \int_{\{\sum_{j=1}^{k} y_j \le T\} \cap \mathbb{R}_+^{k}} \exp\left(-\frac{3k_H|\xi|^2}{2} \sum_{j=1}^{k} y_j^{2H}\right) \prod_{j=1}^{k} y_j^{2H-2}\, dy_1 \cdots dy_k,$$

where

$$c_k = \int_{\{\sum_{j=1}^{k} y_j \le T\} \cap \mathbb{R}_+^{k}} \prod_{j=2}^{k} y_j^{2H-2}\, dy_1 \cdots dy_k.$$

By the change of variable $z_j = |\xi| y_j^H$ we obtain

$$c_k H^{-k} |\xi|^{k/H-2k} \int_{\{\sum_{j=1}^{k} z_j^{1/H} \le |\xi|^{1/H} T\} \cap \mathbb{R}_+^{k}} \exp\left(-\sum_{j=1}^{k} \frac{3k_H z_j^2}{2}\right) \prod_{j=1}^{k} z_j^{1-1/H}\, dz_j$$
$$\le c_k H^{-k} |\xi|^{k/H-2k} \left(\int_0^\infty \exp\left(-\frac{3k_H z^2}{2}\right) z^{1-1/H}\, dz\right)^k.$$

Hence we get

$$A_k \le \frac{c_k (2k)!^2 (aH^{-1})^k}{k!} |\xi|^{k/H-2k},$$

where $a = \int_0^\infty \exp(-3k_H z^2/2) z^{1-1/H}\, dz$, and the proof is now complete. □

4. Decomposition of the interaction energy. Flandoli (2002) defined the energy $\mathbb{H}$ via a decomposition of the Stratonovich integral (6) into a backward Itô integral plus some (easily controlled) correction terms. Although this method seems to have some serious drawbacks with respect to Fourier transform techniques developed in Flandoli and Gubinelli (2002) and in our Section 3, it still gives some insight on the behavior of the interaction energy $\mathbb{H}_{xy}$, since it exhibits an intersection local time type term as the main contribution to the divergence of $\mathbb{H}_{xy}$ when x tends to y [see Theorem 9 in Flandoli (2002)].

We take up this program in this section, getting a decomposition of the Stratonovich integral $\mathbb{H}_{xy}$ into a double Skorohod integral with respect to B plus some additional terms, among which the main contribution is a double integral that can be interpreted as a weighted intersection local time for the three-dimensional fractional Brownian motion. The intersection local time with respect to the fBm was studied in Rosen (1987) in the two-dimensional case and in Hu (2001) using Wiener chaos expansions.

However, our decomposition only holds for $\frac{1}{2} < H < \frac{2}{3}$, and we also prove that both double Skorohod integrals and the weighted intersection local time are divergent whenever $H \geq \frac{2}{3}$.

The main result of this section is the following:

PROPOSITION 11. *Assume* $\frac{1}{2} < H < \frac{2}{3}$. *For any* $x \neq y$, *set*

$$\widehat{\mathbb{H}_{xy}} = \sum_{i=1}^{3} \int_0^T \int_0^t \frac{1}{|x + B_t - y - B_s|} \, dB_s^i \, dB_t^i. \tag{45}$$

Then $\widehat{\mathbb{H}_{xy}}$ *exists as the limit in* $L^2(\Omega)$ *of the sequence* $\widehat{\mathbb{H}^n_{xy}}$ *defined using the approximation* $\sigma_n(x)$ *of* $|x|^{-1}$ *introduced in* (26) *and the following decomposition holds*:

$$\begin{aligned}
\widehat{\mathbb{H}_{xy}} = {} & \sum_{i=1}^{3} \int_0^T \int_0^t \frac{1}{|x - y + B_t - B_r|} \, \delta B_r^i \, \delta B_t^i \\
& - H^2 \int_0^T \int_0^t \delta_0(x - y + B_t - B_r)(t - r)^{2(2H-1)} \, dr \, dt \\
& + H(2H - 1) \int_0^T \left(\int_0^t \frac{1}{|x - y + B_t - B_r|} (t - r)^{2H-2} \, dr \right) dt \\
& + H \int_0^T \left(\frac{1}{|x - y + B_T - B_r|} (T - r)^{2H-1} + \frac{1}{|x - y + B_r|} r^{2H-1} \right) dr.
\end{aligned}$$

Notice that to stick to the notation of Flandoli (2002), we chose to deal here with the half integral $\widehat{\mathbb{H}_{xy}}$ over the domain

$$\{0 \leq s \leq t \leq T\}$$

and to simplify the notation, we have omitted the constant $\Gamma^2/8\pi$. Nevertheless, it holds that $\mathbb{H}_{xy} = (\Gamma^2/8\pi)(\widehat{\mathbb{H}_{xy}} + \widehat{\mathbb{H}_{yx}})$ and we have proved using Fourier analysis that $\mathbb{H}_{xy}$ has moments of any order. Finally, for the sake of simplicity, we also chose to work with fixed x and y.

Section 4.1 is devoted to the decomposition of $\widehat{\mathbb{H}_{xy}}$ when $1/|z|$ is replaced by a smooth function $\sigma : \mathbb{R}^3 \to \mathbb{R}$ in (45). We then apply an approximation argument to obtain our decomposition in Section 4.2.

4.1. *An Itô–Stratonovich correction.* Consider a smooth function $\sigma : \mathbb{R}^3 \to \mathbb{R}$ with bounded derivatives. We set $\partial_i \sigma$ for $\partial \sigma / \partial x_i$.The aim of this section is to justify the existence of the double Stratonovich integral

$$\begin{aligned}
J(x) &= \int_0^T \left(\int_0^t \sigma(x + B_t - B_s) \, dB_s \right) dB_t \\
&:= \sum_{i=1}^{3} \int_0^T \left(\int_0^t \sigma(x + B_t - B_s) \, dB_s^i \right) dB_t^i.
\end{aligned}$$

Furthermore, we express this integral in terms of a double Skorohod integral, which is more suitable for getting estimates, plus some correction terms. The main result of this section is contained in the following proposition.

PROPOSITION 12. *For any smooth function σ with bounded derivatives, the integral $J(x)$ is well defined and*

$$\begin{aligned} J(x) = &\int_0^T \left(\int_0^t \sigma(x + B_t - B_s)\, \delta B_s \right) \delta B_t \\ &- H^2 \int_0^T \int_0^t \Delta\sigma(x + B_t - B_r)(t-r)^{2(2H-1)}\, dr\, dt \\ &+ 2H(2H-1) \int_0^T \int_0^t \sigma(x + B_t - B_r)(t-r)^{2H-2}\, dr\, dt \\ &+ H \int_0^T \left(\sigma(x + B_T - B_r)\,(T-r)^{2H-1} - \sigma(x + B_r)\, r^{2H-1} \right) dr. \end{aligned}$$

PROOF. For $i = 1, 2, 3$ and $t \in [0, T]$, set $\alpha_t^i = \int_0^t \sigma(x + B_t - B_s)\, dB_s^i$. Then $J(x) = \sum_{i=1}^3 \int_0^T \alpha_t^i\, dB_t^i$. We divide the proof into several steps.

STEP 1 (Decomposition of α^i). Since

$$D_r^i[\sigma(x + B_t - B_s)] = \partial_i \sigma(x + B_t - B_s)\mathbf{1}_{[s,t]}(r),$$

using relationship (21), we can write $\alpha_t^i = \beta_t^i + \gamma_t^i$, where

$$\beta_t^i = \int_0^t \sigma(x + B_t - B_s)\, \delta B_s^i$$

and

$$\begin{aligned} \gamma_t^i &= \alpha_H \int_0^t \int_0^T D_r^i[\sigma(x + B_t - B_s)]|s-r|^{2H-2}\, dr\, ds \\ &= H \int_0^t \partial_i \sigma(x + B_t - B_s)(t-s)^{2H-1}\, ds. \end{aligned}$$

We now write $J(x) = A_1 + A_2$, where $A_1 = \sum_{i=1}^3 \int_0^T \gamma_t^i\, dB_t^i$ and $A_2 = \sum_{i=1}^3 \int_0^T \beta_t^i\, dB_t^i$, and decompose A_1 and A_2 separately.

STEP 2 (Decomposition of A_1). An easy application of the stochastic Fubini theorem gives

$$A_1 = H \sum_{i=1}^3 \int_0^T \left(\int_s^T \partial_i \sigma(x + B_t - B_s)(t-s)^{2H-1}\, dB_t^i \right) ds. \tag{46}$$

Then, by a slight modification of the Itô type formula (22) applied to the function $\phi(t,y)=(t-s)^{2H-1}\sigma(z+y)$ and to the process $\{B_t-B_s, t\in[s,T]\}$, we obtain

$$\begin{aligned}\sum_{i=1}^{3}\int_s^T \partial_i\sigma(x+B_t-B_s)(t-s)^{2H-1}\,dB_t^i \\ = \sigma(x+B_T-B_s)(T-s)^{2H-1} \\ -(2H-1)\int_s^T \sigma(x+B_t-B_s)(t-s)^{2H-2}\,dt.\end{aligned}\tag{47}$$

Substituting (47) into (46) we have $A_1=R_1+R_2$ with

$$R_1 = H\int_0^T \sigma(x+B_T-B_s)(T-s)^{2H-1}\,ds,$$

$$R_2 = -\alpha_H\int_0^T\int_0^t \sigma(x+B_t-B_s)(t-s)^{2H-2}\,ds\,dt.$$

STEP 3 (Decomposition of A_2). By relationship (21) we have $A_2=R_3+A_3$, with

$$R_3=\sum_{i=1}^{3}\int_0^T \beta_t^i\,\delta B_t^i=\int_0^T\left(\int_0^t \sigma(x+B_t-B_s)\,\delta B_s\right)\delta B_t,$$

$$A_3=\alpha_H\sum_{i=1}^{3}\int_0^T\int_0^t D_s^i\beta_t^i(t-s)^{2H-2}\,ds\,dt.$$

We compute the derivative of β_t^i for a fixed $t\in[0,T]$ and $i\in\{1,2,3\}$. By the commutation relationship between D^i and δ^i, given for instance in Nualart (1995), we have

$$D_s^i\beta_t^i=\sigma(x+B_t-B_s)+\int_0^s \partial_i\sigma(x+B_t-B_r)\,\delta B_r^i.$$

Hence $A_3=R_4+R_5$, where

$$R_4=3\alpha_H\int_0^T\int_0^t \sigma(x+B_t-B_s)(t-s)^{2H-2}\,ds\,dt$$

and

$$\begin{aligned}R_5&=\alpha_H\sum_{i=1}^{3}\int_0^T\int_0^t (t-s)^{2H-2}\left(\int_0^s \partial_i\sigma(x+B_t-B_r)\,\delta B_r^i\right)ds\,dt\\ &=H\sum_{i=1}^{3}\int_0^T\left(\int_0^t (t-r)^{2H-1}\,\partial_i\sigma(x+B_t-B_r)\,\delta B_r^i\right)dt.\end{aligned}\tag{48}$$

STEP 4 (Computation of R_5). The stochastic integral with respect to B^i can be evaluated again using Itô's formula, but since $t\geq r$, we have to compute first the correction term due to the anticipating term B_t. This can be performed using the substitution formula of Proposition 4. Set $F = x + B_t$, and for $i = 1, 2, 3$ and $0 \le r \le t$, put $\zeta_r^i(z) = \partial_i \sigma(z - B_r)(t - r)^{2H-1}$. Then the process ζ^i and the random variable F satisfy the hypothesis of Proposition 4, and

$$\begin{aligned}
&\sum_{i=1}^{3} \int_0^t (t-r)^{2H-1}\, \partial_i \sigma(x + B_t - B_r)\, \delta B_r^i \\
&\qquad = \sum_{i=1}^{3} \int_0^t \zeta_r^i(F)\, \delta B_r^i \\
&\qquad = \sum_{i=1}^{3} \int_0^t \zeta_r^i(z)\, \delta B_r^i \Big|_{z=F} \\
&\qquad\qquad - \alpha_H \sum_{i=1}^{3} \int_0^T \int_0^t \partial_{z_i} \zeta_r^i(F) D_v^i F^i |r - v|^{2H-2}\, dr\, dv.
\end{aligned} \tag{49}$$

Taking into account the relationship $\partial_{z_i} \zeta_r^i(z) = \partial_{ii}^2 \sigma(z - B_r)(t - r)^{2H-1}$, the last term of this expression is equal to

$$\begin{aligned}
&\alpha_H \int_0^t \int_0^t \Delta\sigma(x + B_t - B_r)(t - r)^{2H-1} |r - v|^{2H-2}\, dr\, dv \\
&\qquad = H \int_0^t \Delta\sigma(x + B_t - B_r)(t - r)^{2H-1} \big[r^{2H-1} + (t - r)^{2H-1} \big]\, dr.
\end{aligned} \tag{50}$$

Furthermore, for a fixed $z \in \mathbb{R}$, the Itô type formula (21) gives

$$\begin{aligned}
\sum_{i=1}^{3} \int_0^t \zeta_r(z)\, \delta B_r^i &= \sum_{i=1}^{3} \int_0^t \partial_i \sigma(z - B_r)(t - r)^{2H-1}\, \delta B_r^i \\
&= -\sigma(z) t^{2H-1} - (2H - 1) \int_0^t \sigma(z - B_r)(t - r)^{2H-2}\, dr \\
&\qquad + H \int_0^t \Delta\sigma(z - B_r)(t - r)^{2H-1} r^{2H-1}\, dr.
\end{aligned} \tag{51}$$

Plugging (49) into (48), and using (50) and (51) yields

$$\begin{aligned}
R_5 = &-H \int_0^T \sigma(x + B_t)\, t^{2H-1}\, dt \\
&- H(2H - 1) \int_0^T \int_0^t \sigma(x + B_t - B_r)(t - r)^{2H-2}\, dr\, dt \\
&- H^2 \int_0^T \int_0^t \Delta\sigma(x + B_t - B_r)(t - r)^{2(2H-1)}\, dr\, dt.
\end{aligned}$$

Finally, $J(x) = \sum_{i=1}^{5} R_i$, and this gives the desired result. $\square$

4.2. *Proof of Proposition* 11. Consider the function σ_n defined in (26), which satisfies the estimate

$$\sigma_n(x) \le c_1|x|^{-1}. \tag{52}$$

In fact, by Lemma 17 we have

$$\sigma_n(x) = E(|x+Z|^{-1}) \le c_1|x|^{-1},$$

where Z is a three-dimensional random variable with law $N(0, 1/n)$. Moreover,

$$\partial_i|x|^{-1} = -|x|^{-3}x_i \tag{53}$$

and, again by Lemma 17,

$$|\partial_i\sigma_n| \le c_2|x|^{-2}. \tag{54}$$

The second partial derivatives of σ are

$$\partial_j\partial_i|x|^{-1} = -\delta_{ij}|x|^{-3} + 3|x|^{-5}x_ix_j$$

and we have

$$\Delta|x|^{-1} = \frac{4}{\pi}\delta_0$$

in the sense of distributions. This implies

$$\Delta\sigma_n(x) = \frac{4}{\pi}p_{1/n}(x). \tag{55}$$

The function σ_n is infinitely differentiable with bounded derivatives. As a consequence, by Proposition 12 the double Stratonovich integral

$$J_n(x) = \int_0^T\left(\int_0^t \sigma_n(x+B_t-B_s)\,dB_s\right)dB_t$$

exists and can be decomposed into the following terms:

$$\begin{aligned} J_n(x) &= \int_0^T\left(\int_0^t \sigma_n(x+B_t-B_s)\,\delta B_s\right)\delta B_t \\ &\quad - H^2\int_0^T\left(\int_0^t \Delta\sigma_n(x+B_t-B_r)(t-r)^{2(2H-1)}\,dr\right)dt \\ &\quad + H(2H-1)\int_0^T\left(\int_0^t \sigma_n(x+B_t-B_r)(t-r)^{2H-2}\,dr\right)dt \\ &\quad + H\int_0^T\left(\sigma_n(x+B_T-B_r)(T-r)^{2H-1}+\sigma_n(x+B_r)r^{2H-1}\right)dr \\ &:= A_n+B_n+C_n+D_n. \end{aligned} \tag{56}$$

Notice that $\widehat{\mathbb{H}^n_{xy}} = J_n(x-y)$. Then the proof of Proposition 11 is a consequence of the following lemmas.

LEMMA 13. *Assume $\frac{1}{2} < H < \frac{2}{3}$ and $x \neq 0$. The term A_n in* (56) *converges in $L^2(\Omega)$ to the double Skorohod integral*

$$\sum_{i=1}^{3} \int_0^T \int_0^t \frac{1}{|x + B_t - B_s|} \, \delta B_s^i \, \delta B_t^i.$$

PROOF. The proof is done in several steps.

STEP 1. Let us first show that $\sigma_n(x + B_t - B_s)$ converges in $L^2([0,T]^2 \times \Omega)$ to $|x + B_t - B_s|^{-1}$. By the estimate (52) and using the dominated convergence theorem, it suffices to show that

$$E \int_0^T \int_0^t |x + B_t - B_s|^{-2} \, ds \, dt < \infty,$$

and, again, this is a consequence of Lemma 17.

STEP 2. By the closability of the double Skorohod integral it remains only to show that A_n converges in $L^2(\Omega)$ to some random variable. For this we need to prove that the limit $\lim_{n,m\to\infty} E(A_n A_m)$ exists.

STEP 3. Let us compute $E(A_n A_m)$. Set $u_t^{(n)} = \int_0^t \sigma_n(x + B_t - B_s) \, \delta B_s$. So, by formula (25), we need to compute $E(u_r^{(n),i} u_{r'}^{(m),i})$ and $E(D_\theta^j u_r^{(n),i} D_{r'}^i u_{\theta'}^{(m),j})$. We have, again by (25),

$$\begin{aligned} &E\big(u_r^{(n),i} u_{r'}^{(m),i}\big) \\ &\quad = E\langle \sigma_n(x + B_r - B_\cdot)\mathbf{1}_{[0,r]}, \sigma_m(x + B_{r'} - B_\cdot)\mathbf{1}_{[0,r']}\rangle_{\mathcal{H}} \\ &\qquad + E\langle D_\cdot^i[\sigma_n(x + B_r - B_\cdot)]\mathbf{1}_{[0,r]}, \big(D_\cdot^i[\sigma_m(x + B_{r'} - B_\cdot)]\mathbf{1}_{[0,r']}\big)^*\rangle_{\mathcal{H}\otimes\mathcal{H}} \\ &\quad = \beta_1(r,r') + \beta_2(r,r'), \end{aligned}$$

where

$$\begin{aligned} \beta_1(r,r') = \alpha_H \int_0^{r'} \int_0^r & E[\sigma_n(x + B_r - B_\theta)\sigma_m(x + B_{r'} - B_{\theta'})] \\ & \times |\theta - \theta'|^{2H-2} \, d\theta \, d\theta' \end{aligned} \tag{57}$$

and

$$
\begin{aligned}
\beta_2^i(r,r') &= \alpha_H^2 \int_{[0,T]^4} E\big(D_\theta^i[\sigma_n(x+B_r-B_\eta)]D_{\eta'}^i[\sigma_m(x+B_{r'}-B_{\theta'})]\big)\\
&\qquad \times \mathbf{1}_{\{\eta\le r,\theta'\le r'\}}|\eta-\eta'|^{2H-2}\\
&\qquad \times |\theta-\theta'|^{2H-2}\,d\eta\,d\eta'\,d\theta\,d\theta'\\
&= \alpha_H^2 \int_{[0,T]^4} E\big(\partial_i\sigma_n(x+B_r-B_\eta)\,\partial_i\sigma_m(x+B_{r'}-B_{\theta'})\big)\\
&\qquad \times \mathbf{1}_{\{\eta\le\theta\le r\}}\mathbf{1}_{\{\theta'\le\eta'\le r'\}}|\eta-\eta'|^{2H-2}\\
&\qquad \times |\theta-\theta'|^{2H-2}\,d\eta\,d\eta'\,d\theta\,d\theta'.
\end{aligned} \tag{58}
$$

On the other hand, using again the commutation relationship between D and δ,

$$
D_\theta^j u_r^{(n),i} = \delta_{ij}\mathbf{1}_{\{\theta\le r\}}\sigma_n(x+B_r-B_\theta) + \int_0^r D_\theta^j[\sigma_n(x+B_r-B_s)]\,\delta B_s^i
$$

and

$$
\begin{aligned}
&E\big(D_\theta^j u_r^{(n),i} D_{r'}^i u_{\theta'}^{(m),j}\big)\\
&\quad = \delta_{ij}\mathbf{1}_{\{\theta\le r,r'\le\theta'\}}E\big[\sigma_n(x+B_r-B_\theta)\sigma_m(x+B_{\theta'}-B_{r'})\big]\\
&\qquad + \delta_{ij}\mathbf{1}_{\{\theta\le r\}}E\bigg[\sigma_n(x+B_r-B_\theta)\int_0^{\theta'} D_{r'}^i[\sigma_m(x+B_{\theta'}-B_s)]\,\delta B_s^j\bigg]\\
&\qquad + \delta_{ij}\mathbf{1}_{\{r'\le\theta'\}}E\bigg[\sigma_m(x+B_{\theta'}-B_{r'})\int_0^{r} D_{\theta}^j[\sigma_m(x+B_{r}-B_s)]\,\delta B_s^i\bigg]\\
&\qquad + E\bigg[\bigg(\int_0^r D_\theta^j[\sigma_n(x+B_r-B_s)]\,\delta B_s^i\bigg)\\
&\qquad\qquad \times \bigg(\int_0^{\theta'} D_{r'}^i[\sigma_m(x+B_{\theta'}-B_s)]\,\delta B_s^j\bigg)\bigg]\\
&\quad := (\beta_3^{ij}+\beta_4^{ij}+\beta_5^{ij}+\gamma^{ij})(r,r',\theta,\theta').
\end{aligned} \tag{59}
$$

Using the duality relationship (15) between the divergence and derivative operators we can write

$$
\begin{aligned}
&\beta_4^{ij}(r,r',\theta,\theta')\\
&\quad = \delta_{ij}\mathbf{1}_{\{\theta\le r\}}E\big\langle D_\cdot^j[\sigma_n(x+B_r-B_\theta)],\mathbf{1}_{[0,\theta']}(\cdot)D_{r'}^i[\sigma_m(x+B_{\theta'}-B_\cdot)]\big\rangle_{\mathcal{H}}\\
&\quad = \alpha_H\delta_{ij}\int_0^T\int_0^T E\big(\partial_j\sigma_n(x+B_r-B_\theta)\partial_i\sigma_m(x+B_{\theta'}-B_{s'})\big)\\
&\qquad\qquad \times \mathbf{1}_{\{\theta\le s\le r\}}\mathbf{1}_{\{s'\le r'\le\theta'\}}|s-s'|^{2H-2}\,ds\,ds'.
\end{aligned} \tag{60}
$$

Similarly

$$\begin{aligned}
&\beta_5^{ij}(r,r',\theta,\theta') \\
&\quad = \delta_{ij}\mathbf{1}_{\{r'\le\theta'\}}E\langle D^i_{\cdot}[\sigma_m(x+B_{\theta'}-B_{r'})],\mathbf{1}_{[0,r]}(\cdot)D^j_\theta[\sigma_n(x+B_r-B_{\cdot})]\rangle_{\mathcal{H}} \\
&\quad = \alpha_H\delta_{ij}\int_0^T\int_0^T E\big(\partial_i\sigma_m(x+B_{\theta'}-B_{r'})\,\partial_j\sigma_n(x+B_r-B_s)\big) \\
&\qquad\qquad \times \mathbf{1}_{\{r'\le s'\le\theta'\}}\mathbf{1}_{\{s\le\theta\le r\}}|s-s'|^{2H-2}\,ds\,ds'
\end{aligned}\tag{61}$$

and, using the isometry property of the Skorohod integral (25), we can write

$$\begin{aligned}
&\gamma^{ij}(r,r',\theta,\theta') \\
&\quad = \delta_{ij}E\langle D^j_\theta[\sigma_n(x+B_r-B_{\cdot})]\mathbf{1}_{[0,r]}(\cdot),D^i_{r'}[\sigma_m(x+B_{\theta'}-B_{\cdot})]\mathbf{1}_{[0,\theta']}(\cdot)\rangle_{\mathcal{H}} \\
&\qquad + E\langle D^j_\circ D^j_\theta[\sigma_n(x+B_r-B_{\cdot})]\mathbf{1}_{[0,r]}(\cdot), \\
&\qquad\qquad \big(D^i_{\cdot}D^i_{r'}[\sigma_m(x+B_{\theta'}-B_\circ)]\mathbf{1}_{[0,\theta']}(\circ)\big)^*\rangle_{\mathcal{H}\otimes\mathcal{H}} \\
&\quad := (\beta_6^{ij}+\beta_7^{ij})(r,r',\theta,\theta'),
\end{aligned}$$

where

$$\begin{aligned}
&\beta_6^{ij}(r,r',\theta,\theta') \\
&\quad = \delta_{ij}\alpha_H\int_0^{\theta'}\int_0^r E\big(D^j_\theta[\sigma_n(x+B_r-B_s)]D^i_{r'}[\sigma_m(x+B_{\theta'}-B_{s'})]\big) \\
&\qquad\qquad \times |s-s'|^{2H-2}\,ds\,ds' \\
&\quad = \delta_{ij}\alpha_H\int_0^T\int_0^T E\big(\partial_j\sigma_n(x+B_r-B_s)\,\partial_i\sigma_m(x+B_{\theta'}-B_{s'})\big) \\
&\qquad\qquad \times \mathbf{1}_{\{s\le\theta\le r\}}\mathbf{1}_{\{s'\le r'\le\theta'\}}|s-s'|^{2H-2}\,ds\,ds'
\end{aligned}\tag{62}$$

and

$$\begin{aligned}
&\beta_7^{ij}(r,r',\theta,\theta') \\
&\quad = \alpha_H^2\int_{[0,T]^4}E\big(D^j_sD^j_\theta[\sigma_n(x+B_r-B_\eta)]D^i_{\eta'}D^i_{r'}[\sigma_m(x+B_{\theta'}-B_{s'})]\big) \\
&\qquad\qquad \times \mathbf{1}_{\{s'\le\theta'\}}\mathbf{1}_{\{\eta\le r\}}|s-s'|^{2H-2}|\eta-\eta'|^{2H-2}\,ds\,ds'\,d\eta\,d\eta' \\
&\quad = \alpha_H^2\int_{[0,T]^4}E\big(\partial_{jj}\sigma_n(x+B_r-B_\eta)\,\partial_{ii}\sigma_m(x+B_{\theta'}-B_{s'})\big)\mathbf{1}_{\{\eta\le s,\theta\le r\}} \\
&\qquad\qquad \times \mathbf{1}_{\{s'\le r',\eta'\le\theta'\}}|s-s'|^{2H-2}|\eta-\eta'|^{2H-2}\,ds\,ds'\,d\eta\,d\eta'.
\end{aligned}\tag{63}$$

As a consequence,

$$E(A_nA_m)=\sum_{i=1}^7 a_i,$$

where the seven terms a_i are computed as follows:

1. The first term is

$$a_1 = \alpha_H \sum_{i=1}^{3} \int_0^T \int_0^T \beta_1(r, r')|r - r'|^{2H-2}\, dr\, dr'$$

and substituting $\alpha_1(r, r')$ from (57) yields

$$a_1 = 3\alpha_H^2 \int_0^T \int_0^T \int_0^{r'} \int_0^r E\big[\sigma_n(x + B_r - B_\theta)\sigma_m(x + B_{r'} - B_{\theta'})\big] \times |\theta - \theta'|^{2H-2}|r - r'|^{2H-2}\, d\theta\, d\theta'\, dr\, dr'.$$

2. For the second term we have

$$a_2 = \alpha_H \sum_{i=1}^{3} \int_0^T \int_0^T \beta_2^i(r, r')|r - r'|^{2H-2}\, dr\, dr'$$

and substituting $\beta_2^i(r, r')$ from (58) yields

$$a_2 = \alpha_H^3 \sum_{i=1}^{3} \int_{[0,T]^6} E\big(\partial_i\sigma_n(x + B_r - B_\eta)\, \partial_i\sigma_m(x + B_{r'} - B_{\theta'})\big) \times \mathbf{1}_{\{\eta \le \theta \le r\}} \mathbf{1}_{\{\theta' \le \eta' \le r'\}} |\eta - \eta'|^{2H-2} \times |\theta - \theta'|^{2H-2}|r - r'|^{2H-2}\, d\eta\, d\eta'\, d\theta\, d\theta'\, dr\, dr'.$$

3. The remaining terms are, for $k = 3, 4, 5, 6, 7$,

$$a_k = \alpha_H^2 \sum_{i,j=1}^{3} \int_{[0,T]^4} \beta_k^{ij}(r, r', \theta, \theta')|r - r'|^{2H-2}|\theta - \theta'|^{2H-2}\, dr\, dr'\, d\theta\, d\theta'.$$

Substituting $\beta_k^{ij}(r, r', \theta, \theta')$ by its value from (59)–(63) yields

$$a_3 = 3\alpha_H^2 \int_{[0,T]^4} E\big[\sigma_n(x + B_r - B_\theta)\sigma_m(x + B_{\theta'} - B_{r'})\big] \times \mathbf{1}_{\{\theta \le r, r' \le \theta'\}}|r - r'|^{2H-2}|\theta - \theta'|^{2H-2}\, dr\, dr'\, d\theta\, d\theta',$$

$$a_4 = \alpha_H^3 \sum_{i=1}^{3} \int_{[0,T]^6} E\big(\partial_i\sigma_n(x + B_r - B_\theta)\, \partial_i\sigma_m(x + B_{\theta'} - B_{s'})\big) \times \mathbf{1}_{\{\theta \le s \le r\}} \mathbf{1}_{\{s' \le r' \le \theta'\}} |s - s'|^{2H-2} \times |r - r'|^{2H-2}|\theta - \theta'|^{2H-2}\, dr\, dr'\, d\theta\, d\theta'\, ds\, ds',$$

$$a_5 = \alpha_H^3 \sum_{i=1}^{3} \int_{[0,T]^6} E\big(\partial_i\sigma_m(x + B_{\theta'} - B_{r'})\, \partial_j\sigma_n(x + B_r - B_s)\big)$$

$$\times \mathbf{1}_{\{r'\le s'\le\theta'\}}\mathbf{1}_{\{s\le\theta\le r\}}|s-s'|^{2H-2}$$
$$\times |r-r'|^{2H-2}|\theta-\theta'|^{2H-2}\,dr\,dr'\,d\theta\,d\theta'\,ds\,ds',$$

$$a_6=\alpha_H^3\sum_{i=1}^3\int_{[0,T]^6}E\big(\partial_i\sigma_n(x+B_r-B_s)\,\partial_i\sigma_m(x+B_{\theta'}-B_{s'})\big)$$
$$\times \mathbf{1}_{\{s\le\theta\le r\}}\mathbf{1}_{\{s'\le r'\le\theta'\}}|s-s'|^{2H-2}$$
$$\times |r-r'|^{2H-2}|\theta-\theta'|^{2H-2}\,dr\,dr'\,d\theta\,d\theta'\,ds\,ds'$$

and

$$a_7=\alpha_H^4\int_{[0,T]^8}E\big(\Delta\sigma_n(x+B_r-B_\eta)\Delta\sigma_m(x+B_{\theta'}-B_{s'})\big)$$
$$\times \mathbf{1}_{\{\eta\le s,\theta\le r\}}\mathbf{1}_{\{s'\le r',\eta'\le\theta'\}}|s-s'|^{2H-2}|\eta-\eta'|^{2H-2}$$
$$\times |r-r'|^{2H-2}|\theta-\theta'|^{2H-2}\,dr\,dr'\,d\theta\,d\theta'\,ds\,ds'\,d\eta\,d\eta'.$$

The above computations lead to the result

$$E(A_nA_m)=\alpha_{n,m}+\beta_{n,m}+\gamma_{n,m},$$

where

$$\alpha_{n,m}=3\alpha_H^2\int_{[0,T]^4}E\big[\sigma_n(x+B_r-B_\theta)\sigma_m(x+B_{r'}-B_{\theta'})\big]\mathbf{1}_{\{\theta\le r,\theta'\le r'\}}$$
$$\times\big[|\theta-\theta'|^{2H-2}|r-r'|^{2H-2}$$
$$+|r-\theta'|^{2H-2}|\theta-r'|^{2H-2}\big]\,d\theta\,d\theta'\,dr\,dr',$$

$$\beta_{n,m}=\alpha_H^3\sum_{i=1}^3\int_{[0,T]^6}E\big(\partial_i\sigma_n(x+B_r-B_\theta)\,\partial_i\sigma_m(x+B_{r'}-B_{\theta'})\big)$$
$$\times \mathbf{1}_{\{\theta\le s\le r\}}\mathbf{1}_{\{\theta'\le s'\le r'\}}$$
$$\times\big[(|\theta-s'|\,|s-\theta'|\,|r-r'|)^{2H-2}$$
$$+(|s-\theta'|\,|r-s'|\,|\theta-r'|)^{2H-2}$$
$$+(|\theta-s'|\,|r-\theta'|\,|s-r'|)^{2H-2}$$
$$+(|\theta-\theta'|\,|r-s'|\,|s-r'|)^{2H-2}\big]\,ds\,ds'\,d\theta\,d\theta'\,dr\,dr' \tag{64}$$

and

$$\gamma_{n,m}=\alpha_H^4\int_{[0,T]^8}E\big(\Delta\sigma_n(x+B_r-B_\theta)\Delta\sigma_m(x+B_{r'}-B_{\theta'})\big)$$
$$\times \mathbf{1}_{\{\theta\le s,\eta\le r\}}\mathbf{1}_{\{\theta'\le s',\eta'\le r'\}}$$
$$\times|s-\theta'|^{2H-2}|\theta-\eta'|^{2H-2}$$
$$\times|r-s'|^{2H-2}|\eta-r'|^{2H-2}\,dr\,dr'\,d\theta\,d\theta'\,ds\,ds'\,d\eta\,d\eta'. \tag{65}$$

STEP 4. Let us check the convergence of the term $\alpha_{n,m}$. For each $x \neq 0$, $\theta < r$ and $\theta' < r'$, the expression

$$\sigma_n(x + B_r - B_\theta)\sigma_m(x + B_{r'} - B_{\theta'})$$

converges as n and m tend to infinity to

$$|x + B_r - B_\theta|^{-1}|x + B_{r'} - B_{\theta'}|^{-1}.$$

Then we can conclude by dominated convergence and the estimate [see (52)]

$$\sigma_n(x + B_r - B_\theta) \le c_1|x + B_r - B_\theta|^{-1},$$

and the fact that, by Lemma 17,

$$\begin{aligned} E\big(&|x + B_r - B_\theta|^{-1}|x + B_{r'} - B_{\theta'}|^{-1}\big) \\ &\le \big[E|x + B_r - B_\theta|^{-2}E|x + B_{r'} - B_{\theta'}|^{-2}\big]^{1/2} \\ &\le c_2|x|^{-2}. \end{aligned}$$

STEP 5. Let us check the convergence of the term $\beta_{n,m}$. For each $x \neq 0$, $\theta < r$ and $\theta' < r'$, by (53) the expression

$$\sum_{i=1}^{3} \partial_i\sigma_n(x + B_r - B_\theta)\,\partial_i\sigma_m(x + B_{r'} - B_{\theta'})$$

converges as n and m tend to infinity to

$$\frac{\langle x + B_r - B_\theta, x + B_{r'} - B_{\theta'}\rangle}{|x + B_r - B_\theta|^3|x + B_{r'} - B_{\theta'}|^3}.$$

By the estimate (54) we have

$$|\partial_i\sigma_n(x + B_r - B_\theta)| \le c_2|x + B_r - B_\theta|^{-2}. \tag{66}$$

For each $a < b$ the following inequality holds:

$$\int_a^b |x - y|^{2H-2}\,dx \le k_H|a - b|^{2H-1}. \tag{67}$$

In fact, if $a \le y \le b$, we obtain

$$\begin{aligned} \int_a^b |x - y|^{2H-2}\,dx &= \frac{1}{2H-1}(|b - y|^{2H-1} + |y - a|^{2H-1}) \\ &\le \frac{2}{2H-1}|a - b|^{2H-1} \end{aligned}$$

and if $y \le a$, since $(u + v)^\alpha \le u^\alpha + v^\alpha$, when $u, v > 0$ and $0 < \alpha < 1$,

$$\begin{aligned} \int_a^b |x - y|^{2H-2}\,dx &= \frac{1}{2H-1}(|b - y|^{2H-1} - |a - y|^{2H-1}) \\ &\le \frac{1}{2H-1}|a - b|^{2H-1}, \end{aligned}$$

the case $y \geq b$ being analogous, and (67) follows.

Taking into account (66) and (67), it suffices to show, by dominated convergence, that the following integral is finite:

$$\begin{aligned}\int_{[0,T]^4} & E\big(|x + B_r - B_\theta|^{-2}|x + B_{r'} - B_{\theta'}|^{-2}\big) \\ & \times \big(|r - r'|^{2H-2} + |r - \theta'|^{2H-2} + |r' - \theta|^{2H-2} + |\theta - \theta'|^{2H-2}\big) \\ & \times |r - \theta|^{2H-1}|r' - \theta'|^{2H-1}\, dr\, dr'\, d\theta\, d\theta'.\end{aligned} \tag{68}$$

Consider the decomposition

$$B_{r'} - B_{\theta'} = \lambda(B_r - B_\theta) + Z, \tag{69}$$

where

$$\begin{aligned}\lambda &= \gamma|r - \theta|^{-2H}, \\ \gamma &= E\big((B_r - B_\theta)(B_{r'} - B_{\theta'})\big) \\ &= \tfrac{1}{2}\big(|r - \theta'|^{2H} - |r - r'|^{2H} + |r' - \theta|^{2H} - |\theta - \theta'|^{2H}\big)\end{aligned} \tag{70}$$

and Z is a centered three-dimensional Gaussian random variable independent of $B_r - B_\theta$ with variance

$$\sigma_Z^2 = \frac{|r' - \theta'|^{2H}|r - \theta|^{2H} - \gamma^2}{|r - \theta|^{2H}}.$$

Substituting (69) into (68) and applying Lemma 17 yields

$$\begin{aligned}& E\big(|x + B_r - B_\theta|^{-2}|x + B_{r'} - B_{\theta'}|^{-2}\big) \\ &\quad = E\big(|x + B_r - B_\theta|^{-2}|x + \lambda(B_r - B_\theta) + Z|^{-2}\big) \\ &\quad \leq c_2\sigma_Z^{-\gamma_1} E\big(|x + B_r - B_\theta|^{-2}|x + \lambda(B_r - B_\theta)|^{-\gamma_2}\big),\end{aligned}$$

where $\gamma_1 + \gamma_2 = 2$. Fix an exponent $\delta < 3$ close to 3. Hölder's inequality allows us to write

$$\begin{aligned}& E\big(|x + B_r - B_\theta|^{-2}|x + \lambda(B_r - B_\theta)|^{-\gamma_2}\big) \\ &\quad \leq \big[E(|x + B_r - B_\theta|^{-\delta})\big]^{2/\delta}\big[E\big(|x + \lambda(B_r - B_\theta)|^{-\delta\gamma_2/(\delta-2)}\big)\big]^{(\delta-2)/\delta} \\ &\quad \leq c|x|^{-2-\gamma_2},\end{aligned}$$

provided $\delta\gamma_2/(\delta - 2) < 3$. We can write

$$\sigma_Z^{-\gamma_1} = |r - \theta|^{\gamma_1 H} d^{-\gamma_1/2},$$

where

$$d = |r' - \theta'|^{2H}|r - \theta|^{2H} - \gamma^2$$

is the determinant of the covariance matrix of the two-dimensional vector $(B^1_r - B^1_\theta, B^1_{r'} - B^1_{\theta'})$.

To estimate (68) we are going to decompose the integral over the sets

$$G_1 = \{\theta \le \theta' \le r \le r'\},$$
$$G_2 = \{\theta \le r \le \theta' \le r'\}$$

and

$$G_3 = \{\theta \le \theta' \le r' \le r\}.$$

For $i = 1, 2, 3$, we denote by I^i the integral (64) over the set G_i.

We distinguish three cases:

(i) Consider first the integral over the set G_1. We know [see equation (3.9) in Hu (2001)] that there exists a constant k_H such that on G_1,

$$d \ge k_H\big(|r-\theta|^{2H}|r-r'|^{2H} + |r'-\theta'|^{2H}|\theta-\theta'|^{2H}\big). \tag{71}$$

Using the estimate (67) with $[a, b] = [\theta, r]$ and $[a, b] = [\theta', r']$, we obtain

$$\begin{aligned}|I^1| \le c_{H,x} \int_{G_1} |r-\theta|^{\gamma_1 H} d^{-\gamma_1/2} \\ &\times \big(|r-r'|^{2H-2} + |\theta-\theta'|^{2H-2}\big) \\ &\times |r-\theta|^{2H-1}|r'-\theta'|^{2H-1}\, d\theta\, d\theta'\, dr\, dr'.\end{aligned}$$

Now we apply (71) and we get

$$\begin{aligned}|I^1| &\le c_{H,x} \int_{G_1} \big(|r-\theta|^{2H}|r-r'|^{2H} + |r'-\theta'|^{2H}|\theta-\theta'|^{2H}\big)^{-\gamma_1/2} \\ &\qquad \times \big(|r-r'|^{2H-2} + |\theta-\theta'|^{2H-2}\big) \\ &\qquad \times |r-\theta|^{2H(1+\gamma_1/2)-1}|r'-\theta'|^{2H-1}\, d\theta\, d\theta'\, dr\, dr' \\ &\le c_{H,x} \int_{G_1} |\theta-\theta'|^{-\gamma_1 H}|r-r'|^{2H-2} \\ &\qquad \times |r-\theta|^{2H(1+\gamma_1/2)-1}|r'-\theta'|^{2H(1-\gamma_1/2)-1}\, d\theta\, d\theta'\, dr\, dr' \\ &\quad + c_{H,x} \int_{G_1} |\theta-\theta'|^{2H-2}|r-r'|^{-\gamma_1 H} \\ &\qquad \times |r-\theta|^{2H-1}|r'-\theta'|^{2H-1}\, d\theta\, d\theta'\, dr\, dr'.\end{aligned}$$

Hence, making the change of variables $\theta' - \theta = x$, $r - \theta' = y$ and $r - r' = z$, we obtain

$$\begin{aligned}|I^1| &\le c_{H,x} \int_{[0,T]^3} x^{-\gamma_1 H} z^{2H-2}(x+y)^{2H(1+\gamma_1/2)-1} \\ &\qquad \times (y+z)^{2H(1-\gamma_1/2)-1}\, dx\, dy\, dz \\ &\quad + c_{H,x} \int_{[0,T]^3} x^{2H-2} z^{-\gamma_1 H}(x+y)^{2H-1}(y+z)^{2H-1}\, dx\, dy\, dz,\end{aligned}$$

which is finite provided $\gamma_1 < 2$. The restrictions

$$2 < \delta < 3, \qquad \frac{\delta\gamma_2}{\delta - 2} < 3, \qquad \gamma_1 + \gamma_2 = 2$$

imply $0 < \gamma_2 < 1$ and, hence, $1 < \gamma_2 < 2$.

(ii) Consider now the integral over the set G_2. We know [see equation (3.11) in Hu (2001)] that there exists a constant k_H such that on G_2,

$$d \geq k_H |r - \theta|^{2H} |r' - \theta'|^{2H}. \tag{72}$$

Using the estimate (67) we obtain

$$\begin{aligned} |I^2| \leq c_{H,x} \int_{G_2} & |r - \theta|^{\gamma_1 H} d^{-\gamma_1/2} \\ & \times \big(|r - r'|^{2H-2} + |\theta - \theta'|^{2H-2}\big) \\ & \times |r - \theta|^{2H-1} |r' - \theta'|^{2H-1} \, d\theta \, d\theta' \, dr \, dr'. \end{aligned}$$

Now we apply (72) and we get

$$\begin{aligned} |I^2| \leq c_{H,x} \int_{G_2} & \big(|r - r'|^{2H-2} + |\theta - \theta'|^{2H-2}\big) \\ & \times |r - \theta|^{2H-1} |r' - \theta'|^{2H(1-\gamma_1/2)-1} \, d\theta \, d\theta' \, dr \, dr'. \end{aligned}$$

Hence, making the change of variables $r - \theta = x$, $\theta' - r = y$ and $r' - \theta' = z$, we obtain

$$\begin{aligned} |I^2| \leq c_{H,x} \int_{[0,T]^3} & \big[(y+z)^{2H-2} + (x+y)^{2H-2}\big] \\ & \times x^{2H-1} z^{2H(1-\gamma_1/2)-1} \, dx \, dy \, dz, \end{aligned}$$

which is finite provided $\gamma_1 < 2$.

(iii) Consider now the integral over the set G_3. We know [see equation (3.10) in Hu (2001)] that there exists a constant k_H such that on G_3,

$$d \geq k_H |r - \theta|^{2H} |r' - \theta'|^{2H}. \tag{73}$$

Using the estimate (67) we obtain

$$\begin{aligned} |I^3| \leq c_{H,x} \int_{G_3} & |r - \theta|^{\gamma_1 H} d^{-\gamma_1/2} \\ & \times \big(|r - r'|^{2H-2} + |\theta - \theta'|^{2H-2}\big) \\ & \times |r - \theta|^{2H-1} |r' - \theta'|^{2H-1} \, d\theta \, d\theta' \, dr \, dr'. \end{aligned}$$

Now we apply (73) and we get

$$\begin{aligned} |I^3| \leq c_{H,x} \int_{G_3} & \big(|r - r'|^{2H-2} + |\theta - \theta'|^{2H-2}\big) \\ & \times |r - \theta|^{2H-1} |r' - \theta'|^{2H(1-\gamma_1/2)-1} \, d\theta \, d\theta' \, dr \, dr'. \end{aligned}$$

Hence, making the change of variables $r-\theta=x$, $\theta'-r=y$ and $r'-\theta'=z$, we obtain

$$|I^3|\le c_{H,x}\int_{[0,T]^3}\bigl[(y+z)^{2H-2}+(x+y)^{2H-2}\bigr]\\ \times x^{2H-1}z^{2H(1-\gamma_1/2)-1}\,dx\,dy\,dz,$$

which is finite provided $\gamma_1<2$.

STEP 6. Let us show the convergence of the term $\gamma_{n,m}$ defined by (65). We have, by relationship (55),

$$\begin{aligned}
&E\bigl(\Delta\sigma_n(x+B_r-B_\theta)\,\Delta\sigma_m(x+B_{r'}-B_{\theta'})\bigr)\\
&\quad=\frac{16}{\pi^2}E\bigl[p_{1/n}(x+B_r-B_\theta)p_{1/m}(x+B_{r'}-B_{\theta'})\bigr]\\
&\quad=2^7\pi\delta_{m,n}^{-3/2}\exp\biggl(-\frac{3|x^2|\nu_{m,n}}{2\delta_{m,n}}\biggr),
\end{aligned}$$

where

$$\delta_{m,n}=\biggl(|r-\theta|^{2H}+\frac{1}{m}\biggr)\biggl(|r'-\theta'|^{2H}+\frac{1}{n}\biggr)-\gamma^2,$$

$$\nu_{m,n}=|r-\theta|^{2H}+\frac{1}{m}+|r'-\theta'|^{2H}+\frac{1}{n}-2\gamma.$$

Set

$$\Psi_{m,n}:=2^7\pi\delta_{m,n}^{-3/2}\exp\biggl(-\frac{3|x^2|\nu_{m,n}}{2\delta_{m,n}}\biggr)|r-\theta|^{4H-2}|r'-\theta'|^{4H-2}.$$

As n and m tend to infinity, $\Psi_{m,n}$ converges to

$$\Psi:=2^7\pi\delta^{-3/2}\exp\biggl(-\frac{3|x^2|\nu}{2\delta}\biggr)|r-\theta|^{4H-2}|r'-\theta'|^{4H-2},$$

where

$$\delta=|r-\theta|^{2H}|r'-\theta'|^{2H}-\gamma^2, \tag{74}$$

$$\nu=|r-\theta|^{2H}+|r'-\theta'|^{2H}-2\gamma, \tag{75}$$

where γ is defined in (70). Moreover, for all $\alpha,c>0$, we have $y^\alpha\exp(-cy)\le k_\alpha c^{-\alpha}$. Hence, for $0\le\alpha\le\frac{3}{2}$,

$$\Psi_{m,n}\le\frac{c_\alpha|r-\theta|^{4H-2}|r'-\theta'|^{4H-2}}{|x|^{2\alpha}\delta_{m,n}^{3/2-\alpha}\nu_{m,n}^{\alpha}}\le\frac{c_\alpha|r-\theta|^{4H-2}|r'-\theta'|^{4H-2}}{|x|^{2\alpha}\delta^{3/2-\alpha}\nu^{\alpha}}:=\overline{\Psi}.$$

Then it suffices to show that the integral of the function $\overline{\Psi}$ on $[0,T]^4$ is finite. Note first that as in Step 5, we can now split the integral of $\overline{\Psi}$ into three pieces.

(i) On the set $G_1 = \{\theta \le \theta' \le r \le r'\}$, setting $w = \theta' - \theta$, $y = r - \theta'$ and $z = r' - r$, inequality (71) can be read

$$\delta \ge k_H\big[(w+y)^{2H} z^{2H} + (y+z)^{2H} w^{2H}\big].$$

Moreover, using the local nondeterminism of the fractional Brownian motion (see Lemma 1), we get

$$\nu \ge k_H(w^{2H} + z^{2H}).$$

Take now $1 \le \alpha < \frac{1}{H}$ and write $\alpha = \frac{1}{H} - \beta$ with $\beta > 0$. Then

$$\overline{\Psi} \le \frac{c_{H,\alpha}(w+y)^{4H-2}(y+z)^{4H-2}}{((w+y)^{2H}z^{2H} + (y+z)^{2H}w^{2H})^{3/2-\alpha}(w^{2H}+z^{2H})^{\alpha}}$$

and, since $u^2 + v^2 > 2uv$ for all $u, v \ge 0$,

$$\overline{\Psi} \le \frac{c_{H,\beta}((y+z)(w+y))^{5H/2-1-\beta H}}{(wz)^{3H/2}} \le \frac{c_{H,\beta}}{(wz)^{3H/2}} \equiv \overline{\Psi}^{(1)},$$

where we have used the fact that $\frac{5H}{2} - 1 - \beta H > 0$ if $H > \frac{1}{2}$ and β is small enough. Performing the change of variable $w = \theta' - \theta$, $y = r - \theta'$, $z = r' - r$ and $u = T - r'$, it is now easily seen that if $H < \frac{2}{3}$,

$$\int_{G_1} \overline{\Psi}^{(1)}(r, r', \theta, \theta')\, dr\, dr'\, d\theta\, d\theta' < \infty. \tag{76}$$

(ii) On $G_2 = \{\theta \le r \le \theta' \le r'\}$, setting $r - \theta = w$, $\theta' - r = y$ and $r' - \theta' = z$, inequality (72) can be written

$$\delta \ge k_H w^{2H} z^{2H}$$

and Lemma 1 yields

$$\nu \ge k_H(w^{2H} + z^{2H}).$$

Hence

$$\overline{\Psi} \le c_{H,\alpha}(wz)^{-(2-H(1+\alpha))} \equiv \overline{\Psi}^{(2)}$$

and, taking $1 < \alpha$, we get

$$\int_{G_2} \overline{\Psi}^{(2)}(r, r', \theta, \theta')\, dr\, dr'\, d\theta\, d\theta' < \infty. \tag{77}$$

(iii) On $G_3 = \{\theta \le \theta' \le r' \le r\}$, set $w = \theta' - \theta$, $y = r' - \theta'$ and $z = r - r'$. Then (73) reads

$$\delta \ge k_H(w+y+z)^{2H} y^{2H}.$$

Furthermore, using the fact that $w + y + z \ge w + z \ge (1/T^2)wz$, we get

$$\delta \ge k_H(wyz)^{2H}.$$

On the other hand, Lemma 1 gives

$$\nu \geq k_H(w^{2H} + z^{2H}).$$

Hence

$$\overline{\Psi} \leq c_{H,\alpha} y^{-(2-H(1+2\alpha))}(wz)^{-(2-H(1+\alpha))} \equiv \overline{\Psi}^{(3)}$$

and taking $1 < \alpha$ leads to

$$\int_{G_3} \overline{\Psi}^{(3)}(r, r', \theta, \theta')\, dr\, dr'\, d\theta\, d\theta' < \infty. \tag{78}$$

Putting together relationships (76)–(78) completes the proof of the lemma. □

LEMMA 14. *Assume* $\frac{1}{2} < H < \frac{2}{3}$ *and* $x \neq 0$. *The term* B_n *in* (56) *converges in* $L^2(\Omega)$ *to the weighted intersection local time*

$$-H^2 \int_0^T \int_0^t \delta_0(x + B_t - B_r)(t-r)^{2(2H-1)}\, dr\, dt.$$

PROOF. Notice that $B_n = -(4H^2/\pi)L_n$ with

$$L_n = \int_0^T \left(\int_0^t p_{1/n}(x + B_t - B_\theta)(t-\theta)^{2(2H-1)}\, d\theta \right) dt.$$

It is enough to check that the limit $\lim_{n,m\to\infty} E(L_n L_m)$ exists. However, using the results proved in Step 6 of the previous proposition, we get

$$E(L_n L_m) = (2\pi)^6 \int_0^T \int_0^T \int_0^t \int_0^t \delta_{m,n}^{-3/2} \exp\left(-\frac{3|x^2|\nu_{m,n}}{2\delta_{m,n}}\right) \times (t-\theta)^{2(2H-1)}(t'-\theta')^{2(2H-1)}\, d\theta\, d\theta'\, dt\, dt',$$

which we have proved converges to

$$(2\pi)^6 \int_0^T \int_0^T \int_0^t \int_0^t \delta^{-3/2} \exp\left(-\frac{3|x^2|\nu}{2\delta}\right) \times (t-\theta)^{2(2H-1)}(t'-\theta')^{2(2H-1)}\, d\theta\, d\theta'\, dt\, dt'$$

with

$$\delta = |r-\theta|^{2H}|r'-\theta'|^{2H} - \gamma^2,$$
$$\nu = |r-\theta|^{2H} + |r'-\theta'|^{2H} - 2\gamma.$$

□

Similar computations lead to the following two results, which in turn end the proof of our decomposition.

LEMMA 15. *Assume $\frac{1}{2} < H < \frac{2}{3}$ and $x \neq 0$. The term C_n in* (56) *converges in* $L^2(\Omega)$ *to*

$$H(2H-1)\int_0^T \left(\int_0^t \frac{1}{|x+B_t-B_r|}(t-r)^{2H-2}\,dr\right)dt.$$

LEMMA 16. *Assume $\frac{1}{2} < H < \frac{2}{3}$ and $x \neq 0$. The term D_n in* (56) *converges in* $L^2(\Omega)$ *to*

$$H\int_0^T \left(\frac{1}{|x+B_T-B_r|}(T-r)^{2H-1} + \frac{1}{|x+B_r|}r^{2H-1}\right)dr.$$

REMARK. If we assume that $H \geq \frac{2}{3}$, then the function $\lim_{n,m\to\infty}\Psi_{m,n}$ is not integrable in $[0,T]^4$ due to Lemma 18. This fact shows that the restriction $H < \frac{2}{3}$ cannot be avoided.

APPENDIX

In this section we show two technical lemmas that have been used in this article.

LEMMA 17. *Let Y be a three-dimensional Gaussian random variable, the components of which are independent, centered and have variance σ^2. Then, for any positive number $\gamma < 3$, there exists a constant c_γ such that for any $x \in \mathbb{R}^3$,*

$$E|x+Y|^{-\gamma} \leq c_\gamma \min(|x|^{-\gamma}, \sigma^{-\gamma}).$$

PROOF. We have

$$\begin{aligned} E|x+Y|^{-\gamma} &= (2\pi\sigma^2)^{-3/2}\int_{\mathbb{R}^3} |x+y|^{-\gamma} e^{-|y|^2/2\sigma^2}\,dy \\ &\leq \left(\frac{2}{|x|}\right)^{\gamma} + (2\pi\sigma^2)^{-3/2}\int_{|x+y|\leq|x|/2} |x+y|^{-\gamma} e^{-|y|^2/2\sigma^2}\,dy. \end{aligned}$$

For the second summand in the above expression there are the estimates,

$$\begin{aligned} &\sigma^{-3}\int_{|z|\leq|x|/2} |z|^{-\gamma} e^{-|x-z|^2/2\sigma^2}\,dz \\ &\quad\leq \sigma^{-3} e^{-|x|^2/8\sigma^2}\int_{|z|\leq|x|/2} |z|^{-\gamma}\,dz \\ &\quad\leq c_\gamma \sigma^{-3} e^{-|x|^2/8\sigma^2}|x|^{3-\gamma} \\ &\quad\leq c'_\gamma |x|^{-\gamma} \end{aligned}$$

and we deduce

$$E|x+Y|^{-\gamma} \leq c_\gamma |x|^{-\gamma}.$$

On the other hand, we have

$$E|x+Y|^{-\gamma} = \sigma^{-\gamma} E\left|\frac{x}{\sigma} + Z\right|^{-\gamma},$$

where Z is a three-dimensional standard Gaussian vector. Finally, it suffices to observe that

$$\lim_{\sigma\to 0} E\left|\frac{x}{\sigma} + Z\right|^{-\gamma} = 0$$

and

$$\lim_{\sigma\to \infty} E\left|\frac{x}{\sigma} + Z\right|^{-\gamma} = E|Z|^{-\gamma} < \infty. \qquad \square$$

LEMMA 18. *Let δ and ν be the functions defined in* (74) *and* (75), *respectively. Then the integral*

$$\int_{[0,T]^4} \delta^{-3/2} \exp\left(-\frac{3|x^2|\nu}{2\delta}\right)|r-\theta|^{4H-2}|r'-\theta'|^{4H-2}\, d\theta\, d\theta'\, dr\, dr'$$

is infinite if $H \geq \frac{2}{3}$.

PROOF. On the set $G_1 = \{\theta \leq \theta' \leq r \leq r'\}$ and setting, as before, $w = \theta' - \theta$, $y = r - \theta'$ and $z = r' - r$, we can write

$$\begin{aligned}
\delta &= (w+y)^{2H}(y+z)^{2H} - \gamma^2,\\
\nu &= (w+y)^{2H} + (y+z)^{2H} - 2\gamma,\\
\gamma &= \tfrac{1}{2}\big(y^{2H} + (w+y+z)^{2H} - z^{2H} - w^{2H}\big).
\end{aligned}$$

Using the Taylor expansions

$$\begin{aligned}
(w+y)^{2H} &= y^{2H} + 2Hy^{2H-1}w + H(2H-1)\eta_w^{2H-2}w^2,\\
(y+z)^{2H} &= y^{2H} + 2Hy^{2H-1}z + H(2H-1)\eta_z^{2H-2}z^2,\\
(w+y+z)^{2H} &= y^{2H} + 2Hy^{2H-1}(w+z) + H(2H-1)\eta_{w+z}^{2H-2}(w+z)^2,
\end{aligned}$$

where $\eta_w \in [y, y+w]$, $\eta_z \in [y, y+z]$ and $\eta_{w+z} \in [y, y+w+z]$, we can write

$$\gamma = y^{2H} + Hy^{2H-1}(w+z) + \frac{H}{2}(2H-1)\eta_{w+z}^{2H-2}(w+z)^2 - \frac{1}{2}(z^{2H} + w^{2H}).$$

Then it is easy to get

$$\nu = z^{2H} + w^{2H} + H(2H-1)\big[\eta_w^{2H-2}w^2 + \eta_z^{2H-2}z^2 - \eta_{z+w}^{2H-2}(z+w)^2\big]$$

and an expression for δ of the type

$$\delta = y^{2H}(z^{2H} + w^{2H}) + w^2 g_{1,y} + z^2 g_{2,y} + wz g_{3,y},$$

where, for fixed y, $g_{1,y}, g_{2,y}$ and $g_{3,y}$ are polynomial functions of w, z, w^{2H}, z^{2H}, w^{2H-1}, z^{2H-1}, η_w^{2H-2}, η_z^{2H-2} and η_{w+z}^{2H-2}. Moreover, for a fixed y, these functions are bounded on the set G_1. Notice that since $r-\theta \ge y$ and $r'-\theta' \ge y$, we have

$$\begin{aligned}
&\int_0^T\int_0^T\int_0^t\int_0^t \delta^{-3/2}\exp\biggl(-\frac{3|x^2|\nu}{2\delta}\biggr)(r-\theta)^{2(2H-1)}(r'-\theta')^{2(2H-1)}\,d\theta\,d\theta'\,dr\,dr'\\
&\quad\ge \int_{G_1} y^{4(2H-1)}\delta^{-3/2}\exp\biggl(-\frac{3|x^2|\nu}{2\delta}\biggr)\,d\theta\,d\theta'\,dr\,dr'\\
&\quad\ge \frac{1}{4}\int_{1/4}^{3/4} y^{4(2H-1)}\biggl(\int_{\{z^2+w^2\le\varepsilon^2, z,w\ge 0\}}\delta^{-3/2}\exp\biggl(-\frac{3|x^2|\nu}{2\delta}\biggr)\,dz\,dw\biggr)\,dy
\end{aligned}$$

for ε small enough.

Set now $z = r\cos(\xi)$ and $w = r\sin(\xi)$. Then

$$\begin{aligned}
&\int_{\{z^2+w^2\le\varepsilon^2, z,w\ge 0\}}\delta^{-3/2}\exp\biggl(-\frac{3|x^2|\nu}{2\delta}\biggr)\,dz\,dw\\
&\qquad = \int_0^{\pi/2}\int_0^{\varepsilon} r\delta^{-3/2}\exp\biggl(-\frac{3|x^2|\nu}{2\delta}\biggr)\,dr\,d\xi.
\end{aligned}$$

Using the above expressions of ν and δ we can check, for $y \in [\frac{1}{4}, \frac{3}{4}]$, that

$$\lim_{r\to 0}\exp\biggl(-\frac{3|x^2|\nu}{2\delta}\biggr) = \exp\biggl(-\frac{3|x^2|}{2y^{2H}}\biggr) \ge c_x.$$

So, for ε small enough,

$$\begin{aligned}
&\int_0^{\pi/2}\int_0^{\varepsilon} r\delta^{-3/2}\exp\biggl(-\frac{3|x^2|\nu}{2\delta}\biggr)\,dr\,d\xi\\
&\quad\ge \frac{c_x}{2}\int_0^{\pi/2}\int_0^{\varepsilon} r\delta^{-3/2}\,dr\,d\xi\\
&\quad= \frac{c_x}{2y^{2H}}\int_0^{\pi/2}\int_0^{\varepsilon}\frac{1}{r^{3H-1}((\cos(\xi)^{2H}+\sin(\xi)^{2H})+f(r,\xi))^{3/2}}\,dr\,d\xi,
\end{aligned}$$

where

$$f(r,\xi) = r^{2-2H}\sin(\xi)^2 g_1 + r^{2-2H}\cos(\xi)^2 g_2 + r^{2-2H}\sin(\xi)\cos(\xi) g_3,$$

and this integral is divergent when $H \ge \frac{2}{3}$. □

REFERENCES

ALÒS, E. and NUALART, D. (2003). Stochastic integration with respect to fractional Brownian motion. Preprint. *Stochastics Stochastics Rep.* **75** 129–152.

ALÒS, E., MAZET, O. and NUALART, D. (2001). Stochastic calculus with respect to Gaussian processes. *Ann. Probab.* **29** 766–801.

BERMAN, S. (1973). Local nondeterminism and local times of Gaussian processes. *Indiana Univ. Math. J.* **23** 69–94.

BOCHNER, S. and CHANDRASEKHARAN, K. (1949). Fourier transforms. *Ann. Math. Stud.* **19**.

CARMONA, P. and COUTIN, L. (2000). Stochastic integration with respect to fractional Brownian motion. *C. R. Acad. Sci. Paris Sér. I. Math.* **330** 231–236.

CHORIN, A. (1994). *Vorticity and Turbulence*. Springer, Berlin.

DAI, W. and HEYDE, C. C. (1996). Itô's formula with respect to fractional Brownian motion and its application. *J. Appl. Math. Stochastic Anal.* **9** 439–448.

DECREUSEFOND, L. and ÜSTÜNEL, A. S. (1998). Stochastic analysis of the fractional Brownian motion. *Potential Anal.* **10** 177–214.

FLANDOLI, F. (2002). On a probabilistic description of small scale structures in 3D fluids. *Ann. Inst. H. Poincaré Probab. Statist.* **38** 207–228.

FLANDOLI, F. and GUBINELLI, M. (2002). The Gibbs ensemble of a vortex filament. *Probab. Theory Related Fields* **122** 317–340.

FLANDOLI, F. and MINELLI, I. (2001). Probabilistic models of vortex filaments. *Czechoslovak Math. J.* **51** 713–731.

HU, Y. (2001). Self-intersection local time of fractional Brownian motions via chaos expansion. *J. Math. Kyoto Univ.* **41** 233–250.

HU, Y. and ØKSENDAL, B. (2003). Fractional white noise calculus and applications to finance. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **6** 1–32.

LIN, S. J. (1995). Stochastic analysis of fractional Brownian motions. *Stochastics Stochastics Rep.* **55** 121–140.

MÉMIN, J., MISHURA, Y. and VALKEILA, E. (2001). Inequalities for the moments of Wiener integrals with respect to fractional Brownian motions. *Statist. Probab. Lett.* **51** 197–206.

NUALART, D. (1995). *The Malliavin Calculus and Related Topics*. Springer, Berlin.

PIPIRAS, V. and TAQQU, M. S. (2000). Integration questions related to fractional Brownian motion. *Probab. Theory Related Fields* **118** 251–291.

ROSEN, J. (1987). The intersection local time of fractional Brownian motion in the plane. *J. Multivariate Anal.* **23** 37–46.

RUSSO, F. and VALLOIS, P. (1993). Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields* **97** 403–421.

YOUNG, L. C. (1936). An inequality of the Hölder type connected with Stieltjes integration. *Acta Math.* **67** 251–282.

D. NUALART
C. ROVIRA
FACULTAT DE MATEMÀTIQUES
UNIVERSITAT DE BARCELONA
GRAN VIA 585
08007 BARCELONA
SPAIN
E-MAIL: nualart@mat.ub.es
rovira@mat.ub.es

S. TINDEL
DÉPARTEMENT DE MATHÉMATIQUES
INSTITUT GALILÉE—UNIVERSITÉ PARIS 13
99 AVENUE J. B. CLÉMENT
93430 VILLETANEUSE
FRANCE
E-MAIL: tindel@math.univ-paris13.fr