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BROWNIAN MOTION AND DIRICHLET PROBLEMS AT INFINITY[1]

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We discuss angular convergence of Riemannian Brownian motion on a Cartan–Hadamard manifold and show that the Dirichlet problem at infinity for such a manifold is uniquely solvable under the curvature conditions $-Ce^{(2-\eta)ar(x)} \le K_M(x) \le -a^2$ $(\eta > 0)$ and $-Cr(x)^{2\beta} \le K_M(x) \le -\alpha(\alpha-1)/r(x)^2$ $(\alpha > \beta + 2 > 2)$, respectively.

1. Introduction. A Cartan–Hadamard manifold is a complete, simply connected Riemannian manifold with nonpositive sectional curvature. We fix a reference point $o \in M$ once and for all. It is well known that the exponential map $\exp : T_oM \to M$ from the tangent space T_oM based at o is a diffeomorphism. This defines a polar coordinate system (r, θ) on M. Two geodesic rays γ_1 and γ_2 on M are called equivalent if there is a constant C such that $d(\gamma_1(t), \gamma_2(t)) \le C$ for all $t \ge 0$. It can be shown that this is an equivalence relation on the set of geodesic rays. The set of equivalence classes is the sphere at infinity $S_\infty(M)$. A basic fact of Cartan–Hadamard manifolds is that $\widehat{M} = M \cup S_\infty(M)$ with a properly defined topology (called the cone topology) is a compactification of M. For each $o \in M$, the sphere at infinity $S_\infty(M)$ can be identified homeomorphically with the unit sphere in the tangent space T_oM. If (r, θ) are the polar coordinates based at o, then a sequence of points $z_n \in M$ converges to a boundary point $\theta_0 \in S_\infty(M)$ if and only if $r(z_n) \to \infty$ and $\theta(z_n) \to \theta_0$ (see [5]).

Given a continuous function f on $S_\infty(M)$, the Dirichlet problem at infinity is to find a function $u_f \in C^\infty(M) \cap C(\widehat{M})$ that is harmonic on M and equal to f on $S_\infty(M)$. We say that the Dirichlet problem at infinity is solvable for M if for every $f \in C(S_\infty(M))$ there is a unique solution u_f. This property of a Cartan–Hadamard manifold can be obtained under certain conditions on the curvature of M and can be approached analytically or probabilistically. For analytic methods, see [1, 3, 4, 6, 7]; for probabilistic methods, see [8–10, 14–16, 18]. The more difficult problem of identifying the Martin boundary with the boundary at infinity was discussed in [4] and [13]. We are mainly concerned with a probabilistic approach to the problem, which involves basically proving the angular convergence of transient Brownian motion.

In this paper, we will combine an improved version of the method used in [9] and an idea from [14] to prove the solvability of the Dirichlet problem at infinity

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under certain curvature growth conditions more generous than previously known. We consider two typical situations. In the first case, the sectional curvature is assumed to be bounded by a negative constant: $\mathrm{Sect}_x \le -a^2$. In the second case, we assume that $\mathrm{Sect}_x \le -c/r^2$ $[r = r(x) = d(x, o)]$. This second case is significant because it vanishes as $r \to \infty$. Let us now state our main theorems.

THEOREM 1.1. *Let M be a Cartan–Hadamard manifold. Suppose that there exist a positive constant a and a positive and nonincreasing function h with $\int_0^\infty rh(r)\,dr < \infty$ such that*

$$-h(r)^2 e^{2ar} \le \mathrm{Ric}_x \quad \textit{and} \quad \mathrm{Sect}_x \le -a^2.$$

Then the Dirichlet problem at infinity for M is solvable.

Early lower bounds of the form $Ce^{\lambda ar}$ were obtained in [6] with $\lambda < 1/3$ and in [14] with $\lambda < 1/2$. Our result represents a significant improvement in this respect.

THEOREM 1.2. *Let M be a Cartan–Hadamard manifold. Suppose that there exist positive constants r_0, $\alpha > 2$ and $\beta < \alpha - 2$ such that*

$$-r^{2\beta} \le \mathrm{Ric}_x \quad \textit{and} \quad \mathrm{Sect}_x \le -\frac{\alpha(\alpha-1)}{r^2}$$

for all $r = r(x) \ge r_0$. Then the Dirichlet problem at infinity for M is solvable.

Hsu and March [9] proved a lower bound of the form $-r^{2\beta}$ with $\beta < 1 - 2/\alpha < 1$. Our new result opens the possibility of $\beta \ge 1$.

The rest of this paper has three sections. In Section 2, we state some preliminary results needed for the proof of our main theorems. In Sections 3 and 4, we deal with the constant upper bound case and the vanishing upper bound case, respectively.

2. Preliminary results. Let M be a Riemannian manifold and $\widetilde{M} = M \cup \{\Delta\}$ its one-point compactification. The path space $W(M)$ based on M is the space of continuous maps $X \in C([0, \infty); \widetilde{M})$ with the following property: if $X_t = \Delta$ for some t, then $X_s = \Delta$ for all $s \ge t$. The lifetime $e(X)$ is defined by $e(X) = \inf\{t : X_t = \Delta\}$. The path space $W(M)$ is equipped with the standard filtration $\mathscr{B}_* = \{\mathscr{B}_t\}$ and the lifetime $e : W(M) \to \mathbb{R}_+$ is a $\mathscr{B}_*$-stopping time. We use $\mathbb{P}_x$ to denote the law of Brownian motion on M starting from x. It is a probability measure on $W(M)$.

Now let M be a Cartan–Hadamard manifold and $\widehat{M} = M \cup S_\infty(M)$ its compactification by the sphere at infinity. A Brownian motion X can be decomposed into the radial process $r_t = r(X_t)$ and the angular process $\theta_t = \theta(X_t)$. The probabilistic approach to the Dirichlet problem is based on the following well-known fact.

THEOREM 2.1. *Let M be a Cartan–Hadamard manifold. Suppose that, for any $x \in M$,*

$$\mathbb{P}_x\left\{X_e = \lim_{t\uparrow e} X_t \text{ exists}\right\} = 1$$

(in the cone topology of $\widehat{M}$) and, for any $\theta_0 \in S_\infty(M)$ and any neighborhood U of θ_0 in $S_\infty(M)$,

$$\lim_{x\to\theta_0} \mathbb{P}_x\{X_e \in U\} = 1.$$

Then the Dirichlet problem at infinity for M is solvable. For any $f \in C(S_\infty(M))$, the function $u_f(x) = \mathbb{E}_x f(X_e)$ is the unique solution of the Dirichlet problem with boundary function f.

PROOF. Since $u_f(x) = \mathbb{E}_x u_f(X_{\tau_D})$ for any relatively compact open set D containing x, where τ_D is the first exit time of D, we see that u is harmonic on M. For any $\varepsilon > 0$ and $\theta_0 \in S_\infty(M)$, choose a neighborhood U of θ_0 such that $|f(\theta) - f(\theta_0)| \le \varepsilon$ for $\theta \in U$. Then

$$\begin{aligned}|u_f(x) - f(\theta_0)| &\le \mathbb{E}_x |f(X_e) - f(\theta_0)| \\ &\le \varepsilon \mathbb{P}_x\{X_e \in U\} + 2\|f\|_\infty \mathbb{P}_x\{X_e \notin U\}.\end{aligned}$$

Letting $x \to \theta_0$, we have $\limsup_{x\to\theta_0} |u_f(x) - f(\theta_0)| \le \varepsilon$. This shows that $\lim_{x\to\theta_0} u_f(x) = f(\theta_0)$, as desired.

To prove the uniqueness, let $\{D_n\}$ be an exhaustion of M and u a solution of the Dirichlet problem at infinity with boundary function f. Then $\{u_f(X_{t\wedge\tau_{D_n}}), t \ge 0\}$ is a uniformly bounded martingale under $\mathbb{P}_x$; hence, $u(x) = \mathbb{E}_x u(X_{t\wedge\tau_{D_n}})$. Letting $t \uparrow \infty$ and then $n \uparrow \infty$, we have

$$u(x) = \mathbb{E}_x u(X_e) = \mathbb{E}_x f(X_e) = u_f(x). \qquad \square$$

REMARK 2.2. Ancona [2] constructed a Cartan–Hadamard manifold such that Brownian motion converges to a single point on the boundary at infinity. For such manifolds, the Dirichlet problem at infinity is clearly not solvable.

We end this section with a description of the general method for proving angular convergence of Brownian motion. Define a sequence of stopping times $\{\tau_n\}$ by $\tau_0 = 0$ and

$$\tau_n = \inf\{t \ge \tau_{n-1} : d(X_t, X_{\tau_{n-1}}) = 1\}.$$

Let $\Delta\tau_n = \tau_n - \tau_{n-1}$ be the amount of time for the nth step. The angular oscillation during the time interval $[\tau_{n-1}, \tau_n]$ is

$$\Delta\theta_n = \max_{\tau_{n-1}\le t\le\tau_n} \angle\big(\theta(X_{\tau_{n-1}}), \theta(X_t)\big).$$

PROPOSITION 2.3. *Let M be a Cartan–Hadamard manifold on which Brownian motion is transient, that is,*

$$\mathbb{P}_x\{r_t \to \infty \text{ as } t \uparrow e\} = 1.$$

The Dirichlet problem at infinity is solvable if, for any positive ε and δ, there is an R such that, for all $z \in M$ with $r(z) \geq R$,

$$\mathbb{P}_z\left\{\sum_{n=1}^{\infty} \Delta\theta_n \leq \delta\right\} \geq 1 - \varepsilon. \tag{2.1}$$

PROOF. First, we note that $\sum_{n=1}^{\infty} \Delta\theta_n < \infty$ implies that $\lim_{t\uparrow e} X_t = X_e$ exists. Let $x \in M$ and $\varepsilon > 0$. Choose $R \geq r(x)$ such that (2.1) holds (for $\delta = 1$, say). Let $\tau_R = \inf\{t : r_t = R\}$. Then

$$\begin{aligned}
\mathbb{P}_x\left\{X_e = \lim_{t\uparrow e} X_t \text{ exists}\right\} &\geq \mathbb{P}_x\left\{\sum_{n=1}^{\infty} \Delta\theta_n < \infty\right\} \\
&= \mathbb{E}_x \mathbb{P}_{X_{\tau_R}}\left\{\sum_{n=1}^{\infty} \Delta\theta_n < \infty\right\} \\
&\geq 1 - \varepsilon.
\end{aligned}$$

Since ε is arbitrary, this shows that $\mathbb{P}_x\{X_e = \lim_{t\uparrow e} X_t \text{ exists}\} = 1$.

Let $\theta_0 \in S_\infty(M)$ and U a neighborhood of θ_0 on $S_\infty(M)$ containing θ_0. There is a $\delta > 0$ such that

$$\{\theta \in S_\infty(M) : \angle(\theta, \theta_0) \leq 2\delta\} \subset U.$$

We have

$$\angle\bigl(\theta_0, \theta(X_e)\bigr) \leq \angle\bigl(\theta_0, \theta(X_0)\bigr) + \sum_{n=0}^{\infty} \Delta\theta_n.$$

For any $\varepsilon > 0$, choose $R > 0$ such that (2.1) holds. Then, for all $x \in M$ such that $r(x) \geq R$ and $\angle(\theta(x), \theta_0) \leq \delta$, we have

$$\mathbb{P}_x\{X_e \in U\} \geq \mathbb{P}_x\bigl\{\angle\bigl(\theta_0, \theta(X_e)\bigr) \leq 2\delta\bigr\} \geq \mathbb{P}_x\left\{\sum_{n=0}^{\infty} \Delta\theta_n \leq \delta\right\} \geq 1 - \varepsilon.$$

This shows that

$$\lim_{x\to\theta_0} \mathbb{P}_x\{X_e \in U\} = 1.$$

By Theorem 2.1, the Dirichlet problem at infinity for M is solvable. $\square$

We use the following result to estimate the amount of time the Brownian motion spends for each step. Let

$$\tau_1 = \inf\{t > 0 : d(X_t, X_0) = 1\}.$$

PROPOSITION 2.4. *There are positive constants C_1, C_2 such that if the Ricci curvature on the geodesic ball $B(x;1)$ of radius 1 centered at x is bounded from below by a negative constant $-L^2 \le -1$, then*

$$\mathbb{P}_x\left\{\tau_1 \le \frac{C_1}{L}\right\} \le e^{-C_2 L}.$$

In fact, we can take $C_1 = 1/8d$ and $C_2 = 1/2$.

PROOF. This is Lemma 4 of [9]. We give a simpler proof here. Let $r_t = d(X_t, x)$ be the radial process. According to [11], there is a Brownian motion β such that

$$r_t = \beta_t + \frac{1}{2}\int_0^t \Delta r(X_s)\,ds - L_t,$$

where L is nondecreasing and increases only when X_t is on the cut locus of o. By Itô's formula, we have

$$r_t^2 = 2\int_0^t r_s\,dr_s + \langle r\rangle_t.$$

Hence,

$$r_t^2 \le 2\int_0^t r_s\,d\beta_s + \int_0^t r_s\,\Delta r(X_s)\,ds + t. \tag{2.2}$$

By the Laplacian comparison theorem, we have, for all $z \in B(x;1)$,

$$\Delta r(z) \le (d-1)L\coth Lr(z).$$

On the other hand, $l\coth l \le 1 + l$ for all $l \ge 0$. Hence, if $s \le \tau_1$, we have

$$r_s\,\Delta r(X_s) \le (d-1)Lr_s\coth Lr_s \le (d-1)(1+L).$$

We now let $t = \tau_1$ in (2.2) and obtain

$$1 \le 2\int_0^{\tau_1} r_s\,d\beta_s + 2\,dL\tau_1.$$

From the above inequality, we see that the event $\tau_1 \le 1/8dL$ implies

$$\int_0^{\tau_1} r_s\,d\beta_s \ge \frac{3}{8}.$$

By Lévy's criterion, there is a Brownian motion W such that

$$\int_0^{\tau_1} r_s\,d\beta_s = W_\eta, \qquad \eta = \int_0^{\tau_1} r_s^2\,ds \le \frac{1}{8dL}.$$

Hence, $\tau_1 \le 1/8dL$ implies

$$\max_{0\le s\le 1/8dL} W_s \ge W_\eta \ge \frac{3}{8}.$$

The random variable on the left-hand side is distributed as $\sqrt{1/8dL}|W_1|$. It follows that

$$\mathbb{P}_x\left[\tau_1 \le \frac{1}{8dL}\right] \le \mathbb{P}_x\left[|W_1| \ge \sqrt{\frac{9L}{8}}\right] \le e^{-L/2}. \qquad \square$$

We will use the following geometric result to estimate the angle in a Cartan–Hadamard manifold. It is essentially Lemma 2 of [9], but we include a complete proof to clarify a few points.

LEMMA 2.5. *Let M be a Cartan–Hadamard manifold. Suppose that there are positive constants $\alpha \ge 1$ and $r_0 \ge 1$ such that*

$$\mathrm{Sect}_x \le -\frac{\alpha(\alpha-1)}{r(x)^2}, \qquad r(x) \ge r_0.$$

Let $x, y \in M$ be such that

$$r(x) \ge 2r_0, \qquad r(y) \ge 2r_0, \qquad d(x, y) \le 1.$$

Then there is a constant C independent of x and y such that the angle between the geodesic rays to x and y satisfies

$$\angle\big(\theta(x), \theta(y)\big) \le \frac{C}{r(x)^\alpha}.$$

PROOF. Without loss of generality, we assume $r(x) \le r(y)$. Let

$$K(r) = \min\left\{-\sup_{r(x)\le r} \mathrm{Sect}_x, \frac{\alpha(\alpha-1)}{r^2}\right\}.$$

Let G be the unique solution of the Jacobi equation

$$G''(r) - K(r)G(r) = 0, \qquad G(0) = 0, \qquad G'(0) = 1.$$

Since $K(r) = \alpha(\alpha-1)/r^2$ for $r \ge r_0$, we have $G(r) = c_1 r^\alpha + c_2 r^{1-\alpha}$. Hence,

$$G(r) \sim c_1 r^\alpha, \qquad \frac{G'(r)}{G(r)} \sim \frac{\alpha}{r} \qquad \text{as } r \uparrow \infty. \tag{2.3}$$

In particular, $G(r) \ge C^{-1} r^\alpha$ for some C and all $r \ge r_0$. Now let N be the rotationally symmetric manifold with the metric $ds_N^2 = dr^2 + G(r)^2\,d\theta^2$. In N, consider the geodesic triangle AOB such that

$$d(O, A) = r(x), \qquad d(O, B) = r(y), \qquad \angle\big(\theta(A), \theta(B)\big) = \angle\big(\theta(x), \theta(y)\big).$$

By the Rauch comparison theorem, we have $d_N(A, B) \le d(x, y)$. Hence,

$$1 \ge d_N(A, B) \ge G(r(x))\angle\big(\theta(A), \theta(B)\big) = G(r(x))\angle\big(\theta(x), \theta(y)\big).$$

This implies that $\angle(\theta(x), \theta(y)) \le C/r(x)^\alpha$. $\square$

When the sectional curvature is bounded from above by a negative constant, we have the following analogue of the above lemma.

LEMMA 2.6. *Let M be a Cartan–Hadamard manifold. Suppose that there is a positive constant a such that* $\mathrm{Sect}_x \le -a^2$. *Let $x, y \in M$ be such that $r(x) \le r(y)$ and $d(x, y) \le 1$. Then*

$$\angle\big(\theta(x), \theta(y)\big) \le \frac{a}{\sinh ar(x)} \le \left[\frac{1}{r(x)} + 2a\right] e^{-ar(x)}.$$

PROOF. Let $G(r) = \sinh ar/a$ and follow the proof of the preceding lemma. □

3. Constant upper bound. In this section, we consider the case of a constant upper bound on the sectional curvature of M. We first give an estimate on the probability that Brownian motion starting at $r(x) = R$ will ever return to $r = R \le r(x)$.

LEMMA 3.1. *Suppose that* $\mathrm{Sect}_x \le -a^2$. *For any $R \ge 0$, we have, for $r(x) \ge R$,*

$$\mathbb{P}_x\{r_t \le R \textit{ for some } t \ge 0\} \le \cosh^{1-d} a(r - R). \tag{3.1}$$

PROOF. There is a Brownian motion β such that

$$r_t = r_0 + \beta_t + \tfrac{1}{2} \int_0^t \Delta r(X_t)\, dt.$$

By the Laplacian comparison theorem, we have $\Delta r \ge (d - 1)a \coth ar$. If we define r^* by

$$r_t^* = r_0 + \beta_t + \frac{d-1}{2} \int_0^t a \coth ar_s^*\, ds,$$

then a comparison theorem for stochastic differential equations shows that $r_t \ge r_t^*$. Thus, it is enough to prove the estimate for r^*.

The following argument is well known. Let

$$l(r) = \int_r^\infty (\sinh au)^{1-d}\, du$$

and $\sigma_R = \inf\{t : r_t^* = R\}$. If $r(x) \ge R$, then $\{l(r_{t\wedge\sigma_R}^*)\}$ is a uniformly bounded martingale. Letting $t \uparrow \infty$, we have

$$l(r) = \mathbb{E}_x l\big(r_{t\wedge\sigma_R}^*\big) = l(R)\mathbb{P}_x\{\sigma_R < \infty\}.$$

Hence,

$$\mathbb{P}_x\{r_t^* \le R \text{ for some } t \ge 0\} = \mathbb{P}_x\{\sigma_R < \infty\} = \frac{l(r)}{l(R)}.$$

On the other hand,

$$\frac{l(r(x))}{l(R)} = \frac{\int_r^\infty (\sinh au)^{1-d}\,du}{\int_R^\infty (\sinh au)^{d-1}\,du}$$
$$\leq \sup_{u\geq R}\left[\frac{\sinh a(u+r-R)}{\sinh au}\right]^{1-d}$$
$$\leq \cosh^{1-d} a(r-R).$$

In the last step, we have used

$$\frac{\sinh(x+y)}{\sinh x} = \frac{\sinh x\cosh y + \cosh x\sinh y}{\sinh x} \geq \cosh y.$$

The result follows. $\square$

Next, we consider the rate of escape for Brownian motion.

LEMMA 3.2. *Suppose that* $\mathrm{Sect}_x \leq -a^2$. *For any* $\lambda < (d-1)a/2$, *we have*

$$\lim_{r(x)\to\infty} \mathbb{P}_x\bigl\{r_t \geq \max\{\lambda t, r(x)/2\},\ \forall t\geq 0\bigr\} = 1.$$

PROOF. Again, it is enough to show the result for the r_t^* in the proof of the preceding lemma. Fix a $\lambda_1 \in (\lambda, (d-1)a/2)$ and take R such that

$$[(d-1)a/2]\coth ar \geq \lambda_1, \qquad r \geq R/2.$$

Suppose that $\varepsilon > 0$. By Lemma 3.1, we can take R even larger such that, for all $x \in M$ with $r(x) \geq R$,

$$\mathbb{P}_x\{r_t^* \geq r(x)/2,\ \forall t \geq 0\} \geq 1-\varepsilon. \tag{3.2}$$

By the law of iterated logarithm,

$$\liminf_{t\uparrow\infty} \frac{\beta_t}{\sqrt{2t\log\log t}} = -1.$$

Hence, there is an even larger R (independent of x) such that

$$\mathbb{P}_x\{\beta_t \geq -(\lambda-\lambda_1)t - R,\ \forall t\geq 0\} \geq 1-\varepsilon. \tag{3.3}$$

If the events in (3.2) and (3.3) happen simultaneously, then

$$r_t^* = r_0^* + \beta_t + \frac{d-1}{2}\int_0^t a\coth ar_s^*\,ds$$
$$\geq R - (\lambda_1-\lambda)t - R + \lambda_1 t$$
$$= \lambda t.$$

It follows that for all $x \in M$ with $r(x) \geq R$ we have

$$\mathbb{P}_x\{r_t^* \geq \max\{\lambda_1 t, r(x)/2\},\ \forall t \geq 0\} \geq 1 - 2\varepsilon.$$

This proves the lemma. $\square$

We now estimate the total angular variation. Suppose that $r_t \geq r(x)/2$ for all $t \geq 0$ with large $r(x)$. Recall that in Section 2 we have defined

$$\begin{aligned}
\tau_n &= \inf\{t \geq \tau_{n-1} : d(X_t, X_{\tau_{n-1}}) = 1\}, \qquad \tau_0 = 0,\\
\Delta\tau_n &= \tau_n - \tau_{n-1},\\
\Delta\theta_n &= \max_{\tau_{n-1} \leq t \leq \tau_n} \angle\big(\theta(X_{\tau_{n-1}}), \theta(X_t)\big).
\end{aligned}$$

From Lemma 2.6, we have $\Delta\theta_n \leq Ce^{-ar_{\tau_n}}$. Hence,

$$\sum_{n=1}^{\infty} \Delta\theta_n \leq C \sum_{n=1}^{\infty} e^{-ar_{\tau_n}}.$$

Next, let J_k be the total number of steps in the geodesic ball of radius k, that is,

$$J_k = \#\{n : r_{\tau_n} \leq k\}.$$

We have

$$\sum_{n=1}^{\infty} \Delta\theta_n \leq C \sum_{k=1}^{\infty} (J_k - J_{k-1}) e^{-a(k-1)} \leq C_0 \sum_{k=1}^{\infty} J_k e^{-ak}. \tag{3.4}$$

Thus, the problem is reduced to finding a good estimate for J_k.

REMARK 3.3. The idea of studying J_k is due to Leclercq [14].

THEOREM 3.4. *Let M be a Cartan–Hadamard manifold whose sectional curvature is bounded from above by $-a^2$. Suppose that the Ricci curvature satisfies the lower bound*

$$\mathrm{Ric}_x \geq -h(r)^2 e^{2ar},$$

where h is a positive and nonincreasing function such that $\int_0^\infty r h(r)\, dr < \infty$. Then the Dirichlet problem at infinity for M is solvable.

PROOF. Fix a constant $\lambda < (d-1)a/2$ and let

$$A = \{r_t \geq \max\{\lambda t, r(x)/2\},\ \forall t \geq 0\}.$$

By Lemma 3.2, there is an R such that, for $r(x) \geq R$,

$$\mathbb{P}_x\{A\} \geq 1 - \frac{\varepsilon}{2}.$$

Let τ_{n_l} be the lth time such that $r_{\tau_{n_l}} \le k-1$. Then

$$\{\tau_{n_l} \le t\} = \left\{ \sum_{n=1}^{\infty} I_{\{r_{\tau_n} \le k-1, \tau_n \le t\}} \ge l \right\},$$

from which it is clear that τ_{n_l} is a stopping time.

For a fixed k, denote for the time being

$$L_k = C_1 h(k) e^{ak}, \qquad N_k = \frac{(k+1)L_k}{\lambda C_1}.$$

Without loss of generality, we may assume that $h(k) \ge e^{-ak/2}$ [otherwise, just add $e^{-ar/2}$ to $h(r)$] and $L_k \ge 1$. Consider the length of time $\Delta\tau_{n_l}$ for the next step. Let

$$B_l = \left\{ \Delta\tau_{n_l} \le \frac{C_1}{L_k}, \tau_{n_l} < \infty \right\}, \qquad C_{N_k} = B_1 \cup B_2 \cup \cdots \cup B_{N_k}.$$

By Proposition 2.4 and the fact that τ_{n_l} is a stopping time,

$$\mathbb{P}_x B_l = \mathbb{E}_x \left\{ \mathbb{P}_{X_{\tau_{n_l}}} \left[\tau_1 \le \frac{C_1}{L_k} \right], \tau_{n_l} < \infty \right\} \le e^{-C_2 L_k}. \tag{3.5}$$

Recall that J_{k-1} is the total number of steps such that $r_{\tau_n} \le k-1$. We have $\{J_{k-1} \ge N_k\} = \{\tau_{n_{N_k}} < \infty\}$. Now

$$\{J_{k-1} \ge N_k\} \cap A = \left\{\tau_{n_{N_k}} < \infty\right\} \cap A \cap C_{N_k} + \left\{\tau_{n_{N_k}} < \infty\right\} \cap A \cap C_{N_k}^c. \tag{3.6}$$

On A, we have $r_t \ge \lambda t$ for all $t \ge 0$. This means that

$$|\{t : r_t \le k\}| \le \frac{k}{\lambda}.$$

But on $\{\tau_{n_{N_k}} < \infty\} \cap C_{N_k}^c$,

$$|\{t : r_t \le k\}| \ge \sum_{l=1}^{N_k} \Delta\tau_{n_l} \ge N_k \frac{C_1}{L_k} = \frac{k+1}{\lambda}.$$

This shows that $\{\tau_{n_{N_k}} < \infty\} \cap A \cap C_{N_k}^c = \varnothing$ and we have, from (3.6),

$$\{J_{k-1} \ge N_k\} \cap A \subseteq C_{N_k} = B_1 \cup B_2 \cup \cdots \cup B_{N_k}.$$

By (3.5),

$$\mathbb{P}_x\{J_{k-1} \ge N_k, A\} \le N_k e^{-C_2 L_k} \le C_3 k e^{ak - C_2 e^{ak/2}}.$$

Using the definition of L_k, we see from the above inequality that, for any $\varepsilon > 0$, there is a sufficiently large R such that, for $r(x) \ge R$,

$$\sum_{k \ge r(x)/2}^{\infty} \mathbb{P}_x\left\{J_k \ge C_4 k h(k) e^{ak}, A\right\} \le \frac{\varepsilon}{2}.$$

On A, we have $r_t \ge r(x)/2$ for all t. This means that $J_k = 0$ for $k \le r(x)/2$. It follows that, for $r(x) \ge R$,

$$\mathbb{P}_x\left\{J_k = 0, k \le \frac{r(x)}{2}; J_k \le C_4 k h(k) e^{ak}, k \ge \frac{r(x)}{2}\right\} \ge \mathbb{P}_x A - \frac{\varepsilon}{2} \ge 1 - \varepsilon.$$

If the event in the above inequality holds, then, by (3.4),

$$\sum_{n=1}^{\infty} \Delta\theta_n \le C_4 \sum_{k \ge r(x)/2} k h(k).$$

This can be made arbitrarily small because the $\sum_{k=1}^{\infty} k h(k)$ converges by hypothesis. Therefore, we have shown that for any positive ε and δ, there is an R such that, for all $x \in M$ with $r(x) \ge R$,

$$\mathbb{P}_x\left\{\sum_{n=1}^{\infty} \Delta\theta_n \le \delta\right\} \ge 1 - \varepsilon.$$

By Proposition 2.3, this implies the solvability of the Dirichlet problem at infinity for M. $\square$

4. Vanishing upper bound. In this section, we assume that M is a Cartan–Hadamard manifold whose curvature satisfies the following condition: there are positive constant r_0, $\alpha > 2$ and $\beta < \alpha - 2$ such that, for all $r(x) \ge r_0$,

$$-r(x)^{2\beta} \le \mathrm{Ric}_x \quad \text{and} \quad \mathrm{Sect}_x \le -\frac{\alpha(\alpha-1)}{r(x)^2}.$$

The proof for this case is completely parallel to that in the previous section, so we will be brief.

LEMMA 4.1. *There is a constant C such that, for all $R \ge 1$ and $x \in M$ with $r(x) \ge R$,*

$$\mathbb{P}_x\{r_t \le R \textit{ for some } t \ge 0\} \le C\left[\frac{R}{r(x)}\right]^{(d-1)\alpha-1}.$$

PROOF. Define the function G as in the proof of Lemma 2.5. As before, we may assume that M is rotationally symmetric with metric $ds^2 = dr^2 + G(r)^2\, d\theta^2$. In this case, by the same argument as in Lemma 3.1, we have

$$\mathbb{P}_x\{r_t \le R \text{ for some } t \ge 0\} = \frac{\int_{r(x)}^{\infty} G(s)^{1-d}\, ds}{\int_R^{\infty} G(s)^{1-d}\, ds}.$$

The result follows immediately from the fact that $G(r) \sim c_1 r^{\alpha}$ as $r \uparrow \infty$. $\square$

In the proof of the next lemma, we need the following fact (see [17]): let Y^a be the Bessel process of index $q > 1$ from $a \geq 0$:

$$Y_t^a = a + \beta_t + \frac{q}{2}\int_0^t \frac{ds}{Y_s^a}, \tag{4.1}$$

where β is a one-dimensional Brownian motion. Then for any $\lambda > 0$ we have

$$\mathbb{P}\left\{\lim_{t\uparrow\infty} \frac{Y_t^a}{t^{1/2-\lambda}} = \infty\right\} = 1. \tag{4.2}$$

Note that $Y_t^a \leq Y_t^b$ if $a \leq b$.

LEMMA 4.2. *For any $\lambda > 0$, we have*

$$\lim_{r(x)\to\infty} \mathbb{P}_x\{r_t \geq \max\{t, r(x)\}^{1/2-\lambda},\ \forall t \geq 0\} = 1.$$

PROOF. Again, it is enough to assume that M is rotationally symmetric, as in Lemma 4.1. The radial process is given by

$$r_t = r_0 + \beta_t + \frac{d-1}{2}\int_0^t \frac{G'(r_s)}{G(r_s)}\,ds.$$

Now take a $q \in (1, (d-1)\alpha)$. By (2.3), there is an $r_1 \geq 1$ such that

$$(d-1)\frac{G'(r)}{G(r)} \geq \frac{q}{r}, \qquad r \geq r_1.$$

Let Y^a be the Bessel process of index q defined by (4.1). If $r(x) \geq r_1$, then we have

$$r_t \geq Y_t^{r(x)} \geq Y_t^{r_1} \geq Y_t^1, \qquad t \leq \sigma_{r_1},$$

where σ_{r_1} is the first time r_t reaches r_1. For any $\varepsilon > 0$, there is an $R \geq r_1$ (independent of x) such that

$$\mathbb{P}_x\{Y_t^1 \geq t^{1/2-\lambda},\ \forall t \geq R\} \geq 1 - \varepsilon.$$

Hence, using Lemma 4.1, we have, for $r(x) \geq R \geq 1$,

$$\begin{aligned}
&\mathbb{P}_x\{r_t \geq \max\{t, r(x)\}^{1/2-\lambda},\ \forall t \geq 0\}\\
&\quad \geq \mathbb{P}_x\{r_t \geq t^{1/2-\lambda},\ \forall t \geq r(x)\} - \mathbb{P}_x\{r_t \leq r(x)^{1/2-\lambda} \text{ for some } t \geq 0\}\\
&\quad \geq \mathbb{P}_x\{Y_t^1 \geq t^{1/2-\lambda},\ \forall t \geq R\} - Cr(x)^{-(\lambda+1/2)[(d-1)\alpha-1]}\\
&\quad \geq 1 - \varepsilon - Cr(x)^{-(\lambda+1/2)[(d-1)\alpha-1]}.
\end{aligned}$$

It follows that for all sufficiently large $r(x)$ we have

$$\mathbb{P}_x\{r_t \geq \max\{t, r(x)\}^{1/2-\lambda},\ \forall t \geq 0\} \geq 1 - 2\varepsilon. \qquad \square$$

THEOREM 4.3. *Suppose that M is a Cartan–Hadamard manifold. Suppose that there exist positive constants r_0, $\alpha > 2$ and $\beta < \alpha - 2$ such that*

$$-r(x)^{2\beta} \le \mathrm{Ric}_x \quad \textit{and} \quad \mathrm{Sect}_x \le -\frac{\alpha(\alpha-1)}{r(x)^2} \qquad \textit{for } r \ge r_0.$$

Then the Dirichlet problem at infinity is solvable for M.

PROOF. We define τ_n, $\Delta\tau_n$, $\Delta\theta_n$, τ_{n_l} and J_k as in the previous section. Under the current upper bound of the sectional curvature, we have $\Delta\theta_n \le C/r_{\tau_n}^\alpha$ by Lemma 2.5. Hence,

$$\begin{aligned}\sum_{n=1}^\infty \Delta\theta_n &\le C_0 J_1 + C_0 \sum_{k=1}^\infty \frac{J_{k+1}-J_k}{k^\alpha} \\ &\le C_0 J_1 + C_1 \sum_{k=1}^\infty \frac{J_k}{k^{\alpha+1}} + C_0 \liminf_{k\uparrow\infty} \frac{J_k}{k^\alpha}.\end{aligned} \tag{4.3}$$

We will now estimate the size of J_k. By Proposition 2.4, we have

$$\mathbb{P}_x\{\Delta\tau_{n_l} \le C_1 k^{-\beta}, \tau_{n_l} < \infty\} \le e^{-C_1 k^\beta}.$$

Choose a positive λ such that $\beta + 2/(1-2\lambda) < \alpha$. Let

$$A = \{r_t \ge \max\{t, r(x)\}^{1/2-\lambda}, \ \forall t \ge 0\}.$$

Fix an arbitrary $\varepsilon > 0$. By Lemma 4.2, $\mathbb{P}_x A \ge 1 - \varepsilon/2$ for sufficiently large $r(x)$. By the same argument as in Theorem 3.4, we have

$$\mathbb{P}_x\{J_k \ge (C_1+1)k^{\beta+2/(1-2\lambda)}, A\} \le C_3 k^{\beta+2/(1-2\lambda)} e^{-C_2 k^\beta}.$$

On A, we have $|\{t : r_t \le k\}| \le k^{2/(1-2\lambda)}$ and $J_k = 0$ for $k \le r(x)^{1/2-\lambda}$. Hence, as in the proof of Theorem 3.4, we have, for sufficiently large $r(x)$,

$$\begin{aligned}&\mathbb{P}_x\{J_k = 0, k \le r(x)^{1/2-\lambda}; J_k \le C_4 k^{\beta+2/(1-2\lambda)}, k \ge r(x)^{1/2-\lambda}\} \\ &\quad \ge \mathbb{P}_x A - C_3 \sum_{k \ge r(x)^{1/2-\lambda}} k^{\beta+2/(1-2\lambda)} e^{-C_2 k^\beta} \\ &\quad \ge 1 - \varepsilon.\end{aligned}$$

If the event in the above inequality is true, then $J_k/k^\alpha \to 0$ as $k \uparrow \infty$ and, by (4.3),

$$\begin{aligned}\sum_{n=1}^\infty \Delta\theta_n &\le C_4 \sum_{k \ge r(x)^{1/2-\lambda}} k^{-(\alpha+1)+\beta+2/(1-2\lambda)} \\ &\le C_5 r(x)^{-(\alpha-\beta)(1-2\lambda)/2+1}.\end{aligned}$$

By our choice of λ, the exponent is negative. Hence, we have shown that for any positive ε and δ, there is an R such that, for $r(x) \geq R$,

$$\mathbb{P}_x\left\{\sum_{n=1}^{\infty} \Delta\theta_n \leq \delta\right\} \geq 1 - \varepsilon.$$

The theorem now follows from Proposition 2.3. □

REMARK 4.4. For the Bessel process Y^a in (4.1), we have

$$\mathbb{P}\left\{\liminf_{t\to\infty} \frac{Y_t^a}{\sqrt{t}\psi(t)} \geq 1\right\} = 1$$

if ψ is a positive nonincreasing function such that $\int_0^\infty \psi(t)^{q-1}\,dt < \infty$. Using this rate instead of $t^{1/2-\lambda}$ in (4.2), we can improve the lower bound in the above theorem. For example, it can be shown that the Dirichlet problem is solvable if the Ricci curvature is bounded from below by $-r^{2(\alpha-2)}/(\ln r)^{2l}$ for $l > (d\alpha - \alpha + 1)/(d\alpha - \alpha - 1)$.

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