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GEODESICS IN TWO-DIMENSIONAL FIRST-PASSAGE PERCOLATION

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We consider standard first-passage percolation on $\mathbb{Z}^2$. Geodesics are nearest-neighbor paths in $\mathbb{Z}^2$, each of whose segments is time-minimizing. We prove part of the conjecture that doubly infinite geodesics do not exist. Our main tool is a result of independent interest about the coalescing of semi-infinite geodesics.

1. Introduction. Standard first-passage percolation [3–5] may be regarded as a family of models, each of which yields a random metric $T(u,v)$ on the sites u,v of $\mathbb{Z}^d$. The metric, usually thought of as a passage time, is constructed out of i.i.d. nonnegative random variables $\tau(e)$, with common distribution μ, indexed by the nearest-neighbor edges e of $\mathbb{Z}^d$. The passage time $T(r)$ for a finite path r consisting of edges $e_1,\dots,e_n$ is simply $\sum_i \tau(e_i)$ and $T(u,v)$ is the infimum of $T(r)$ over all finite paths r between u and v. *We shall assume throughout that μ is continuous*, in which case there is (a.s.) a unique path $M(u,v)$ such that $T(u,v) = T(M(u,v))$.

We will call a finite or infinite path r a *geodesic* if, for every pair of sites u', v' touched by r, the segment $r(u',v')$ of r between u' and v' is exactly $M(u',v')$. Clearly, a finite r is a geodesic if and only if it is some $M(u,v)$. This paper is concerned with the nature and existence of infinite geodesics. There always exist semiinfinite geodesics (which we shall call *unigeodesics*). To see this, define $R(x)$ to be the union of $M(x,y)$ over all y's in $\mathbb{Z}^d$. For each x, $R(x)$ is easily shown to be a tree (i.e., it has no loops and is connected) spanning all of $\mathbb{Z}^d$ and hence there exists at least one unigeodesic starting from x.

The existence of doubly infinite geodesics (which we shall call *bigeodesics*) is another matter entirely. Indeed, originating from the physics literature on disordered Ising models (as we discuss below) is a conjecture that, at least for $d = 2$, bigeodesics should not exist. In this paper, we give a result in that direction by focusing on what we call $(\hat{x}, \hat{y})$-bigeodesics, whose two ends have the definite asymptotic directions $\hat{x}$ and $\hat{y}$ (unit vectors in $\mathbb{R}^d$). One motivation for this focus comes from [6]. Theorem 2.1 of that paper shows that under certain assumptions, there cannot exist any bigeodesics other than $(\hat{x}, \hat{y})$ ones, and further arguments, related to Proposition 3.2 there, can be used to

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rule out all cases except $\hat{y} = -\hat{x}$. In this paper (see Theorem 2 of Section 2), we partly rule out such geodesics for $d = 2$ by proving that, for Lebesgue almost every $\hat{x}$, there is a.s. no $(\hat{x}, -\hat{x})$-bigeodesic. Note, however, that, according to the full conjecture, such geodesics should be ruled out for every (not just almost every) $\hat{x}$ and furthermore should be ruled out not just for every deterministic $\hat{x}$ but also for $\hat{x}(\omega)$, depending on the ω in the underlying probability space of the $\tau(e)$'s. It should also be noted that one assumption of Theorem 2.1 and Proposition 3.2 of [6] concerns uniform curvature properties of the first-passage asymptotic shape and this assumption, although plausible, has not been verified for any distribution μ. The results of this paper thus leave much to be done for a verification of the full conjecture. We remark that a different step toward the conjecture was made by Wehr [8], who showed that the number of bigeodesics (of any type) can a.s. only be 0 or ∞.

We conclude this section with a brief discussion of some physics background. Let $\mathbb{Z}^{d*}$ denote the lattice $\mathbb{Z}^d + (1/2, 1/2, \dots, 1/2)$. It is dual to $\mathbb{Z}^d$ in the sense that each elementary plaquette (i.e., one of the codimension-1 faces of an elementary cube) of $\mathbb{Z}^d$ is the perpendicular bisector of a nearest-neighbor edge e^* of $\mathbb{Z}^{d*}$. In a nearest-neighbor Ising model on $\mathbb{Z}^{d*}$, there are real coupling constants J_{e^*} indexed by the nearest-neighbor edges $e^* = \langle x^*, y^* \rangle$ of $\mathbb{Z}^{d*}$ and a formal energy function on $\mathscr{S} = \{-1, +1\}^{\mathbb{Z}^{d*}}$:

$$H(S) = -\frac{1}{2} \sum_{\langle x^*, y^* \rangle} J_{\langle x^*, y^* \rangle} S_{x^*} S_{y^*}. \tag{1.1}$$

Although this is only a formal series, the *change* in H, $\Delta H_A(S)$, when $S \in \mathscr{S}$ is changed by "flipping" (i.e., multiplying by a minus sign) the S_{x^*}'s for x^* in the *finite* $A \subset \mathbb{Z}^{d*}$, is well defined:

$$\Delta H_A(S) = \sum_{\partial A} J_{\langle x^*, y^* \rangle} S_{x^*} S_{y^*}, \tag{1.2}$$

where ∂A denotes the set of nearest-neighbor edges of $\mathbb{Z}^{d^*}$ touching both A and its complement. Thus one defines an (infinite volume) *ground state* of H to be any $S \in \mathscr{S}$ such that (1.2) is nonnegative for every finite A.

Ground states always exist (by a compactness argument) and they always come in pairs (related by a global flip). Furthermore, in the ferromagnetic case where $J_{e^*} \geq 0$ for every e^*, the constant configurations, $S_{x^*} \equiv 1$ and $S_{x^*} \equiv -1$, are, of course, ground states. In the physically interesting case of *disordered* Ising models, where the J_{e^*}'s are (say) i.i.d. random variables, little else is known rigorously about ground states, even for nonnegative J_{e^*}'s. However, nonrigorous scaling arguments (see, e.g., Section 5 of [2]) suggest the conjecture that in low dimensions (including $d = 2$), these disordered ferromagnets (with, say, a common continuous distribution for the J_{e^*}'s) have *no* nonconstant ground states.

For $d = 2$, the relation to first-passage percolation is easily understood. Consider the first-passage model where $\tau(e)$ for an edge e in $\mathbb{Z}^2$ is equal to J_{e^*} for e^* the perpendicular bisector edge of e. If there existed a bigeodesic, then the pair of configurations with one constant value on one side of the geodesic and the opposite value on the other side would be nonconstant ground states. A converse argument is also not difficult. One concludes that the existence of nonconstant ground states for such disordered Ising ferromagnets is equivalent to the existence of bigeodesics for the dual-lattice first-passage model. Analogous statements are valid for $d > 2$, but first-passage geodesics must be replaced by minimal (codimension one) surfaces constructed out of plaquettes.

2. Results. We begin by defining more precisely a term from the last section. A unigeodesic consisting of the edges $\langle x_0, x_1 \rangle, \langle x_1, x_2 \rangle, \dots$ is an $\hat{x}$-unigeodesic if $x_n / \|x_n\| \to \hat{x}$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. An $(\hat{x}, \hat{y})$-bigeodesic is a bigeodesic which is the union of an $\hat{x}$-unigeodesic and a $\hat{y}$-unigeodesic.

There are two main results in this paper, presented as Theorems 1 and 2 in this short section. Both results are restricted to $d = 2$. Theorem 1 concerns the coalescing of $\hat{x}$-unigeodesics, and Theorem 2 concerns the nonexistence of $(\hat{x}, \hat{y})$-bigeodesics. The proof of Theorem 2 is based on Theorem 1 and is given at the end of this section. The proof of Theorem 1 is given in the next section. A preliminary version of Theorem 1 was stated in [6] (as Theorem 2.3 there) but the hypotheses on μ were stronger than for Theorem 1 and only a brief sketch of the proof was given. In both Theorems 1 and 2, we deal with deterministic $\hat{x}$'s and $\hat{y}$'s belonging to a set $\mathscr{U}$ which we now define. We denote by $(\Omega, \mathscr{F}, P)$ the underlying probability space for the $\tau(e)$'s; a specific choice will not be made until the next section.

For any unit vector $\hat{x}$ in $\mathbb{R}^d$, let $D_U(\hat{x})$ denote the event that, for every x in $\mathbb{Z}^d$, there is *at most* one $\hat{x}$-unigeodesic starting at x. We define $\mathscr{U}$ as the set of $\hat{x}$ such that $P(D_U(\hat{x})) = 1$. The next theorem, taken from [6], shows that Lebesgue almost every $\hat{x}$ is in $\mathscr{U}$. We include the proof for completeness.

THEOREM 0. *Suppose $d = 2$ and ν is any continuous (Borel) probability measure on the unit sphere of $\mathbb{R}^2$. Then $\nu(\mathscr{U}) = 1$.*

PROOF. Let $\tilde{e} = (u, v)$ be a *directed* nearest-neighbor edge in $\mathbb{Z}^2$. As noted in the Introduction, $R(u)$, the union of all finite geodesics starting from u, is (a.s.) a spanning tree with at least one (infinite) unigeodesic starting from u. Let us regard $R(u)$ as a kind of family tree with u representing the initial ancestor. Suppose that v is a child of u [i.e., the edge $\tilde{e} = (u, v)$ appears in $R(u)$] and that v has infinitely many descendants; this is equivalent to assuming that there exists at least one unigeodesic whose first step is $\tilde{e}$. We will next define $r^+(\tilde{e})$ to be the particular unigeodesic, starting with $\tilde{e}$, obtained from a counterclockwise search algorithm, as follows. [An analogous clockwise search defines a unigeodesic $r^-(\tilde{e})$.] For each $x \neq u$, let $Y_0(x)$ denote the parent of x and $\{Y_i(x)\colon 1 \le i \le J(x)\}$ denote the children of x. Then $J(x)$, the number of children, is either 0, 1, 2 or 3 and we have chosen a specific clockwise ordering of the children so that the angle $\theta_i(x)$ from

$x - Y_0(x)$ to $Y_i(x) - x$ (which can be either $\pi/2$, 0 or $-\pi/2$) is decreasing in i. The unigeodesic $r^+(\tilde{e})$ consists of the steps $(x_0, x_1), (x_1, x_2), \dots$ with $(x_0, x_1) = \tilde{e}$ and with x_{n+1} for $n \geq 1$ taken as the $Y_i(x_n)$ with the smallest i such that $Y_i(x_n)$ has infinitely many descendants.

If there are two distinct $\hat{x}$-unigeodesics r_1 and r_2 starting at some x, then they have to bifurcate at some u going respectively to v_1 and v_2 in their next steps. Because $d = 2$, any other unigeodesics from u caught "between" r_1 and r_2 must be $\hat{x}$-unigeodesics as well. Now either r_1 is asymptotically counterclockwise to r_2 or vice versa. In the former case, $r^+((u, v_2))$ [as well as $r^-((u, v_1))$] is an $\hat{x}$-unigeodesic and in the latter case $r^+((u, v_1))$ [as well as $r^-((u, v_2))$] is an $\hat{x}$-unigeodesic. Thus $D_U(\hat{x})$ must occur unless the event $G(\tilde{e}, \hat{x})$, that $r^+(\tilde{e})$ is an $\hat{x}$-unigeodesic, occurs for some $\tilde{e}$, and so

$$1 \geq P(D_U(\hat{x})) \geq 1 - \sum_{\tilde{e}} P(G(\tilde{e}, \hat{x})). \tag{2.1}$$

For each $\tilde{e}$ (and P-almost every ω), $r^+(\tilde{e})$ cannot be an $\hat{x}$-unigeodesic for more than one $\hat{x}$; thus by Fubini's theorem and by continuity of ν,

$$\int P(G(\tilde{e}, \hat{x}))\nu(d\hat{x}) = \int \left[\int I_{G(\tilde{e}, \hat{x})}(\omega)\nu(d\hat{x}) \right] P(d\omega) = 0. \tag{2.2}$$

Integrating (2.1) against $\nu(d\hat{x})$ then implies that $P(D_U(\hat{x})) = 1$ for ν-a.e. $\hat{x}$. $\square$

REMARKS. It seems reasonable to conjecture that $\mathscr{U}$ is the entire unit sphere, but we do not even know that the coordinate vectors belong to $\mathscr{U}$. Our proof of Theorem 0 is based on the fact that, for P-a.e. ω, the set of $\hat{x}$'s for which $D_U(\hat{x})$ does not occur must be countable. This should not be the case for $d > 2$, but some version of Theorem 0 could well be valid.

For $\hat{x} \in \mathscr{U}$, we denote by $s_u = s_u(\hat{x})$ the unique (if it exists) $\hat{x}$-unigeodesic starting from $u \in \mathbb{Z}^d$. Although the existence of such unigeodesics is not needed in this paper, we note that their existence was proved in [6], Theorem 2.1, under certain assumptions (including the unverified curvature hypothesis mentioned in the Introduction). A crucial consequence of $\hat{x}$ belonging to $\mathscr{U}$ is that, if s_u and s_v ever meet (i.e., if they are not site-disjoint), then (a.s.) they coalesce. The next theorem shows that, for $d = 2$, s_u and s_v must meet, and thus coalesce. Its proof is given in Section 3. Although our proof is two-dimensional, we see no reason why Theorem 1 should be invalid for $d > 2$.

THEOREM 1. *For $d = 2$ and $\hat{x} \in \mathscr{U}$, there is zero probability that there exist disjoint $\hat{x}$-unigeodesics.*

In this paper, we apply Theorem 1 to obtain the following theorem on the nonexistence of bigeodesics. Note that, if $\hat{x} \in \mathscr{U}$, then also $-\hat{x} \in \mathscr{U}$ and so the theorem rules out $(\hat{x}, -\hat{x})$-bigeodesics. A rather different application of (a preliminary version of) Theorem 1 to surface microstructure in first-passage percolation may be found in [6].

THEOREM 2. *For $d = 2$ and $\hat{x}, \hat{y} \in \mathscr{U}$, there is zero probability that there exist $(\hat{x}, \hat{y})$-bigeodesics.*

PROOF. By the definition of $\mathscr{U}$, we may assume $\hat{x} \neq \hat{y}$. If there were two distinct $(\hat{x}, \hat{y})$-bigeodesics with $\hat{x}, \hat{y} \in \mathscr{U}$, then two applications of Theorem 1 show that the two bigeodesics would have to meet at some x and at some $y \neq x$ while being distinct in between, resulting in nonuniqueness for $M(x, y)$. Thus there can be at most one $(\hat{x}, \hat{y})$-bigeodesic. Let A be the event that there exists a unique $(\hat{x}, \hat{y})$-bigeodesic and for Λ a finite subset of $\mathbb{Z}^2$, let A_Λ be the intersection of A with the event that the unique $(\hat{x}, \hat{y})$-bigeodesic passes through Λ. By ergodicity (with respect to translations in $\mathbb{Z}^2$), $P(A) = 0$ or 1 and so, to prove the theorem, we need only rule out the case $P(A) = 1$.

If $P(A) = 1$, then $P(A_\Lambda) \geq 1 - \varepsilon$ with $\varepsilon < 1/2$ for some choice of finite Λ (e.g., a large square centered at the origin). Let $z_n \in \mathbb{Z}^2$ be such that $\|z_n\| \to \infty$ and $z_n/\|z_n\| \to \hat{z} \neq \hat{x}$ or $\hat{y}$. Then, for every $\omega \in A_\Lambda$, the unique $(\hat{x}, \hat{y})$-bigeodesic passes through $\Lambda + z_n$ for only finitely many n's. Thus

$$P\left(A_\Lambda \cap A_{\Lambda + z_n}\right) \to 0 \quad \text{as } n \to \infty. \tag{2.3}$$

But by translation invariance and our choice of Λ,

$$P\left(A_\Lambda \cap A_{\Lambda + z_n}\right) \geq 1 - P\left(A_\Lambda^c\right) - P\left(A_{\Lambda + z_n}^c\right) \geq 1 - 2\varepsilon, \tag{2.4}$$

which contradicts (2.3) and proves that $P(A) \neq 1$. $\square$

3. Proof of Theorem 1. For $\hat{x} \in \mathscr{U}$, let $S = S[\hat{x}]$ denote the union of $s_u = s_u(\hat{x})$, the unique (if they exist) $\hat{x}$-unigeodesics starting from any $u \in \mathbb{Z}^d$. More precisely, S is the random graph with vertex set, $\{u \in \mathbb{Z}^d\colon s_u \text{ exists}\}$, and edge set which is the union of the edges of all the s_u's. As mentioned previously, since $\hat{x} \in \mathscr{U}$, s_u and s_v must coalesce if they ever meet. It follows that S is either empty or else is a forest consisting of $N \geq 1$ distinct infinite trees. Theorem 1 is thus equivalent to the statement that, for $d = 2$, $P(N \geq 2) = 0$. The general structure of our proof will parallel that used in [1] to rule out two or more infinite clusters in nearest-neighbor bond percolation. There are three parts. In this section, we take the $\tau(e)$'s to be the coordinate functions, denoted ω_e, on $(\Omega, \mathscr{F}, P)$, the product probability space (over $\mathbb{E}^2$, the edge set of $\mathbb{Z}^2$) of $(\mathbb{R}, \mathscr{B}, \mu)$ with $\mathscr{B}$ the Borel σ-field.

PART 1. *We show that $P(N \geq 2) > 0$ implies $P(N \geq 3) > 0$.* By utilizing the symmetries of $\mathbb{Z}^2$, we may assume, without loss of generality (here and throughout the proof), that $\hat{x}$ has a strictly positive x-coordinate. Then any $\hat{x}$-unigeodesic is eventually to the right of any vertical line. This and translation invariance show that $P(N \geq 2) > 0$ implies that, for some $y_0 > 0$, $P(A(0, y_0)) > 0$, where $A(y_1, y_2, \dots, y_m)$ denotes the event that

$s_{(0,y_1)},\dots,s_{(0,y_m)}$ are disjoint and that, for each $j=1,\dots,m$, every site touched by $s_{(0,y_j)}$, after its initial site at $(0,y_j)$, has strictly positive x-coordinate. Figure 1 gives a schematic diagram of $A(y_1,y_2,y_3)$.

Since $P(A(n,n+y_0))=P(A(0,y_0))>0$, we see that the sequence of events, $A(n,n+y_0)$, occurs infinitely often with positive probability. It follows that, for some $y_1<y_2<y_2'<y_3$ [with $(y_1,y_2)=(n_1,n_1+y_0)$ and $(y_2',y_3)=(n_2,n_2+y_0)$], the intersection $A(y_1,y_2)\cap A(y_2',y_3)$ has positive probability. But, for ω in this intersection, $s_{(0,y_2)}$ must be disjoint from $s_{(0,y_3)}$ since if they were not disjoint they would coalesce and then $s_{(0,y_2')}$, trapped in between them in two dimensions, would also coalesce with them, violating the definition of $A(y_2',y_3)$. Thus $A(y_1,y_2)\cap A(y_2',y_3)$ is contained in $A(y_1,y_2,y_3)$ and so

$$P(A(y_1,y_2,y_3))>0, \tag{3.1}$$

which, of course, implies $P(N\ge 3)>0$.

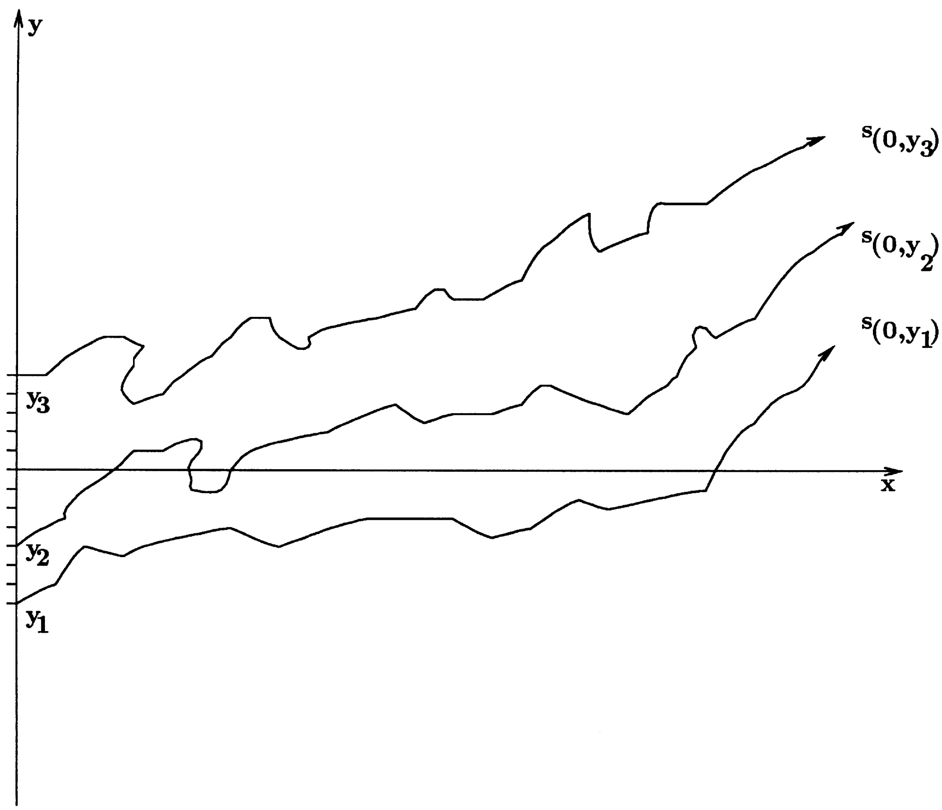

FIG. 1. *A schematic diagram of a configuration from $A(y_1,y_2,y_3)$. The edges in Θ are those indicated just to the left of the y-axis between y_1 and y_3.*

We remark that similar reasoning leads to the conclusion that $P(N = \infty) > 0$ (in fact $= 1$ by ergodicity), but we shall not need this conclusion.

PART 2. *We show that* $P(N \geq 3) > 0$ *implies* $P(F_{m,k}) > 0$ *for some* $m, k \geq 0$. Here $F_{m,k}$ denotes the event that some tree in S touches the rectangle $R_{m,k} = \{u = (\alpha, \beta) \in \mathbb{Z}^2\colon 0 \leq \alpha \leq m,\ -k \leq \beta \leq k\}$ but no other site in the half-plane $\{(\alpha, \beta)\colon \alpha \leq m\}$. Starting from $P(N \geq 3) > 0$ and repeating (some of) the proof of Part 1 or else by simply quoting its conclusion, we have (3.1) valid for some $y_1 < y_2 < y_3$. It turns out that the proof of Part 2 is considerably simpler if the distribution of the $\tau(e)$'s has unbounded support, that is, if $P(\tau(e) > t) > 0$ for every $t < \infty$. So we first give the proof in that restricted case.

Case (*a*)—μ *with unbounded support*. Starting from (3.1), we will show that $P(F_{0,k}) > 0$ with $k = \max(|y_1|, |y_3|)$. Denote by Θ the set of edges (u, v) with $u = (-1, \beta)$, $v = (0, \beta)$ and $y_1 \leq \beta \leq y_3$ (see Figure 1). Denote by B_λ the event that, for each $(u, v) \in \Theta$, there exists a path r between u and v *not* using any edges in Θ with $T(r) < \lambda$. Since $P(B_\lambda) \to 1$ as $\lambda \to \infty$, we have for some large $\tilde{\lambda}$ that

$$P(A(y_1, y_2, y_3) \cap B_{\tilde{\lambda}}) > 0. \tag{3.2}$$

Consider the event $A = A(y_1, y_2, y_3; \tilde{\lambda})$ appearing in (3.2). Suppose an $\omega \in A$ is altered to ω' by changing each $\omega_e < \tilde{\lambda}$ with $e \in \Theta$ to some value at least as large as $\tilde{\lambda}$. Since the ω_e's with $e \in s_{(0, y_i)}$ for $i = 1, 2, 3$, were unchanged while others increased or stayed constant, it follows that, for each i, the path $r_i \equiv s_{(0, y_i)}(\omega)$ continues to be an $\hat{x}$-unigeodesic for ω'. Similarly, ω' continues to belong to $B_{\tilde{\lambda}}$, but, since $\omega'_e \geq \tilde{\lambda}$ for each $e \in \Theta$, it follows that no $e \in \Theta$ can belong to *any* geodesic for ω'. Thus any geodesic r for ω', which starts from $u = (\alpha, \beta)$, with $\alpha < 0$, or $\alpha = 0$ and $\beta \notin [y_1, y_3]$, cannot touch the middle path r_2 without first passing through the lower or upper path, r_1 or r_3. If this r is an $\hat{x}$-unigeodesic, then either it coalesces with r_1 or it coalesces with r_3 or else ω' has the property that some $\hat{x}$-geodesics meet without coalescing.

Now let us consider $\tilde{\Phi}(A)$, the set of all possible ω''s obtained from ω's $\in A$. To be more precise, we define a mapping $\tilde{\Phi}$ on subsets of Ω by first letting $W(\omega) = \{e \in \Theta\colon \omega_e < \tilde{\lambda}\}$ and then setting

$$\tilde{\Phi}(F) = \bigcup_{\omega \in F} \left[\prod_{e \notin W(\omega)} \{\omega_e\} \times \prod_{e \in W(\omega)} [\tilde{\lambda}, \infty) \right]. \tag{3.3}$$

It should be noted that $\tilde{\Phi}$ may map a measurable set (i.e., one in $\mathscr{F}$) to a nonmeasurable set and thus we cannot claim that $P(\tilde{\Phi}(A)) > 0$ since $P(\tilde{\Phi}(A))$ may not be defined. However, since $P(A) > 0$, we can apply the following lemma (which extends Proposition 9 of [6]) to conclude that $\tilde{\Phi}(A) \supset \Phi(A)$, where $\Phi(A) \in \mathscr{F}$ and $P(\Phi(A)) > 0$.

Since $\hat{x} \in \mathscr{U}$, we know that there is zero probability for $\hat{x}$-unigeodesics to meet and not to coalesce. Thus, from the discussion of the last paragraph, almost every $\omega' \in \Phi(A)$ belongs to $A(y_1, y_2, y_3)$ and has the property that

$s_{(\alpha,\beta)}(\omega')$ with $\alpha < 0$, or $\alpha = 0$ and $\beta \notin [y_1, y_3]$, is disjoint from $s_{(0,y_2)}(\omega')$. Thus $s_{(0,y_2)}(\omega')$ belongs to a tree in $S(\omega')$ which touches $(0, y_2)$ but no (α, β) with $\alpha < 0$, or $\alpha = 0$ and $\beta \notin [y_1, y_3]$, and so $\omega' \in F_{0,k}$ with $k = \max(|y_1|, |y_3|)$. We conclude that $P(F_{0,k}) > 0$, as desired.

LEMMA 3.1. *For D any subset of the edge set $\mathbb{E}^2$, denote by $(\Omega_D, \mathscr{F}_D, P_D)$ the product over D of $(\mathbb{R}, \mathscr{B}, \mu)$. Also, for $F \subset \Omega$ and $\omega_1 \in \Omega_{D^c}$, define*

$$F_{D,\omega_1} = \{\omega_2 \in \Omega_D : \omega \equiv (\omega_1, \omega_2) \in F\}. \tag{3.4}$$

For each finite subset w of $\mathbb{E}^2$, let R_w in $\mathscr{F}_w$ be chosen so that $P_w(R_w) > 0$ and then define a map Φ_w on $\mathscr{F}$:

$$\Phi_w(G) = \{\omega_1 \in \Omega_{w^c} : P_w(G_{w,\omega_1}) > 0\} \times R_w. \tag{3.5}$$

Suppose $W(\omega)$ is a random finite subset of $\mathbb{E}^2$ and $\tilde{\Phi}(F)$, for $F \subset \Omega$, is defined as

$$\tilde{\Phi}(F) = \bigcup_{\omega \in F} \left[\left(\prod_{e \notin W(\omega)} \{\omega_e\} \right) \times R_{W(\omega)} \right]. \tag{3.6}$$

Then, for $F \in \mathscr{F}$, $\tilde{\Phi}(F)$ contains $\Phi(F) \in \mathscr{F}$, defined as the following union over finite subsets of $\mathbb{E}^2$:

$$\Phi(F) = \bigcup_w \Phi_w(F \cap \{W = w\}). \tag{3.7}$$

Furthermore, $P(F) > 0$ implies $P(\Phi(F)) > 0$.

PROOF. Let us denote $F \cap \{W = w\}$ by $F(w)$. By decomposing (3.6) according to the value of $W(\omega)$, one sees that

$$\tilde{\Phi}(F) = \bigcup_w \left[\{\omega_1 \in \Omega_{w^c} : F(w)_{w,\omega_1} \text{ is nonempty}\} \times R_w \right], \tag{3.8}$$

which contains $\Phi(F)$, as given by (3.7) and (3.5). Since F is the countable disjoint union of the $F(w)$'s, $P(F) > 0$ implies $P(F(w)) > 0$ for some w. For that w, by Fubini's theorem,

$$P(F(w)) = \int_{\Omega_{w^c}} \left[P_w(F(w)_{w,\omega_1}) \right] P_{w^c}(d\omega_1) > 0, \tag{3.9}$$

and so the set $\hat{F}_w$ of ω_1's such that the integrand in the square brackets is strictly positive, has $P_{w^c}(\hat{F}_w) > 0$. But, according to the definition (3.5) of Φ_w, $\Phi_w(F(w)) = \hat{F}_w \times R_w$ and so

$$P(\Phi_w(F(w))) = P_{w^c}(\hat{F}_w) P_w(R_w) > 0. \tag{3.10}$$

Since $\Phi(F)$ contains $\Phi_w(F(w))$, we conclude that $P(\Phi(F)) > 0$, as desired. □

We have now completed the proof in case (a) of Part 2, that is, μ with unbounded support. We continue with the proof for the case of bounded support. The only place where boundedness enters the proof is in the use of the shape theorem, but for that it would suffice for μ to have, for example, a finite second moment.

Case (b)—μ *with bounded support*. As in case (a), we start from (3.1). Our unit vector $\hat{x} = (\hat{x}_1, \hat{x}_2)$ has been assumed (without loss of generality) to have $\hat{x}_1 > 0$. We can and will also assume that $\hat{x}_2 \geq 0$. Let $\bar{\lambda}$ denote the sup of the support of μ and let $\gamma(\hat{x})$ denote the time constant in the $\hat{x}$ direction. That is, if v_n is a sequence of sites in $\mathbb{Z}^2$, with $\|v_n\| \to \infty$ and $v_n/\|v_n\| \to \hat{x}$, then, for any fixed u, $T(u, v_n)/\|v_n\| \to \gamma(\hat{x})$ a.s. and in L^1 (see, e.g., [4] and [5]). Since there exist paths from u to v_n with (approximately) $\|v_n\|(\hat{x}_1 + \hat{x}_2)$ edges, it easily follows that

$$\gamma(\hat{x}) < \bar{\lambda}(\hat{x}_1 + \hat{x}_2). \tag{3.11}$$

For the fixed y_1, y_2, y_3, any positive integer m and any $\varepsilon' > 0$, let $C_m^{\varepsilon'}$ denote the event that the $\hat{x}$-unigeodesics $s_{(0, y_j)}$ (if they exist) touch the line $\{(\alpha, \beta)\colon \alpha = m\}$ for the first time [coming from $(0, y_j)$] within the vertical line segment $Q_{m, \varepsilon'}$, of length $\varepsilon' m$ centered at the point $(m/\hat{x}_1)\hat{x}$ (see Figure 2). We denote the distance $m/\hat{x}_1$ of this point from the origin by L_m. From the definition of an $\hat{x}$-unigeodesic, it follows that, for any $\varepsilon' > 0$, $P(C_m^{\varepsilon'}) \to 1$ as $m \to \infty$.

Next, for $\varepsilon, \varepsilon' > 0$, let $B_m^{\varepsilon, \varepsilon'}$ denote the event that, for every site $z = (0, y)$ with $y_1 \leq y \leq y_3$ and every site $w \in Q_{m, \varepsilon'}$,

$$T(z, w) < (\gamma(\hat{x}) + \varepsilon) L_m. \tag{3.12}$$

From the first-passage shape theorem (see, e.g., [4] and [5]), it follows that $\gamma(\hat{x}') \to \gamma(\hat{x})$ as $\hat{x}' \to \hat{x}$, and also that, for any $\varepsilon > 0$ and all $\varepsilon' > 0$ sufficiently small, $P(B_m^{\varepsilon, \varepsilon'}) \to 1$ as $m \to \infty$.

Finally, for any positive integer m and k, let $C_{m,k}$ denote the event that the three $\hat{x}$-unigeodesics $s_{(0, y_j)}$ (if they exist) do not intersect $\{(\alpha, \beta)\colon 0 \leq \alpha \leq m, |\beta| > k\}$. From the definition of an $\hat{x}$-unigeodesic, it follows that, for any fixed m, $P(C_{m,k}) \to 1$ as $k \to \infty$.

Combining all these results, we see that, for *any* $\varepsilon > 0$, then for sufficiently small $\varepsilon' > 0$, then for sufficiently large m and finally for sufficiently large k,

$$\tilde{C}(\varepsilon, \varepsilon', m, k) \equiv C_m^{\varepsilon'} \cap B_m^{\varepsilon, \varepsilon'} \cap C_{m,k} \tag{3.13}$$

has probability as close to 1 as desired—in particular close enough so that

$$P\big(A(y_1, y_2, y_3) \cap \tilde{C}(\varepsilon, \varepsilon', m, k)\big) \geq \tfrac{1}{2} P(A(y_1, y_2, y_3)) > 0. \tag{3.14}$$

Let A denote the event whose probability appears on the left-hand side of (3.14). Let $\tilde{r}_j$ denote the segment of $r_j \equiv s_{(0, y_j)}$ between $(0, y_j)$ and the first place r_j touches $Q_{m, \varepsilon'}$ (see Figure 2). For $\omega \in A$, we have by (3.12) that $T(\tilde{r}_j) < (\gamma(\hat{x}) + \varepsilon) L_m$ and thus between any points z and w as in (3.12), one can choose a path $r(z, w)$ which first moves vertically (down), then follows $\tilde{r}_1$, then moves vertically again, with

$$T(r(z, w)) < \bar{\lambda}(y_3 - y_1) + (\gamma(\hat{x}) + \varepsilon) L_m + \bar{\lambda}\varepsilon' m. \tag{3.15}$$

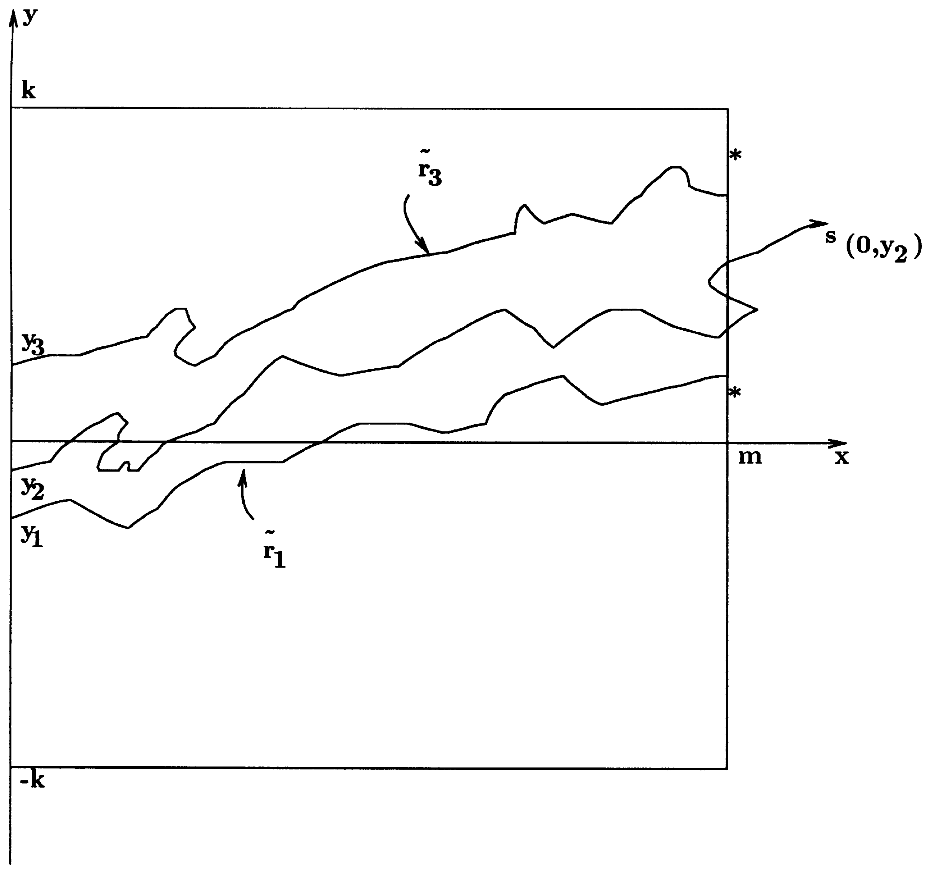

FIG. 2. *A schematic diagram of a configuration from* $A(y_1, y_2, y_3) \cap C_m^{\varepsilon'} \cap C_{m,k}$. *The set* Θ *consists of those edges between* $\tilde{r}_1, \tilde{r}_3$ *and the vertical lines at* $x = 0$ *and* $x = m$. *The asterisks on the vertical line at* $x = m$ *denote the upper and lower endpoints of* $Q_{m,\varepsilon'}$.

None of the edges of $r(z, w)$ belong to Θ, the random finite set of edges bounded by $\tilde{r}_1$, $\tilde{r}_3$, $Q_{m,\varepsilon'}$ and $\{(0, y)\colon y_1 \le y \le y_3\}$ (see Figure 2). On the other hand, *if* an alternate path $r'(z, w)$ using only edges of Θ had $\tau(e) \ge \bar{\lambda} - \varepsilon$ for every edge, then, since r' would contain at least $L_m(\hat{x}_1 + \hat{x}_2) - (y_3 - y_1) - \varepsilon' m$ edges,

$$T(r'(z,w)) \ge (\bar{\lambda} - \varepsilon)\left[L_m(\hat{x}_1 + \hat{x}_2) - (y_3 - y_1) - \varepsilon' m\right]. \tag{3.16}$$

We can proceed as follows. By (3.11), we can choose ε so that

$$\gamma(\hat{x}) + \varepsilon < (\bar{\lambda} - \varepsilon)(\hat{x}_1 + \hat{x}_2). \tag{3.17}$$

Noting that $m = L_m \hat{x}_1$, we then choose ε' so small that

$$\gamma(\hat{x}) + \varepsilon + \bar{\lambda}\hat{x}_1\varepsilon' < (\bar{\lambda} - \varepsilon)(\hat{x}_1 + \hat{x}_2) - \varepsilon'(\bar{\lambda} - \varepsilon)\hat{x}_1. \tag{3.18}$$

Then we choose m so large that, for our fixed $y_3 - y_1$, the right-hand side of (3.15) is strictly smaller than that of (3.16). We also pick ε' small enough and m large enough so that for k sufficiently large (3.14) is valid.

Note that, for any $\varepsilon > 0$, $\mu([\bar{\lambda} - \varepsilon, \infty)) > 0$ since $\bar{\lambda}$ is the sup of the support. As in case (a) (but with $\tilde{\lambda} = \bar{\lambda} - \varepsilon$), we now define $W(\omega)$ (for $\omega \in A$) to be $\{e \in \Theta\colon \omega_e < \tilde{\lambda}\}$ and define $\tilde{\Phi}(A)$ exactly as in case (a). We now apply Lemma 3.1 again and conclude that $P(\Phi(A)) > 0$ with $\Phi(A) \subset \tilde{\Phi}(A)$. The point now is that, for any $\omega \in \Phi(A)$, any $\hat{x}$-unigeodesic originating in the half-plane $\{(\alpha, \beta)\colon \alpha \le m\}$ other than in the rectangle $R_{m,k}$ could not touch the middle geodesic r_2 without either touching $\tilde{r}_1$ or $\tilde{r}_3$ (in which case, it would a.s. coalesce with r_1 or r_3) or else having a segment $r'(z, w)$ entirely using edges of Θ and thus satisfying (3.16). But, by the choices of the last paragraph, no such segment can be a (finite) geodesic because $r(z, w)$ still satisfies (3.15).

Thus $\Phi(A)$ has positive probability and is a subset of $F_{m,k}$ so that $P(F_{m,k}) > 0$, as desired.

PART 3. *We show that $P(F_{m,k}) > 0$ is impossible*. Consider a rectangular array of nonintersecting translates $R^u_{m,k}$ of the basic rectangle $R_{m,k}$ indexed by $u \in \mathbb{Z}^2$ (in a natural way) and consider the corresponding translated events $F^u_{m,k}$. If $F^u_{m,k}$ and $F^v_{m,k}$ both occur, then the corresponding trees in S (from the definitions of these two events) must be disjoint. Let n_L denote the number of $R^u_{m,k}$'s in $[0, L] \times [0, L]$ and N_L the (random) number of the corresponding $F^u_{m,k}$'s which occur. By translation invariance $E(N_L) = n_L P(F_{m,k})$. Clearly, $n_L \ge cL^2$ for some $c > 0$ and large L and thus $P(F_{m,k}) > 0$ implies that, with $c' = cP(F_{m,k}) > 0$,

$$P\left(N_L \ge c'L^2\right) > 0 \quad \text{for all large } L. \tag{3.19}$$

But $N_L \le$ the number of disjoint trees in S which touch $[0, L] \times [0, L]$. Since each tree in S is infinite, this number cannot exceed the number of boundary sites in $[0, L] \times [0, L]$ which is less than or equal to $c''L$ for large L. For L so large that $c''L < c'L^2$, the assumption that $P(F_{m,k}) > 0$ yields a contradiction, which completes the proof. $\square$

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