The Annals of Probability
1996, Vol. 24, No. 4, 2141–2153

CONVERGENCE IN VARIOUS TOPOLOGIES FOR STOCHASTIC INTEGRALS DRIVEN BY SEMIMARTINGALES[1]

BY ADAM JAKUBOWSKI

Uniwersytet Mikołaja Kopernika

We generalize existing limit theory for stochastic integrals driven by semimartingales and with left-continuous integrands. Joint Skorohod convergence is replaced with joint finite dimensional convergence *plus* an assumption excluding the case when oscillations of the integrand appear immediately before oscillations of the integrator. Integrands may converge in a very weak topology. It is also proved that convergence of integrators implies convergence of stochastic integrals with respect to the same topology.

1. Introduction. Let us begin with a simple example demonstrating one of the central difficulties in limit theory for integrals with discontinuous integrators.

EXAMPLE 1. Let $k^0(t) = \mathbb{1}_{[1/2,\,1]}(t)$, $k^n(t) = \mathbb{1}_{[1/2-1/n,\,1]}(t)$, $n = 1, 2, \dots,$ and let $x^n(t) = \mathbb{1}_{[1/2,\,1]}(t)$, $n = 0, 1, 2, \dots$. Then $k^n \to k^0$ and $x^n \to x^0$ in the Skorohod space $\mathbb{D} = \mathbb{D}([0, 1]\colon \mathbb{R}^1)$, but

$$\int k^n_- \, dx^n \equiv x^0 \not\to \int k^0_- \, dx^0 \equiv 0.$$

[Here—and in the sequel—we consider integrals over the interval excluding 0, that is, $(\int k_- \, dx)(t) = \int_{]0,\,t]} k(s-)\,dx(s)$, with $k(0-) = 0$.]

One can eliminate such pathological situations by assuming *joint* convergence of (k^n, x^n), that is, convergence in $\mathbb{D}([0, 1]\colon \mathbb{R}^2)$. A very general result in this direction was proved by Jakubowski, Mémin and Pagès [(1989), Theorem 2.6].

THEOREM 0. *For each $n \in \mathbb{N}$, let X^n be a semimartingale with respect to the stochastic basis $(\Omega^n, \mathscr{F}^n, \{\mathscr{F}^n_t\}_{t\in[0,1]}, P^n)$ and let K^n be adapted to $\{\mathscr{F}^n_t\}_{t\in[0,1]}$ and with trajectories in $\mathbb{D}$. Assume that*

$$(K^n, X^n) \to_{\mathscr{D}} (K^0, X^0) \qquad (1)$$

Received March 1995; revised August 1995.
[1]Research supported by Komitet Badań Naukowych under Grant 2-1108-91-01 and completed while the author was visiting Université de Rennes I.
AMS 1991 *subject classifications*. Primary 60F17; secondary 60H05, 60B10.
Key words and phrases. Stochastic integral, Skorohod topology, functional convergence, semimartingales.

on the space $\mathbb{D}([0,1]\colon \mathbb{R}^2)$. *Then* X^0 *is a semimartingale with respect to the natural filtration generated by* (K^0, X^0) *and*

$$\int K^n_- \, dX^n \to_{\mathscr{D}} \int K^0_- \, dX^0 \quad \text{on } \mathbb{D}([0,1]\colon \mathbb{R}^1), \tag{2}$$

as well as

$$\left(K^n, X^n, \int K^n_- \, dX^n\right) \to_{\mathscr{D}} \left(K^0, X^0, \int K^0_- \, dX^0\right) \quad \text{on } \mathbb{D}([0,1]\colon \mathbb{R}^3), \tag{3}$$

provided the so-called condition UT holds, that is, provided the family of elementary stochastic integrals $\{\int H^n_- \, dX^n(1)\}$ *with integrands bounded by* 1 *is uniformly tight.*

To be explicit, condition UT means that the family of all random variables of the form

$$\sum_{i=1}^m H^n_{t_{i-1}}(X^n_{t_i} - X^n_{t_{i-1}})$$

is uniformly tight, where $m \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_m = T$, $H^n_{t_i} \le 1$ and $H^n_{t_i}$ is $\mathscr{F}^n_{t_i}$-measurable for $i = 0, 1, \ldots, m$. Condition UT was considered for the first time in Stricker (1985). The reader may find sufficient conditions for condition UT in Jakubowski, Mémin and Pagès (1989) and equivalent reformulations in Mémin and Słomiński (1991). Here we shall mention only that condition UT also plays a crucial role in approximation of solutions of stochastic differential equations. For the corresponding results in this area as well as for interesting examples, we refer to Słomiński (1989, 1994) and Kurtz and Protter (1991).

Theorem 0 suffices for most applications related to stability problems of stochastic differential equations. On the other hand, within limit theory for stochastic integrals there exist phenomena which are not covered by this theorem.

EXAMPLE 2. Normalized sums of moving averages with summable *positive* coefficients of i.i.d. random variables with laws belonging to the domain of attraction of an α-stable law ($\alpha < 2$) in general *do not converge* in a functional manner when $\mathbb{D}$ is equipped with the usual Skorohod J_1 topology. However, they *do converge* to an α-stable Lévy motion if we consider another, weaker topology on $\mathbb{D}$, known as M_1 [see Avram and Taqqu (1992) and Skorohod (1956) for definitions of Skorohod's topologies].

There exists a satisfactory theory of stochastic integration with respect to α-stable processes [see, e.g., Janicki and Weron (1993)]. It follows that for some naturally arising integrators the requirement of convergence in the usual Skorohod topology may be too strong.

EXAMPLE 3. Let, as in Example 1, $x^n(t) = \mathbb{1}_{[1/2,1]}(t)$, $n = 0, 1, 2, \ldots$, and $k^0(t) = \mathbb{1}_{[1/2,1]}(t)$. The difference will be in the choice of k^n:

$$k^n(t) = \mathbb{1}_{[1/2+1/n,1]}(t), \qquad n = 1, 2, \ldots.$$

As before we have $k^n \to k^0$ and $x^n \to x^0$ in $\mathbb{D}$ with the standard topology and $(k^n, x^n) \not\to (k^0, x^0)$ in $\mathbb{D}([0,1]: \mathbb{R}^2)$, but this time

$$\int k^n_- \, dx^n \equiv 0 \to \int k^0_- \, dx^0 \equiv 0.$$

EXAMPLE 4. The preceding example may seem to be artificial and related to the extremely simple structure of the processes involved. To convince the reader that the example illustrates a general rule, let us consider a much less obvious fact.

Let X be a semimartingale on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,1]}, P)$ and let K be adapted to $\{\mathscr{F}_t\}_{t\in[0,1]}$ and with trajectories in $\mathbb{D}$. Choose a partition $\tau = \{0 = t_0 < t_1 < t_2 < \cdots < t_m = 1\}$ of $[0,1]$ and define τ-discretizations of the integrand K:

$$\begin{aligned} &K^\tau(t) = K(t_k) \quad \text{if } t_k \le t < t_{k+1},\ k = 0, 1, \dots, m-1, \\ &K^\tau(1) = K(1). \end{aligned} \tag{4}$$

Then it follows from the "dominated convergence" theorem [see Dellacherie and Meyer (1980), VIII.14] that the elementary stochastic integrals $\int K^\tau_- \, dX$ converge *uniformly in probability* to $\int K_- \, dX$ when τ condenses in a suitable manner. Here again the pair (K^τ, X) does not converge jointly to (K, X) unless the relation between K and X is very special.

The purpose of the present note is to provide a criterion for finite dimensional convergence of stochastic integrals and to demonstrate how this criterion can be used in particular situations to obtain results on functional convergence with respect to various topologies on the space $\mathbb{D}$.

2. The results. Let us denote by $\to_{\mathscr{D}_f(\mathbb{Q})}$ the finite dimensional convergence over the set $\mathbb{Q}$. For example, $(K^n, X^n) \to_{\mathscr{D}_f(\mathbb{Q})} (K^0, X^0)$ means that for every finite subset $\{0 \le t_1 < t_2 < \cdots < t_m \le 1\} \subset \mathbb{Q}$,

$$\begin{aligned} &(K^n(t_1), X^n(t_1), K^n(t_2), X^n(t_2), \dots, K^n(t_m), X^n(t_m)) \\ &\qquad \to_{\mathscr{D}} (K^0(t_1), X^0(t_1), K^0(t_2), X^2(t_2), \dots, K^0(t_m), X^0(t_m)) \end{aligned}$$

on the space $\mathbb{R}^{2m}$.

It is finite dimensional convergence of (K^n, X^n) on a dense set $\mathbb{Q} \subset [0,1]$ and condition UT for $\{X^n\}$ that imply that X^0 is a semimartingale with respect to the natural filtration generated by (K^0, X^0) [see Jakubowski, Mémin and Pagès (1989), Lemma 1.3]. In what follows we will use this fact without further reference.

For $k \in \mathbb{D}$, let $N_\eta(k)$ be the number of η-oscillations of k in the interval $[0,1]$. More precisely, $N_\eta(k) \ge m$ if there are points $0 \le t_1 \le t_2 \le \cdots t_{2m-1} \le t_{2m} \le 1$ such that $|k(t_{2j}) - k(t_{2j-1})| > \eta$, $j = 1, 2, \dots, m$. By reasons to be made clear later we shall say that a sequence $\{K^n\}$ of processes with trajectories in $\mathbb{D}$ is *uniformly S-tight* if both $\{\|K^n\|_\infty = \sup_{t\in[0,1]} K^n(t)\}$ and $\{N_\eta(K^n)\}$, for each $\eta > 0$, are uniformly tight sequences of random variables.

We will also need a variant of the well-known modulus of continuity ω''. For $k,\, x \in \mathbb{D}$, let

$$\omega''_\delta(k, x) = \sup\{\min(|k(s) - k(t)|, |x(t) - x(u)|): \\ 0 \le s < t < u \le (s+\delta) \wedge 1\}.$$

THEOREM 1. *For each $n \in \mathbb{N}$, let X^n be a semimartingale with respect to the stochastic basis $(\Omega^n, \mathcal{F}^n, \{\mathcal{F}^n_t\}_{t\in[0,1]}, P^n)$ and let K^n be adapted to $\{\mathcal{F}^n_t\}_{t\in[0,1]}$ with trajectories in $\mathbb{D}$. Let $\mathbb{Q} \subset [0,1]$, $0, 1 \in \mathbb{Q}$, be dense. Suppose condition UT holds for $\{X^n\}$, $\{K^n\}$ is uniformly S-tight and we have joint finite dimensional convergence over $\mathbb{Q}$. Then*

$$(K^n, X^n) \to_{\mathcal{D}_f(\mathbb{Q})} (K^0, X^0), \tag{5}$$

where both K^0 and X^0 have trajectories in $\mathbb{D}$. Further, suppose there are no oscillations of the K^n preceding oscillations of the X^n. Then

$$\lim_{\delta \searrow 0} \limsup_{n\to\infty} P^n(\omega''_\delta(K^n, X^n) > \varepsilon) = 0, \qquad \varepsilon > 0. \tag{6}$$

Then we have

$$\int K^n_- \, dX^n \to_{\mathcal{D}_f(\mathbb{Q})} \int K^0_- \, dX^0. \tag{7}$$

The proof (as well as proofs of other results below) is deferred to the next section.

REMARK 1. Stricker (1985) proved that condition UT implies uniform S-tightness. Hence all methods of verifying condition UT also apply to uniform S-tightness as well.

It was announced in the Introduction that Theorem 1 may serve as a tool for identification of the limit in various kinds of functional convergence of stochastic integrals. By "functional" we mean convergence in distribution with respect to any topology τ on $\mathbb{D}$ such that relative compactness (in law) and finite dimensional convergence over a dense subset $\mathbb{Q} \subset [0,1]$, $0, 1 \in \mathbb{Q}$, imply convergence in law. By the result due to Topsøe (1969), Skorohod's J_1 topology *is* functional in our sense. On the other hand, the so-called pseudopath topology considered by Meyer and Zheng (1984) is not "functional," for it is known that convergence of sequences in this topology is just the convergence in (Lebesgue) measure, and it is easy to find a sequence $\{x^n\} \subset \mathbb{D}$ which converges in measure to $x^0 \equiv 0$ and is such that $x^n(q) \to 1$ for each rational $q \in [0,1]$.

COROLLARY 2. *Suppose all assumptions of Theorem 1 are in force and we know that the laws of stochastic integrals $\int K^n_- \, dX^n$ are relatively compact when $\mathbb{D}$ is equipped with some topology τ generating "functional" convergence. Then the sequence $\{\int K^n_- \, dX^n\}_{n\in\mathbb{N}}$ converges in law with respect to τ (and the limiting process is $\int K^0_- \, dX^0$).*

Given uniform τ-tightness of X^n's, the task of verifying uniform tightness of $\{\int K^n_- \, dX^n\}_{n\in\mathbb{N}}$ can be quite easy. This is so, for instance, in Theorem 6 below. For metric topologies on $\mathbb{D}$ we have, however, results more direct than Corollary 2. We begin with the classical Skorohod J_1 topology, to emphasize the generality of the present approach.

THEOREM 3. *Let the K^n, X^n and $\mathbb{Q}$ be as in Theorem* 1. *Suppose $\{X^n\}$ satisfies condition UT, $\{K^n\}$ is uniformly S-tight and finite dimensional convergence* (5) *holds. If* (6) *is satisfied and $X^n \to_{\mathscr{D}} X^0$ on the space $(\mathbb{D}, J_1)$, then on the space $(\mathbb{D}([0,1]: \mathbb{R}^2), J_1)$,*

$$(X^n, \int K^n_- \, dX^n) \to_{\mathscr{D}} (X^0, \int K^0_- \, dX^0). \tag{8}$$

REMARK 2. Comparing to Theorem 0, we require very weak convergence of the K^n (in S-topology—see below) and much less information on the joint convergence–finite dimensional convergence and (6).

REMARK 3. In Section 3.3 we provide a direct proof of Theorem 3, based on the a.s. Skorohod representation. There exists, however, a more traditional approach. Theorem 1 identifies finite dimensional distributions of the limit for $\int K^n_- \, dX^n$. Hence *given* Theorem 1, for (8) to be proven we need only uniform J_1-tightness of the sequence of stochastic integrals. This can be obtained, for instance, by use of Proposition 4.3 of Kurtz and Protter (1991), which, in turn, is based on the technique of inverse time-change developed by Kurtz (1991). In fact, an inspection of the latter paper shows that S-tightness is equivalent to the existence of strictly increasing C_n, with C_n $\{\mathscr{F}^n_t\}$-adapted, such that $\{C_n(t)\}$ is uniformly tight for each $t > 0$ and, defining $\gamma^n(t) = C_n^{\to}(t) = \inf\{u: C_n(u) > t\}$, $\{K^n \circ \gamma^n\}$ is relatively compact in the Skorohod J_1 topology.

REMARK 4. The crucial property (6) [or (19) below] holds if either the X^n or K^n are C-tight, that is, are uniformly J_1-tight with all limiting laws concentrated on $C([0,1]: \mathbb{R}^1)$. Hence our Theorem 3 generalizes Theorem 4.7 of Kurtz and Protter (1991) and is a step in a similar direction as their Theorem 4.8, with dramatically simpler formulation.

Without any change in the proof one can obtain limit results for a variety of topologies on $\mathbb{D}$. Let ρ be a metric on $\mathbb{D}$ such that the topology $\mathscr{O}_\rho$ generated by ρ is coarser than Skorohod's J_1 topology, but rich enough to preserve the same family of Borel subsets. This guarantees that all probability measures on $(\mathbb{D}, \mathscr{O}_\rho)$ are tight and that $X^n \to_{\mathscr{D}} X^0$ on $(\mathbb{D}, \mathscr{O}_\rho)$ if and only if X^n, $n = 0, 1, 2, \dots$, admit the almost surely convergent Skorohod representation on the Lebesgue interval $([0,1], \mathscr{B}_{[0,1]}, l)$. In addition we assume that ρ satisfies

$$\rho(x, y) \le C\|x - y\|_\infty, \qquad x, y \in \mathbb{D} \tag{9}$$

for some $C > 0$.

From the point of view of limit theorems it is natural to consider only metrics which are consistent with convergence of elementary integrals. To explain this notion, take a partition $\tau = \{0 = t_0 < t_1 < t_2 < \cdots < t_m = 1\}$, a sequence $a^\tau = (a_0, a_1, \ldots, a_m) \in \mathbb{R}^m$ and $x \in \mathbb{D}$ and define

$$\left(\int a^\tau_- \, dx\right)(t) = \sum_{k=1}^{m} a_{k-1}(x(t_k \wedge t) - x(t_{k-1} \wedge t)).$$

The consistency means that for every τ and every sequence $(a^n)^\tau \to (a^0)^\tau$ (in $\mathbb{R}^m$), $\rho(x^n, x^0) \to 0$ implies

$$\rho\left(\int (a^n)^\tau_- \, dx^n, \int (a^0)^\tau_- \, dx^0\right) \to 0. \tag{10}$$

(At least metrics generating Skorohod's topologies J_1 and M_1 are consistent with convergence of elementary integrals.)

Let us say that ρ is *compatible with integration* if all of the above requirements are satisfied.

THEOREM 4. *Let the K^n, X^n and $\mathbb{Q}$ be as in Theorem* 1 *and let metric ρ on $\mathbb{D}$ be compatible with integration. Suppose $\{X^n\}$ fulfills condition UT, $\{K^n\}$ is uniformly S-tight and finite dimensional convergence* (5) *holds. If* (6) *is satisfied and $X^n \to_{\mathscr{D}} X^0$ on the space $(\mathbb{D}, \mathscr{O}_\rho)$, then on the same space,*

$$\int K^n_- \, dX^n \to_{\mathscr{D}} \int K^0_- \, dX^0. \tag{11}$$

REMARK 5. Suppose all processes are defined on the same probability space $(\Omega, \mathscr{F}, P)$ and in assumptions of Theorem 4 we replace relation (5) with

$$K^n(t) \to_{\mathscr{P}} K^0(t), \qquad X^n(t) \to_{\mathscr{P}} X^0(t), \qquad t \in \mathbb{Q} \tag{12}$$

and $X^n \to_{\mathscr{D}} X^0$ with

$$\lim_{n\to\infty} P(\rho(X^n, X^0) > \varepsilon) = 0, \qquad \varepsilon > 0. \tag{13}$$

Then it follows from the proof of Theorem 4 that (11) changes to

$$\lim_{n\to\infty} P\left(\rho\left(\int K^n_- \, dX^n, \int K^0_- \, dX^0\right) > \varepsilon\right) = 0, \qquad \varepsilon > 0. \tag{14}$$

In a similar way one can transform Theorem 1.

Dealing with metrics ρ generating topologies strictly finer than Skorohod's J_1 topology is difficult since such topologies (if of any interest) may be non-separable. Despite this, we have a result for the convergence uniformly in probability, generalizing Theorem 1.9 of Mémin and Słomiński (1991).

THEOREM 5. *Let K^n's, X^n's and $\mathbb{Q}$ be as in Theorem* 1, *with the additional assumption that all processes are defined on the same probability space*: $(\Omega^n, \mathscr{F}^n, P^n) = (\Omega, \mathscr{F}, P)$. *Suppose condition UT holds for X^n's, the sequence $\{K^n\}$ is uniformly S-tight*, (6) *is satisfied and*

$$\begin{aligned} K^n(t) &\to_{\mathscr{P}} K^0(t), \qquad t \in \mathbb{Q}, \\ \|X^n - X^0\|_\infty &\to_{\mathscr{P}} 0. \end{aligned} \tag{15}$$

Then also

$$\left\| \int K^n_- \, dX^n - \int K^0_- \, dX^0 \right\|_\infty \to_{\mathscr{P}} 0. \tag{16}$$

In fact, in Theorem 1 we have more than finite dimensional convergence only: the stochastic integrals already "functionally" converge with respect to an ultraweak topology on $\mathbb{D}$, introduced in Jakubowski (1994) and called there the S topology. This non-Skorohod sequential topology is not metrizable, but it is still good enough to build a satisfactory theory of convergence in distribution. In particular, we make the following observations:

1. $K \subset \mathbb{D}$ is S-relatively compact iff

$$\sup_{k\in K} \sup_{t\in[0,1]} |k(t)| \le C_K < +\infty, \qquad \sup_{k\in K} N_\eta(k) \le C_\eta < +\infty, \qquad \eta > 0. \tag{17}$$

2. The σ-field $\mathscr{B}_S$ of Borel subsets for S coincides with the usual σ-field generated by projections (or evaluations) on $\mathbb{D}$: $\mathscr{B}_S = \sigma(\pi_t\colon t \in [0,1])$.
3. The set $\mathscr{P}(\mathbb{D}, S)$ of S-*tight* probability measures is exactly the set of distributions of stochastic processes with trajectories in $\mathbb{D}$: $\mathscr{P}(\mathbb{D}, S) = \mathscr{P}(\mathbb{D})$, and for a family $\{K^n\}$ of stochastic processes, uniform tightness with respect to S coincides with the uniform S-tightness introduced at the beginning of this section.
4. The S topology is weaker than Skorohod's J_1 topology. Since the latter is Polish, S is Lusin in the sense of Fernique. Even more is true: $(\mathbb{D}, S)$ is a linear topological space and so is completely regular.
5. On $\mathscr{P}(\mathbb{D})$ there exists a unique sequential topology $\mathscr{O}(\overset{*}{\Rightarrow})$ (where $\overset{*}{\Rightarrow}$ denotes the convergence determining the topology) which is finer than the S-weak topology and for which $\overset{*}{\Rightarrow}$-relative compactness coincides with uniform S-tightness. In particular, for $\mathscr{O}(\overset{*}{\Rightarrow})$ both the direct and the converse Prohorov's theorems are valid.
6. Let $X_n \overset{*}{\to}_{\mathscr{D}} X_0$ mean that the laws of processes X_n converge in our sense: $\mathscr{L}(X_n) \overset{*}{\Rightarrow} \mathscr{L}(X_0)$. Suppose $X_n \overset{*}{\to}_{\mathscr{D}} X_0$. Then in each subsequence $\{X_{n_k}\}_{k\in\mathbb{N}}$ one can find a further subsequence $\{X_{n_{k_l}}\}_{l\in\mathbb{N}}$ such that:
   (a) $\{X_0\} \cup \{X_{n_{k_l}}\colon l = 1, 2, \ldots\}$ admits the usual a.s. Skorohod representation on $([0,1], \mathscr{B}_{[0,1]})$.

(b) Outside some *countable* subset of $[0, 1)$ all finite dimensional distributions of $\{X_{n_{k_l}}\}$ converge to those of X_0.

There are many S-continuous functionals. As examples, mappings of the form $\mathbb{D} \ni x \mapsto \int_0^1 \Phi(x(s))\, d\mu(s) \in \mathbb{R}^1$, where μ is a finite atomless measure on $[0, 1]$ and Φ is continuous, may serve.

THEOREM 6. *Under the assumptions of Theorem* 1 *we have*

$$\int K_-^n \, dX^n \to_{\mathcal{D}} \int K_-^0 \, dX^0$$

on the space $(\mathbb{D}, S)$.

We have noticed that uniform S-tightness implies convergence of finite dimensional distributions outside a countable subset of $[0, 1)$ and for some subsequence. By Remark 1, condition UT also possesses this property. Hence under tightness assumptions only, we can always extract a subsequence (K^{n_k}, X^{n_k}) and a dense set $\mathbb{Q}' \subset \mathbb{R}^1$, $1 \in \mathbb{Q}'$, for which joint finite dimensional convergence over $\mathbb{Q}'$ holds. It is, however, possible that $0 \notin \mathbb{Q}'$, and this fact may influence the convergence of stochastic integrals.

EXAMPLE 5. Let $k^n(t) = 1$, $n = 0, 1, 2, \ldots$, and let $x^n(t) = \mathbb{1}_{[1/n, 1]}(t)$, $n = 1, 2, \ldots$, and $x^0(t) \equiv 1$. Then all assumptions of Theorem 1 are satisfied, except that (5) holds for $\mathbb{Q}' = (0, 1]$. However, for any $t > 0$,

$$\left(\int k_-^n \, dx^n\right)(t) = x^n(t) \not\to 0 = \left(\int k_-^0 \, dx^0\right)(t).$$

[Recall that by our convention $(\int k_- \, dx)(t) = \int_{]0, t]} k(s-)\, dx(s)$.]

There is an easy way to overcome this difficulty. Let us consider an embedding $\mathbb{D}([0, 1]: \mathbb{R}^1) \ni x \mapsto \tilde{x} \in \mathbb{D}([-1, 1]: \mathbb{R}^1)$ given by

$$\tilde{x}(t) = \begin{cases} x(t), & \text{if } t \in [0, 1], \\ 0, & \text{if } t \in [-1, 0), \end{cases} \tag{18}$$

and let

$$\widetilde{\omega}''_\delta(k, x) = \omega''_\delta(\tilde{k}, \tilde{x}),$$

with ω''_δ redefined on $\mathbb{D}([-1, 1]: \mathbb{R}^1)$ in a natural manner.

THEOREM 7. *Let K^n and X^n be as in Theorem* 1. *Suppose that condition UT holds for $\{X^n\}$, the K^n are uniformly S-tight and*

$$\lim_{\delta \searrow 0} \limsup_{n \to \infty} P^n(\widetilde{\omega}''_\delta(K^n, X^n) > \varepsilon) = 0, \qquad \varepsilon > 0. \tag{19}$$

Then along some subsequence $\{n_k\}$,

$$\int K_-^{n_k} \, dX^{n_k} \to_{\mathcal{D}} \int K_-^0 \, dX^0 \tag{20}$$

on the space $(\mathbb{D}, S)$, *where* K^0 *has trajectories in* $\mathbb{D}$ *and* X^0 *is a semimartingale. If, in addition,* $\{X^n\}$ *is uniformly* $\mathscr{O}_\rho$*-tight for a metric* ρ *compatible with integration, then* (20) *may be strengthened to convergence on* $(\mathbb{D}, \mathscr{O}_\rho)$

REMARK 6. Theorem 7 may be viewed as a specific criterion of compactness for sets of stochastic integrals: the closure (in a suitable topology) still contains stochastic integrals only.

3. Proofs.

3.1. *Basic lemma.*

LEMMA 8. *Suppose* $\{K^n\}$ *is uniformly S-tight,* $\{X^n\}$ *satisfies condition UT and* (6) *holds. Then for any sequence* $\tau_m = \{0 = t_{m,0} < t_{m,1} < \cdots < t_{m,k_m} = 1\}$ *of partitions of* $[0, 1]$ *such that*

$$|\tau_m| = \max\{t_{m,k} - t_{m,k-1}\colon k = 1, 2, \ldots, k_m\} \to 0, \tag{21}$$

we have

$$\lim_{m\to\infty} \sup_n P\left(\sup_{t\in[0,1]}\left|\int (K^n)^{\tau_m}_- \, dX^n(t) - \int K^n_- \, dX^n(t)\right| > \varepsilon\right) = 0, \qquad \varepsilon > 0. \tag{22}$$

PROOF. Recall that $(K^n)^{\tau_m}$ is the discretization of K^n given by formula (4). If $n \in \mathbb{N}$ is fixed and $m \to \infty$, then by the dominated convergence theorem,

$$\lim_{m\to\infty} P\left(\sup_{t\in[0,1]}\left|\int (K^n)^{\tau_m}_- \, dX^n(t) - \int K^n_- \, dX^n(t)\right| > \varepsilon\right) = 0, \qquad \varepsilon > 0. \tag{23}$$

It follows that we may replace $\int K^n_- \, dX^n$ with $\int (K^n)^{\tau_{m_n}}_- \, dX^n$ if m_n is large enough. Summarizing, it is enough to prove that for each $\varepsilon > 0$ and m_n such that $|\tau_{m_n}| < |\tau_m|$,

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} P\left(\max_{t\in[0,1]}\left|\int (K^n)^{\tau_m}_- \, dX^n(t) - \int (K^n)^{\tau_{m_n}}_- \, dX^n(t)\right| > \varepsilon\right) = 0. \tag{24}$$

Let us fix $\eta > 0$, $n \in \mathbb{N}$, τ_m and τ_{m_n}, $|\tau_{m_n}| < |\tau_m|$. To make the formulas more readable, let us change the notation slightly and set

$$\tau_m = \{0 = t_0 < t_1 < \cdots < t_{k_m} = 1\},$$
$$\tau_{m_n} = \{0 = s_0 < s_1 < \cdots < s_{k_{m_n}} = 1\}.$$

For $k = 1, \ldots, k_m$ and $j = 0, 1, 2, \ldots$ define

$$\sigma_{k,0} = t_{k-1}, \tag{25}$$

$$\sigma_{k,j+1} = \min\{s_{l-1} \ge \sigma_{k,j}\colon |K^n(s_{l-1}) - K^n(\sigma_{k,j})| > \eta\} \wedge t_k. \tag{26}$$

(We use the convention that $\min\varnothing = 1$.) Then for $t \in [0, 1]$ we can decompose

$$\begin{aligned}
&\int (K^n)_-^{\tau_m}\, dX^n(t) - \int (K^n)_-^{\tau_{m_n}}\, dX^n(t) \\
&\quad = \sum_{i=1}^{k} \sum_{j=1}^{\infty} \sum_{\sigma_{i,\,j-1} \le s_{l-1} < \sigma_{i,\,j}} \big(K^n(\sigma_{i,\,0}) - K^n(s_{l-1})\big) \\
&\qquad\qquad \times \big(X^n(s_l \wedge t) - X^n(s_{l-1} \wedge t)\big).
\end{aligned}$$

Let us observe that

$$\begin{aligned}
&\sum_{\sigma_{i,\,j-1} \le s_{l-1} < \sigma_{i,\,j}} \big(K^n(\sigma_{i,\,0}) - K^n(s_{l-1})\big)\big(X^n(s_l \wedge t) - X^n(s_{l-1} \wedge t)\big) \\
&\quad = \sum_{\sigma_{i,\,j-1} \le s_{l-1} < \sigma_{i,\,j}} \big(K^n(\sigma_{i,\,0}) - K^n(\sigma_{i,\,j-1})\big)\big(X^n(s_l \wedge t) - X^n(s_{l-1} \wedge t)\big) \\
&\qquad + \sum_{\sigma_{i,\,j-1} \le s_{l-1} < \sigma_{i,\,j}} \big(K^n(\sigma_{i,\,j-1}) - K^n(s_{l-1})\big)\big(X^n(s_l \wedge t) - X^n(s_{l-1} \wedge t)\big) \\
&\quad = \big(K^n(\sigma_{i,\,0}) - K^n(\sigma_{i,\,j-1})\big)\big(X^n(\sigma_{i,\,j} \wedge t) - X^n(\sigma_{i,\,j-1} \wedge t)\big) \\
&\qquad + \sum_{\sigma_{i,\,j-1} \le s_{l-1} < \sigma_{i,\,j}} \big(K^n(\sigma_{i,\,j-1}) - K^n(s_{l-1})\big)\big(X^n(s_l \wedge t) - X^n(s_{l-1} \wedge t)\big).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\int (K^n)_-^{\tau_m}\, dX^n(t) - \int (K^n)_-^{\tau_{m_n}}\, dX^n(t) \\
&\quad = \sum_{i=1}^{k} \sum_{j=2}^{\infty} \big(K^n(\sigma_{i,\,0}) - K^n(\sigma_{i,\,j-1})\big)\big(X^n(\sigma_{i,\,j} \wedge t) - X^n(\sigma_{i,\,j-1} \wedge t)\big) \\
&\qquad + \sum_{i=1}^{k} \sum_{j=1}^{\infty} \sum_{\sigma_{i,\,j-1} \le s_{l-1} < \sigma_{i,\,j}} \big(K^n(\sigma_{i,\,j-1}) - K^n(s_{l-1})\big) \\
&\qquad\qquad \times \big(X^n(s_l \wedge t) - X^n(s_{l-1} \wedge t)\big) \\
&\quad = I_\eta^n(t) + J_\eta^n(t).
\end{aligned}$$

Using definitions of $N_\eta(\cdot)$ and $\omega''_\delta(\cdot,\cdot)$ given at the beginning of Section 2 and taking into account that for fixed ω in the sum $I_\eta^n(t, \omega)$ there are no more than $N_\eta(K^n(\omega))$ nonzero summands, we can estimate

$$\begin{aligned}
\sup_{t \in [0,1]} |I_\eta^n(t)| &\le N_\eta(K^n) \sup_{\substack{1 \le i \le k_m \\ j \in \mathbb{N},\, t \in [0,1]}} \big\{ \max\{|K^n(\sigma_{i,\,0}) - K^n(\sigma_{i,\,j-1})|, \\
&\qquad\qquad |X^n(\sigma_{i,\,j} \wedge t) - X^n(\sigma_{i,\,j-1} \wedge t)|\} \\
&\qquad \times \min\{|K^n(\sigma_{i,\,0}) - K^n(\sigma_{i,\,j-1})|, \\
&\qquad\qquad |X^n(\sigma_{i,\,j} \wedge t) - X^n(\sigma_{i,\,j-1} \wedge t)|\}\big\} \\
&\le N_\eta(K^n) 2\big(\|K^n\|_\infty + \|X^n\|_\infty\big)\omega''_{|\tau_m|}(K^n, X^n).
\end{aligned}$$

Since condition UT implies tightness of $\|X^n\|_\infty$ [see, e.g., Jakubowski, Mémin and Pagès (1989), Lemma 1.2] we see that when η is fixed and $|\tau_m| \to 0$, then $\sup_{t\in[0,1]} |I^n_\eta(t)|$ converges in probability to 0 uniformly in n.

It remains to prove that, by the choice of η, the random variables $\sup_{t\in[0,1]} |J^n_\eta(t)|$ can be made as small as desired (in probability, uniformly in n). However, the processes $\eta^{-1} J^n_\eta$ are elementary stochastic integrals appearing in the definition of condition UT. By Jakubowski, Mémin and Pagès [(1989) Lemma 1.1], the family of random variables $\{\sup_{t\in[0,1]} \eta^{-1}|J^n_\eta(t)|: n \in \mathbb{N}, \eta > 0\}$ is uniformly tight and we obtain the required property. $\square$

3.2. *Proof of Theorem* 1. If $\mathbb{Q}$ is not countable, replace it with its proper countable dense subset containing 0 and 1. Let us choose a sequence $\{\tau_m\}$ of partitions of $[0,1]$ such that $|\tau_m| \to 0$, $\tau_m \subset \tau_{m+1} \subset \mathbb{Q}$, $m = 1, 2, \dots,$ and $\cup_{m=1}^{\infty} \tau_m = \mathbb{Q}$.

Let $\mathbb{Q}_0 = \{q_1 < q_2 \cdots < q_r\} \subset \mathbb{Q}$ and let m be so large that $\mathbb{Q}_0 \subset \tau_m = \{0 = t_0 < t_1 < \cdots < t_{k_m}\}$ and

$$\sup_n P\left(\sup_{q\in\mathbb{Q}_0} \left| \int (K^n)^{\tau_m}_- \, dX^n(q) - \int K^n_- \, dX^n(q) \right| > \varepsilon \right) < \varepsilon \tag{27}$$

for $n = 0, 1, 2, \dots$ (the latter by Lemma 8). For each $t \in \tau_m$ the integral $\int (K^n)^{\tau_m}_- \, dX^n(t)$ is a continuous function of the vector

$$(K^n(0), X^n(0), K^n(t_1), X^n(t_1), \dots, K^n(t_{k_m}), X^n(t_{k_m})).$$

Hence (5) implies

$$\int (K^n)^{\tau_m}_- \, dX^n \to_{\mathscr{D}_f(\tau_m)} \int (K^0)^{\tau_m}_- \, dX^0$$

and by $\mathbb{Q}_0 \subset \tau_m$ also

$$\int (K^n)^{\tau_m}_- \, dX^n \to_{\mathscr{D}_f(\mathbb{Q}_0)} \int (K^0)^{\tau_m}_- \, dX^0. \tag{28}$$

The theorem follows now by (27) and (28). $\square$

3.3. *Proofs of Theorems* 3–5. We shall prove Theorem 4 first.

A variant of the Skorohod representation theorem is necessary.

LEMMA 9. *For each subsequence n_k there exists a further subsequence n_{k_l} and random elements $L^0, L^1, \dots$ with values in $\mathbb{R}^{\mathbb{Q}}$ and $Y^0, Y^1, \dots$ with values in $(\mathbb{D}, \mathscr{O}_\rho)$ and defined on the Lebesgue interval such that*

$$\big((L^0(q), Y^0(q))_{q\in\mathbb{Q}}, Y^0\big) \sim \big((K^0(q), X^0(q))_{q\in\mathbb{Q}}, X^0\big)$$

for each $l = 1, 2, \dots,$

$$\big((L^l(q), Y^l(q))_{q\in\mathbb{Q}}, Y^l\big) \sim \big((K^{n_{k_l}}(q), X^{n_{k_l}}(q))_{q\in\mathbb{Q}}, X^{n_{k_l}}\big)$$

and for almost every $\omega \in [0,1]$,

$$L^l(q,\omega) \to L^0(q,\omega), \qquad Y^l(q,\omega) \to Y^0(q,\omega), \qquad q \in \mathbb{Q},$$

and

$$\rho(Y^l(\omega), Y^0(\omega)) \to 0.$$

PROOF. We have separate information on joint finite dimensional convergence and convergence on $(\mathbb{D}, \mathscr{O}_\rho)$. By tightness of both components, in each subsequence we may extract a further subsequence such that the joint convergence holds. Let (U^l, V^l, Z^l) be the Skorohod representation for such a subsequence. In particular, we have

$$(V^l, Z^l) \sim \big(h(X^{n_{k_l}}), X^{n_{k_l}}\big), \tag{29}$$

where $h\colon \mathbb{D} \to \mathbb{R}^{\mathbb{Q}}$ is a measurable mapping given by $h(x) = (x(q))_{q\in\mathbb{Q}}$. Hence (29) implies that $V^l = h(Z^l)$ l-a.s. and so the lemma follows. $\square$

Because of Lemma 9 we may and do assume that $\rho(X^n(\omega), X^0(\omega)) \to 0$ and that $K^n(q, \omega) \to K^0(q, \omega)$, $X^n(q, \omega) \to X^0(q, \omega)$, $q \in \mathbb{Q}$. Using the consistency of ρ with respect to elementary integrals we get for each fixed τ_m,

$$\rho\bigg(\int (K^n)_-^{\tau_m}\, dX^n, \int (K^0)_-^{\tau_m}\, dX^0\bigg) \to 0 \quad \text{a.s.}$$

Further, by (9) and (22) we have for each $\varepsilon > 0$ and as $m \to \infty$,

$$\begin{aligned}\sup_{n\in\mathbb{N}} P\bigg(\rho\bigg(\int (K^n)_-^{\tau_m}\, dX^n, \int K_-^n\, dX^n\bigg) > \varepsilon\bigg)&\\ \le \sup_{n\in\mathbb{N}} P\bigg(\bigg\|\int (K^n)_-^{\tau_m}\, dX^n - \int K_-^n\, dX^n\bigg\|_\infty > \frac{\varepsilon}{C}\bigg) &\to 0.\end{aligned}$$

Similarly, by (23)

$$\begin{aligned}&P\bigg(\rho\bigg(\int (K^0)_-^{\tau_m}\, dX^0, \int K_-^0\, dX^0\bigg) > \varepsilon\bigg)\\ &\quad\le P\bigg(\bigg\|\int (K^0)_-^{\tau_m}\, dX^0 - \int K_-^0\, dX^0\bigg\|_\infty > \tfrac{\varepsilon}{C}\bigg) \to 0 \qquad \text{as } m \to \infty, \varepsilon > 0.\end{aligned}$$

Hence $\int K_-^{n_{k_l}}\, dX^{n_{k_l}} \to_{\mathscr{D}} \int K_-^0\, dX^0$ on $(\mathbb{D}, \mathscr{O}_\rho)$. This concludes the proof, for n_{k_l} was a subsequence of an *arbitrary* subsequence n_k. $\square$

The proof of Theorem 5 is essentially the same (except it does not require reduction via the a.s. Skorohod representation).

3.4. *Proof of Theorem* 6. It has been proved in Jakubowski (1994) that the convergence $\overset{*}{\Rightarrow}$ in $\mathscr{P}(\mathbb{D})$ (induced by the S topology) is "functional" in our sense. That is, finite dimensional convergence and relative $\overset{*}{\Rightarrow}$-compactness imply $\overset{*}{\Rightarrow}$-convergence. Hence in view of (7) it suffices to prove S-uniform tightness of $\{\int K_-^n\, dX^n\}$. Using Lemma 1.6 of Mémin and Słomiński (1991), we check that the processes $\{\int K_-^n\, dX^n\}$ satisfy condition UT. Now the result already mentioned due to Stricker (1985) (see Remark 1) gives us S-uniform tightness of $\{\int K_-^n\, dX^n\}$ and finishes the proof of Theorem 6. $\square$

3.5. *Proof of Theorem* 7. Let us embed our processes into the space $\mathbb{D}([-1,1]\colon \mathbb{R}^1)$ via mapping (18). Then we have finite dimensional convergence on $\mathbb{Q}_1 = \mathbb{Q} \cup [-1, 0)$. Since the left end of the interval of integration belongs to $\mathbb{Q}_1$, it is possible to apply the previous results. It remains to observe that

$$\int_{]-1,\,t]} K^n(s-)\,dX^n(s) = \int_{]0,\,t]} K^n(s-)\,dX^n(s),\ \ n = 1, 2, \dots . \qquad \square$$

Acknowledgments. I thank Jean Mémin and François Coquet for their hospitality during my stays in Rennes. I should also mention that the alternative way of proving Theorem 3 (as shown in Remark 3) was suggested to me by an anonymous referee.

REFERENCES

AVRAM, F. and TAQQU, M. (1992). Weak convergence of sums of moving averages in the α-stable domain of attraction. *Ann. Probab.* **20** 483–503.

DELLACHERIE, C. and MEYER, P. A. (1980). *Probabilités et Potentiel* **2**. Hermann, Paris.

JAKUBOWSKI, A. (1994). A non-Skorohod topology on the Skorohod space. Unpublished manuscript.

JAKUBOWSKI, A., MÉMIN, J. and PAGÈS, G. (1989). Convergence en loi des suites d'intégrales stochastiques sur l'espace $\mathbb{D}^1$ de Skorokhod. *Probab. Theory Related Fields* **81** 111–137.

JANICKI, A. and WERON, A. (1994). *Simulation and Chaotic Behaviour of α-Stable Stochastic Processes*. Marcel Dekker, New York.

KURTZ, T. (1991). Random time changes and convergence in distribution under the Meyer–Zheng conditions. *Ann. Probab.* **19** 1010–1034.

KURTZ, T. and PROTTER, P. (1991). Weak limit theorems for stochastic integrals and stochastic differential equations. *Ann. Probab.* **19** 1035–1070.

MÉMIN, J. and SŁOMIŃSKI, L. (1991). Condition UT et stabilité en loi des solutions d'equations différentieles stochastiques. *Séminaires de Probabilités XXV. Lecture Notes in Math.* **1485** 162–177. Springer, Berlin.

MEYER, P. A. and ZHENG, W. A. (1984). Tightness criteria for laws of semimartingales. *Ann. Inst. H. Poincaré Probab. Statist.* **20** 353–372.

SKOROHOD, A. V. (1956). Limit theorems for stochastic processes. *Theory Probab. Appl.* **1** 261–290.

SŁOMIŃSKI, L. (1989). Stability of strong solutions of stochastic differential equations. *Stochastic Process. Appl.* **31** 173–202.

SŁOMIŃSKI, L. (1994). Stability of stochastic differential equations driven by general semimartingales. Unpublished manuscript.

STRICKER, C. (1985). Lois de semimartingales et critères de compacité. *Séminaires de Probabilités XIX. Lecture Notes in Math.* **1123**. Springer, Berlin.

TOPSØE, F. (1969). A criterion for weak convergence of measures with an application to convergence of measures on $\mathbb{D}[0,1]$. *Math. Scand.* **25** 97–104.

NICHOLAS COPERNICUS UNIVERSITY
FACULTY OF MATHEMATICS AND INFORMATICS
UL. CHOPINA 12/18
87-100 TORUŃ
POLAND
E-MAIL: adjakubo@mat.uni.torun.pl