The Annals of Probability
1998, Vol. 26, No. 3, 1000–1015

SUPERDIFFUSIVE BEHAVIOR OF TWO-DIMENSIONAL BROWNIAN MOTION IN A POISSONIAN POTENTIAL

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We consider d-dimensional Brownian motion in a truncated Poissonian potential conditioned to reach a remote location. If Brownian motion starts at the origin and ends in an hyperplane at distance L from the origin, the transverse fluctuation of the path is expected to be of order L^{ξ}. We are interested in a lower bound for ξ. We first show that $\xi \geq 1/2$ in dimensions $d \geq 2$ and then we prove superdiffusive behavior for $d = 2$, resulting in $\xi \geq 3/5$.

0. Introduction. In the present work we want to focus on a special model in the theory of random motions in a random potential. Throughout this paper we consider Brownian motion in a truncated Poissonian potential. The Brownian motion will start at the origin and will be stopped when reaching a fixed hyperplane at distance L from 0 (see, for instance, [8]). Our purpose is to study certain fluctuation properties of the paths when they feel the presence of a truncated Poissonian potential, as $L \to \infty$.

Let $\mathbb{P}$ stand for the Poisson law with fixed intensity $\nu > 0$ on the space Ω of simple pure point measures ω on $\mathbb{R}^d$, $d \geq 2$. The soft obstacles are generated by a fixed shape function $W(\cdot) \geq 0$, which is measurable, bounded, compactly supported and not a.e. equal to zero. Furthermore we assume that

$$W(\cdot) \text{ is rotationally invariant.} \tag{0.1}$$

For $\omega = \sum_i \delta_{x_i} \in \Omega$ and $x \in \mathbb{R}^d$, we define the truncated Poissonian potential with fixed truncation level $M > 0$ as

$$V(x, \omega) = \left(\sum_i W(x - x_i)\right) \wedge M = \left(\int_{\mathbb{R}^d} W(x - y)\omega(dy)\right) \wedge M. \tag{0.2}$$

Our aim is to have a penalty on the Brownian path when it experiences the potential. For $x \in \mathbb{R}^d$ we denote by P_x the Wiener measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ starting at x, and by $Z_.$ the canonical process on $C(\mathbb{R}_+, \mathbb{R}^d)$. For $\theta \in [0, 2\pi)$, $L > 0$, let $\Lambda(\theta, L)$ be the half-space

$$\Lambda(\theta, L) = \{x \in \mathbb{R}^d; \langle x, \hat{x}(\theta)\rangle \geq L\}, \tag{0.3}$$

where $\hat{x}(\theta) = (\cos\theta, \sin\theta, 0, \ldots, 0)$, and where $\langle\cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^d$. If $\theta = 0$, we write Λ_L for $\Lambda(0, L)$. For $\lambda \geq 0$, $L > 0$,

Received May 1997.
AMS 1991 *subject classifications*. 60K35, 82D30.
Key words and phrases. Brownian motion, Poissonian potential, fluctuation, superdiffusivity.

$\theta \in [0, 2\pi)$ and $\omega \in \Omega$ the new path measure on $C(\mathbb{R}_+, \mathbb{R}^d)$ is then defined by

$$d\hat{P}_0^{\partial\Lambda(\theta, L)} = \frac{1}{e_\lambda(0, \partial\Lambda(\theta, L), \omega)} \exp\left\{-\int_0^{H(\partial\Lambda(\theta, L))} (\lambda + V)(Z_s, \omega)\, ds\right\} dP_0, \tag{0.4}$$

where $e_\lambda(0, \partial\Lambda(\theta, L), \omega)$ is the normalizing constant and where $H(\partial\Lambda(\theta, L)) = \inf\{s \geq 0, Z_s \in \Lambda(\theta, L)\}$ is the entrance time of $Z_\cdot(w)$ in the half-space $\Lambda(\theta, L)$.

From [8], Corollary 1.9, we know that we have a shape theorem: on a set of full $\mathbb{P}$-measure for $\lambda \geq 0$,

$$\lim_{L\to\infty} \frac{1}{L} \log e_\lambda(0, \partial\Lambda(\theta, L)) = -\alpha_\lambda(\hat{x}(\theta)), \tag{0.5}$$

and the Lyapounov coefficients $\alpha_\lambda(\cdot)$ are norms on $\mathbb{R}^d$. In the present article we restrict ourselves to rotationally invariant $W(\cdot)$. Hence the Lyapounov coefficients are proportional to the Euclidean norm. This gives us the possibility to compare $-\log e_\lambda(0, \partial\Lambda(\theta, L))$ to the Euclidean norm.

Our main interest in this article is to study transverse fluctuations. Two natural questions arise: (1) "Where do the paths end in $\partial\Lambda(\theta, L)$?" (2) If we examine the line through the origin perpendicular to $\partial\Lambda(\theta, L)$, "How far do the paths fluctuate from this line before they hit the goal?" These two questions have a similar flavor to questions studied in [6] in the context of first-passage percolation.

For the first question we define a critical exponent $\xi^{(1)}$; it describes the behavior of the hitting distribution of the hyperplane $\partial\Lambda(\theta, L)$ for large L. It is a scale for the concentration of the hitting distribution of $\partial\Lambda(\theta, L)$ for the perturbed Brownian motion. For $\theta \in [0, 2\pi)$, $L > 0$ and $\gamma > 0$, we denote by $B_\theta(L, \gamma)$ the event that the hitting point $Z_{H(\partial\Lambda(\theta, L))}$ is within distance L^γ of $L\hat{x}(\theta) \in \partial\Lambda(\theta, L)$. That is (see Figure 1),

$$B_\theta(L, \gamma) = \left\{w \in C(\mathbb{R}_+, \mathbb{R}^d); \operatorname{dist}\left(Z_{H(\partial\Lambda(\theta, L))}, L\hat{x}(\theta)\right) \leq L^\gamma\right\}. \tag{0.6}$$

The first critical exponent $\xi^{(1)}$ is then defined as follows (thanks to the rotational invariance of our model we can restrict ourselves to the angle $\theta = 0$):

$$\xi^{(1)} = \inf\left\{\gamma \geq 0; \limsup_{L\to\infty} \mathbb{E}\left[\hat{P}_0^{\partial\Lambda_L}[B_0(L, \gamma)]\right] = 1\right\}. \tag{0.7}$$

Observe that $\mathbb{E}[\hat{P}_0^{\partial\Lambda_L}[Z_{H(\partial\Lambda_L)} \in \cdot]]$ is symmetric around $L\hat{x}(0) \in \partial\Lambda_L$; therefore it is meaningful to use symmetric sets around $L\hat{x}(0) \in \partial\Lambda_L$ in the definition (0.6).

For the second question we consider the following construction: for $\theta \in [0, 2\pi)$, $L > 0$ and $\gamma > 0$, we consider l_θ the line $\{\alpha\hat{x}(\theta) \in \mathbb{R}^d; \alpha \in \mathbb{R}\}$ and the truncated cylinder of radius L^γ and symmetry axis l_θ, $Z(\hat{x}(\theta), L^\gamma) = \{z \in \Lambda(\theta, -L); \operatorname{dist}(z, l_\theta) \leq L^\gamma\}$. The truncation is there for technical purposes. The boundary of $Z(\hat{x}(\theta), L^\gamma)$ is denoted by $\partial Z(\hat{x}(\theta), L^\gamma)$ (see Figure 1).

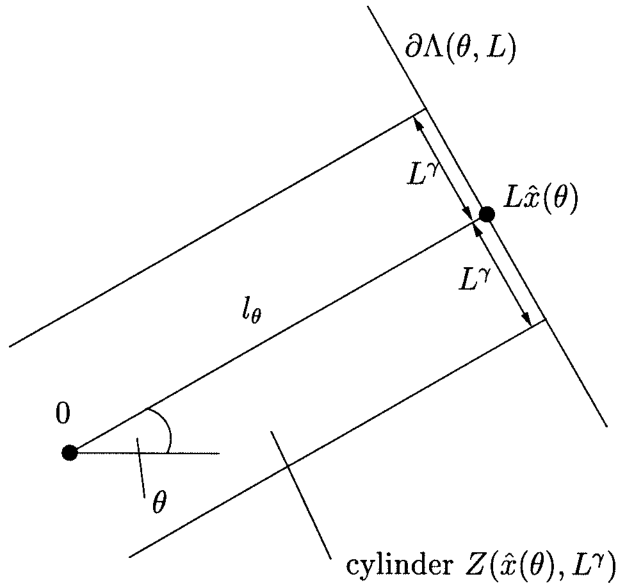

FIG. 1. *Construction of $A_\theta(L, \gamma)$ and $B_\theta(L, \gamma)$.*

Now $A_\theta(L, \gamma)$ is the event that the perturbed Brownian path starting at the origin with goal $\partial\Lambda(\theta, L)$ does *not* leave the cylinder $Z(\hat{x}(\theta), L^\gamma)$. That is,

$$A_\theta(L, \gamma) = \big\{w \in C(\mathbb{R}_+, \mathbb{R}^d); Z_s(w) \in Z(\hat{x}(\theta), L^\gamma) \text{ for all } s \le H(\partial\Lambda(\theta, L))\big\}. \tag{0.8}$$

The second critical exponent is then defined via

$$\xi^{(2)} = \inf\Big\{\gamma \ge 0;\ \limsup_{L\to\infty} \mathbb{E}\big[\hat{P}_0^{\partial\Lambda_L}[A_0(L, \gamma)]\big] = 1\Big\}. \tag{0.9}$$

Since $A_\theta(L, \gamma)$ is a subset of $B_\theta(L, \gamma)$, we have

$$\xi^{(2)} \ge \xi^{(1)}. \tag{0.10}$$

Our main results are the following theorems.

THEOREM 0.1. *In dimensions $d \ge 2$,*

$$\xi^{(1)} \ge 1/2. \tag{0.11}$$

Using (0.10) we already see that $\xi^{(2)} \ge 1/2$ for all dimensions $d \ge 2$. In fact, the following holds.

THEOREM 0.2. *If $d = 2$ and $\lambda > 0$,*

$$\xi^{(2)} \ge 3/5. \tag{0.12}$$

The arguments of the proof of this superdiffusive lower bound (Theorem 0.2) also work in general dimensions $d \geq 3$ (see the proofs of Lemmas 2.1 and 2.2 below). But in this case we do not get a superdiffusive lower bound. For $d \geq 3$, $\xi^{(2)} \geq 3/(d+3)$, which is not new in view of (0.10) and (0.11).

Let us briefly describe the main ideas used to prove the two theorems. At the heart of both proofs lies the following geometric construction. We look at the two hyperplanes $\partial\Lambda(0, L)$ and $\partial\Lambda(\theta_L, L)$ where the choice of the order γ of the angle $\theta_L = L^{-\gamma}$ will be the crucial step in the argument (see Figure 2). We prove Theorem 0.1 by contradiction along the following lines. If $\xi^{(1)}$ were less than 1/2, then the hitting distribution of the hyperplane would be too concentrated. To see this we look at two hyperplanes $\partial\Lambda(0, L)$ and $\partial\Lambda(\theta_L, L)$, with $\theta_L = 8L^{-1/2-\tilde{\varepsilon}}$ where $\tilde{\varepsilon} > 0$ is small [see (1.4) and (1.6) below]. In each of the two "goals" we consider sets $G_0(L) \subset \partial\Lambda(0, L)$ and $G_{\theta_L}(L) \subset \partial\Lambda(\theta_L, L)$, where the hitting probabilities of $Z_{H(\partial\Lambda(\cdot, L))}$ are concentrated. On the one hand the distance between $G_0(L)$ and $G_{\theta_L}(L)$ is very large (it is of order $L^{1/2-\tilde{\varepsilon}}$), but on the other hand $G_0(L)$ [resp., $G_{\theta_L}(L)$] is very close to $\partial\Lambda(\theta_L, L)$ [resp., $\partial\Lambda(0, L)$]. Indeed, with our choice of θ_L, $G_0(L)$ is within a distance $cL^{-2\tilde{\varepsilon}}$ of $\partial\Lambda(\theta_L, L)$ for large L. This leads to a contradiction because it doesn't cost enough to go from $G_0(L)$ to $\partial\Lambda(\theta_L, L)$, that is, the probablity of hitting $\partial\Lambda(\theta_L, L)$ outside of $G_{\theta_L}(L)$ is too large.

The proof of Theorem 0.2 will use the same geometric construction as above but θ_L is chosen to have a different order. Assuming $\xi^{(2)}$ is less than 3/5, we choose $\theta_L = L^{-(1-\gamma)}$ for $\gamma \in (\xi^{(2)}, 3/5)$ [see (2.2)]. Thus the convergence of the angle θ_L to zero is slower than in the first construction. This choice of θ_L enables us to derive an upper and a lower bound on $\mathbb{V}\mathrm{ar}(-\log e_\lambda(0, \partial\Lambda(0, L))+$

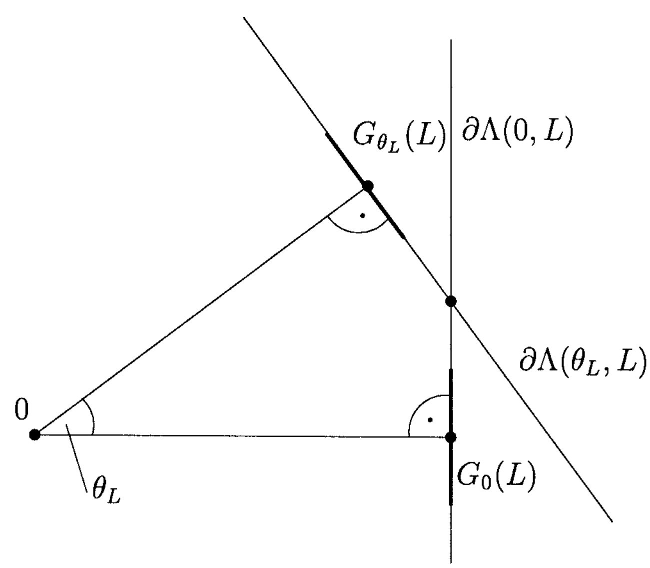

FIG. 2. *The two hyperplanes $\partial\Lambda(0, L)$ and $\partial\Lambda(\theta_L, L)$.*

$\log e_\lambda(0, \partial\Lambda(\theta_L, L)))$. In Lemma 2.1 below we get an upper bound due to the fact that the two cylinders (containing "most" of the paths) are close enough, that is, the distance between the concentration sets $G_0(L)$ and $\partial\Lambda(\theta_L, L)$ is of order $L^{2\gamma-1}$. On the other hand Lemma 2.2 below provides a lower bound relying on the fact that the two cylinders are disjoint enough. This bound is proved by a martingale argument. Comparing the exponents of the lower and the upper bound will provide the superdiffusive claim.

It is worth pointing out that the martingale argument gives a lower bound on $\mathbb{V}\mathrm{ar}(-\log e_\lambda(0, \partial\Lambda(0, L)) + \log e_\lambda(0, \partial\Lambda(\theta_L, L)))$ of the same order $L^{1-\gamma}$ (see Lemma 2.2 below) as the lower bound on $\mathbb{V}\mathrm{ar}(-\log e_\lambda(0, \partial\Lambda(0, L)))$ obtained by the same martingale technique (see Theorem 1.2 of [10]). It is a natural question whether the above variance of the difference itself has the same order as $\mathbb{V}\mathrm{ar}(-\log e_\lambda(0, \partial\Lambda(0, L)))$.

As mentioned above, the results we present have a similar flavor to questions which have been studied in the context of first-passage percolation. However in first-passage percolation on the square lattice the rotational invariance is lost. This turns out to be a serious problem and one has to choose "well-behaved" directions to prove these statements. In our model the directions do not play any role and the geometric ideas are more transparent.

Let us give a general remark on the conjectured behavior of ξ. In the physics literature questions related to fluctuations of growing surfaces and questions about transverse fluctuations have been extensively analyzed (see, e.g., [5]). For a broad variety of models, whose exponent ξ should have the same value as in our model, physics literature predicts $\xi = 2/3$ in dimension $d = 2$ (see [1, 3, 2, 4]), whereas in higher dimensions there are conflicting predictions, but nevertheless ξ should be $\geq 1/2$ (for a discussion see [6]). In [10] we have studied a slightly differently defined critical exponent ξ for a point-to-point model. For ξ we have found an upper bound of 3/4 for all dimensions $d \geq 2$ (Theorem 1.1 in [10]). These bounds on ξ, $\xi^{(1)}$ and $\xi^{(2)}$ are a first approach to the expected behavior of ξ. In the point-to-plane model the geometry is somewhat easier to control. The arguments we use here to prove the diffusive and the superdiffusive lower bound do not work in the point-to-point model. In fact in Theorem 1.3 of [10], we have proved a weaker (subdiffusive) lower bound for the point-to-point model. The point-to-plane model here has a somewhat easier geometric behavior. Using curvature properties we manage to improve the upper bound on the variance [compare (4.2) of [10] to Lemma 2.1], whereas the technique to prove the lower bound on the variance is the same [see (4.3) of [10] and Lemma 2.2].

We close this section with some remarks on how this article is organized and on the notation we use. In Section 1 we prove the diffusive statement and in Section 2 the superdiffusive lower bound. We usually denote positive constants by $c_1, c_2, \ldots$. These constants will depend only on the invariant parameters of our model, namely d, ν, $W(\cdot)$, M and λ.

1. The diffusive lower bound.

In this section we want to prove Theorem 0.1. As mentioned in the introduction, we will see that the hitting distribution of the hyperplane can not be too concentrated. During the proof we will also see the real advantage of using the point-to-plane model. It is an open problem to translate this proof to the point-to-point model.

Before we start with the proof, we introduce some notation for the concentration set: for $\theta \in [0, 2\pi)$, $L > 0$ and $\gamma > 0$, we consider a subset $G_\theta(L, \gamma)$ of $\partial\Lambda(\theta, L)$, which is of diameter $2L^\gamma$ and which is symmetric around $L\hat{x}(\theta) \in \partial\Lambda(\theta, L)$:

$$G_\theta(L, \gamma) = \partial\Lambda(\theta, L) \cap Z(\hat{x}(\theta), L^\gamma). \tag{1.1}$$

When $\theta = 0$, we drop the subscript θ; of course,

$$B_\theta(L, \gamma) = \{Z_{H(\partial\Lambda(\theta, L))} \in G_\theta(L, \gamma)\}. \tag{1.2}$$

PROOF OF THEOREM 0.1. We will prove this theorem by contradiction. Suppose

$$\xi^{(1)} < 1/2. \tag{1.3}$$

We choose

$$\gamma = 1/2 - \tilde{\varepsilon} \in (\xi^{(1)}, 1/2). \tag{1.4}$$

Therefore by the definition of $\xi^{(1)}$ we have that there exists a sequence $(L_n)_n$ with L_n tending to infinity as $n \to \infty$, such that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\mathbb{E}\big[\hat{P}_0^{\partial\Lambda_{L_n}}\big[Z_{H(\partial\Lambda_{L_n})} \in G(L_n, \gamma)\big]\big] \geq 1 - \varepsilon, \tag{1.5}$$

where $G(L_n, \gamma)$ is the deterministic set in the hyperplane $\partial\Lambda(0, L_n)$ defined in (1.1). We define

$$\theta_{L_n} = 8L_n^{-1/2-\tilde{\varepsilon}}, \tag{1.6}$$

$\Lambda_n = \Lambda(0, L_n)$, $\Lambda'_n = \Lambda(\theta_{L_n}, L_n)$, $G_n = G_0(L_n, \gamma)$ and $G'_n = G_{\theta_{L_n}}(L_n, \gamma)$. Here $\mathscr{P}_n$ is the intersection of Λ_n^c, $(\Lambda'_n)^c$, $\Lambda^c(-\pi/2, 0)$ and $\Lambda^c(\theta_{L_n} + \pi/2, 0)$. Then $\mathscr{P}_n$ intersected with the 1-2-hyperplane (called P_n) is a polygon with the following properties (see Figure 3): We want to enumerate its vertices as 0, z_1, z_2, z_3 in a counterclockwise way so that 0 is the origin; P_n is a convex polygon with

$$\mathrm{dist}(0, z_1) = \mathrm{dist}(0, z_3) = L_n,$$

and

$$\mathrm{dist}(z_1, z_2) = \mathrm{dist}(z_2, z_3) = L_n \tan(\theta_{L_n}/2) \geq 2L_n^{1/2-\tilde{\varepsilon}} \quad \text{for all large } n.$$

Therefore, because $\mathrm{diam}(G_n)$ and $\mathrm{diam}(G'_n)$ are equal to $L_n^{1/2-\tilde{\varepsilon}}$ for all n (where $\mathrm{diam}(\cdot)$ is the diameter of the set $\cdot$ restricted to the 1-2-hyperplane) and because G_n (resp., G'_n) is symmetric with respect to z_1 (resp., z_3), we know that

$$\mathrm{dist}(G_n, G'_n) \geq \sqrt{2}L_n^{1/2-\tilde{\varepsilon}} \quad \text{for all large } n \tag{1.7}$$

and

$$\left.\begin{array}{ll} \forall x \in G_n, & \mathrm{dist}(x, \partial\Lambda'_n) \\ \forall x' \in G'_n, & \mathrm{dist}(x', \partial\Lambda_n) \end{array}\right\} \leq 9L_n^{1/2-\tilde{\varepsilon}} \sin(\theta_{L_n}) \leq 72L_n^{-2\tilde{\varepsilon}} \tag{1.8}$$

for all large n.

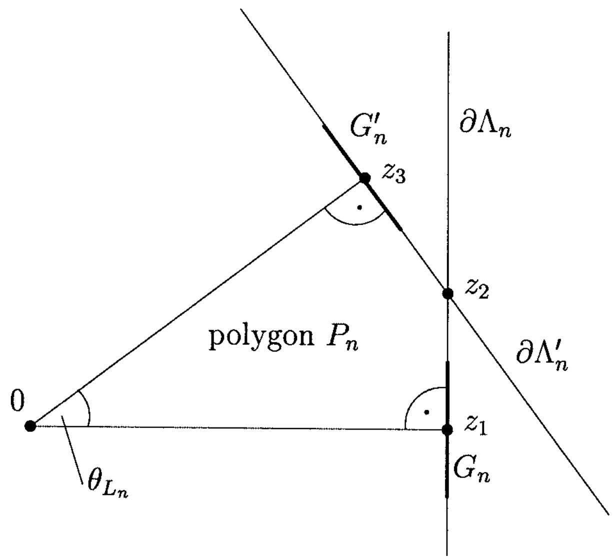

FIG. 3. *Polygon P_n.*

So the distance between G_n and G_n' tends to infinity, but G_n is very close to $\partial\Lambda_n'$ and G_n' is very close to $\partial\Lambda_n$.

Define $\Omega_n = \{\omega \in \Omega; e_\lambda(0, \partial\Lambda_n, \omega) \le e_\lambda(0, \partial\Lambda_n', \omega)\}$. By symmetry we have $\mathbb{P}[\Omega_n] \ge 1/2$ for all n. We define for $\omega \in \Omega$ and $n \in \mathbb{N}$,

$$f_n(\omega) = 1_{\Omega_n}(\omega)\frac{e_\lambda(0, \partial\Lambda_n, \omega)}{e_\lambda(0, \partial\Lambda_n', \omega)} \le 1. \tag{1.9}$$

Further we define

$$\tilde{\delta} = P_0\big[H(\partial\Lambda(0, 1/2)) \le 1\big] > 0. \tag{1.10}$$

Now we try to find a contradiction for arbitrarily large n. In Lemma 1.1 we prove that for all $\varepsilon \in (0, 1/2)$ we have that $\mathbb{E}[\hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda_n')} \in G_n']f_n(\omega)] > \exp\{-(\lambda + M)\}(1/2 - \varepsilon)\tilde{\delta}$ for all large n. This is because the angle θ_{L_n} tends quickly to zero. In Lemma 1.2 we prove that for all $\varepsilon > 0$ we have that $\mathbb{E}[\hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda_n')} \in G_n']f_n(\omega)] < \varepsilon$ for all large n. This is so because the distance between the goals G_n and G_n' is big for large n. Combining these two lemmas leads to a contradiction. This completes the proof of Theorem 0.1. $\square$

LEMMA 1.1. *Assume (1.3), then for all $\varepsilon \in (0, 1/2)$ there exists $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$, we have that*

$$\mathbb{E}\big[\hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda_n')} \in G_n']f_n(\omega)\big] > \exp\{-(\lambda + M)\}(1/2 - \varepsilon)\tilde{\delta}. \tag{1.11}$$

PROOF. We have

$$\hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda'_n)} \in G'_n] \tag{1.12}$$
$$\geq \hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda'_n)} \in G'_n, H(\partial\Lambda_n) \leq H(\partial\Lambda'_n) < \infty]$$
$$+ \hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda'_n)} \in G'_n, H(\partial\Lambda'_n) < H(\partial\Lambda_n) \leq H(\partial\Lambda'_n) + 1].$$

Considering the first term on the right-hand side of (1.12), we find

$$\hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda'_n)} \in G'_n, H(\partial\Lambda_n) \leq H(\partial\Lambda'_n) < \infty]$$
$$\geq \frac{1}{e_\lambda(0, \partial\Lambda_n)} E_0\bigg[\exp\bigg\{-\int_0^{H(\partial\Lambda'_n)} (\lambda + V)(Z_s)\,ds\bigg\},$$
$$Z_{H(\partial\Lambda'_n)} \in G'_n, H(\partial\Lambda_n) \leq H(\partial\Lambda'_n) < \infty\bigg],$$

whereas for the second term on the right-hand side of (1.12), we have the following lower bound (using the strong Markov property):

$$\hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda'_n)} \in G'_n, H(\partial\Lambda'_n) < H(\partial\Lambda_n) \leq H(\partial\Lambda'_n) + 1]$$
$$\geq \frac{\exp\{-(\lambda + M)\}}{e_\lambda(0, \partial\Lambda_n)} \inf_{z \in G'_n} P_z[H(\partial\Lambda_n) \leq 1]$$
$$\times E_0\bigg[\exp\bigg\{-\int_0^{H(\partial\Lambda'_n)} (\lambda + V)(Z_s)\,ds\bigg\},$$
$$Z_{H(\partial\Lambda'_n)} \in G'_n, H(\partial\Lambda'_n) < H(\partial\Lambda_n) < \infty\bigg].$$

Adding these two estimates, and using the fact that for all large n the distance between $z \in G'_n$ and $\partial\Lambda_n$ is less than 1/2, we find [using rotational invariance and (1.5)],

$$\mathbb{E}[\hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda'_n)} \in G'_n] f_n(\omega)]$$
$$\geq \exp\{-(\lambda + M)\}\tilde{\delta}$$
$$\times \mathbb{E}\bigg[\frac{1}{e_\lambda(0, \partial\Lambda_n)} E_0\bigg[\exp\bigg\{-\int_0^{H(\partial\Lambda'_n)} (\lambda + V)(Z_s)\,ds\bigg\},$$
$$Z_{H(\partial\Lambda'_n)} \in G'_n, H(\partial\Lambda'_n) < \infty\bigg] f_n\bigg] \tag{1.13}$$
$$\geq \exp\{-(\lambda + M)\}\tilde{\delta}\mathbb{E}[\hat{P}_0^{\partial\Lambda'_n}[Z_{H(\partial\Lambda'_n)} \in G'_n] 1_{\Omega_n}]$$
$$\geq \exp\{-(\lambda + M)\}\tilde{\delta}(\mathbb{E}[\hat{P}_0^{\partial\Lambda'_n}[Z_{H(\partial\Lambda'_n)} \in G'_n]] - \mathbb{E}[\Omega_n^c])$$
$$\geq \exp\{-(\lambda + M)\}\tilde{\delta}(1 - \varepsilon - 1/2) \quad \text{for all large } n.$$

This completes the proof of the lemma. □

LEMMA 1.2. *Assume (1.3). For any $\varepsilon > 0$ there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have that*

$$\mathbb{E}\big[\hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda'_n)} \in G'_n] f_n(\omega)\big] < \varepsilon. \tag{1.14}$$

PROOF. Using (1.5), we have for all large n,

$$\begin{aligned} &\mathbb{E}\big[\hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda'_n)} \in G'_n] f_n(\omega)\big] \\ &\quad \leq \mathbb{E}\big[\hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda_n)} \in G_n,\ Z_{H(\partial\Lambda'_n)} \in G'_n] f_n(\omega) + \hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda_n)} \notin G_n]\big] \\ &\quad \leq \mathbb{E}\big[\hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda_n)} \in G_n,\ Z_{H(\partial\Lambda'_n)} \in G'_n] f_n(\omega)\big] + \varepsilon/3. \end{aligned} \tag{1.15}$$

We split the first term on the right-hand side of (1.15) into two terms with respect to the event $\{H(\partial\Lambda_n) \leq H(\partial\Lambda'_n)\}$ (resp., $\{H(\partial\Lambda_n) > H(\partial\Lambda'_n)\}$). Using the strong Markov property, and the fact that $Z_.$ behaves as the unperturbed Brownian motion after time $H(\partial\Lambda_n)$, we have

$$\begin{aligned} &\mathbb{E}\big[\hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda_n)} \in G_n, Z_{H(\partial\Lambda'_n)} \in G'_n, H(\partial\Lambda_n) \leq H(\partial\Lambda'_n)] f_n(\omega)\big] \\ &\qquad \leq \sup_{z \in G_n} E_z\big[Z_{H(\partial\Lambda'_n)} \in G'_n\big]. \end{aligned} \tag{1.16}$$

For the second term we use the strong Markov property and the definition of $f_n(\omega)$ to conclude

$$\begin{aligned} &\mathbb{E}\big[\hat{P}_0^{\partial\Lambda_n}[Z_{H(\partial\Lambda_n)} \in G_n, Z_{H(\partial\Lambda'_n)} \in G'_n, H(\partial\Lambda_n) > H(\partial\Lambda'_n)] f_n(\omega)\big] \\ &\quad \leq \mathbb{E}\bigg[\frac{1}{e_\lambda(0, \partial\Lambda'_n)} E_0\bigg[\exp\bigg\{-\int_0^{H(\partial\Lambda'_n)} (\lambda + V)(Z_s)\,ds\bigg\}, \\ &\qquad\qquad Z_{H(\partial\Lambda'_n)} \in G'_n, H(\partial\Lambda'_n) < \infty\bigg] 1_{\Omega_n}\bigg] \\ &\qquad \times \sup_{z \in G'_n} E_z\bigg[\exp\bigg\{-\int_0^{H(\partial\Lambda_n)} (\lambda + V)(Z_s)\,ds\bigg\}, Z_{H(\partial\Lambda_n)} \in G_n\bigg] \\ &\quad \leq \sup_{z \in G'_n} E_z\big[Z_{H(\partial\Lambda_n)} \in G_n\big]. \end{aligned} \tag{1.17}$$

Using standard estimates on the Brownian motion we conclude that the expressions on the right-hand side of (1.16) and (1.17) are smaller than $\varepsilon/3$ for all large n. Combining all these results we find that, for all large n, the claim of the lemma is true. □

2. The superdiffusive lower bound. In this section we prove the superdiffusive lower bound for $d = 2$ and $\lambda > 0$ given in Theorem 0.2. The proof will be a combination of the ideas we have used to prove the subdiffusive lower bound for ξ_0 in Theorem 1.3 of [10] (there we have used two parallel disjoint cylinders) and the ideas to prove the diffusive lower bound $\xi^{(1)} \geq 1/2$. Actually, we refine the technique used in Theorem 1.3 of [10], using also curvature properties. As mentioned in the introduction, we will look again at the

polygon construction used in the proof of Theorem 0.1, but we will choose θ_L to be of a different order ($\theta_L = L^{-(1-\gamma_1)}$, $\gamma_1 > \xi^{(2)}$). This way we find two almost disjoint cylinders such that in Lemma 2.1 we are able to show that for large L, $\mathbb{V}\mathrm{ar}(-\log e_\lambda(0, \partial\Lambda(0, L)) + \log e_\lambda(0, \partial\Lambda(\theta_L, L)) \le c_1 L^{4\gamma_1 - 2}$. This will use the fact that the two cylinders are close together. In Lemma 2.2 we will get a lower bound on the same variance of order $L^{1-\gamma_1}$. Combining these two result will lead us to the claim of Theorem 0.2.

PROOF OF THEOREM 0.2. In view of (0.10) we know that in all dimensions $d \ge 2$, $\xi^{(2)} \ge 1/2$. Suppose $\xi^{(2)} < 3/5$. We will show that this leads to a contradiction. We take

$$\gamma_1 \in (\xi^{(2)}, 3/5). \tag{2.1}$$

Then we use the polygon construction presented in the proof of Theorem 0.1. We define

$$\theta_L = 8L^{-(1-\gamma_1)}, \tag{2.2}$$

$\Lambda_L = \Lambda(0, L)$ and $\Lambda'_L = \Lambda(\theta_L, L)$. Then P_L is the intersection of Λ_L^c, $(\Lambda'_L)^c$, $\Lambda^c(-\pi/2, 0)$ and $\Lambda^c(\theta_L + \pi/2, 0)$ (P_L is the same polygon as in the previous section). We denote its corner as before in a counterclockwise way by 0, z_1, z_2 and z_3, where 0 is the origin. The following properties are true:

$$\mathrm{dist}(0, z_1) = \mathrm{dist}(0, z_3) = L \tag{2.3}$$

and

$$\mathrm{dist}(z_1, z_2) = \mathrm{dist}(z_2, z_3) = L\tan(\theta_L/2) \ge 2L^{\gamma_1} \quad \text{for all large } L. \tag{2.4}$$

Therefore, we see that for all large L,

$$\mathrm{dist}(z_1, z_3) \ge 3L^{\gamma_1}. \tag{2.5}$$

Next we choose

$$\gamma_2 \in (\xi^{(2)}, \gamma_1). \tag{2.6}$$

By the definition of $\xi^{(2)}$, there exists a sequence $(L_n)_{n\ge1}$ with $L_n \to \infty$ such that

$$\mathbb{E}\big[\hat{P}_0^{\partial\Lambda_{L_n}}[A(L_n, \gamma_2)]\big] \to 1 \quad \text{for } n \to \infty. \tag{2.7}$$

Using (2.4) and (2.5) we see that $z_2 \notin Z(\hat{x}(0), L_n^{\gamma_2}) \cup Z(\hat{x}(\theta_{L_n}), L_n^{\gamma_2})$. We will see that these two cylinders are "well separated." That is, that we can apply the same methods as in the proof of Theorem 1.3 of [10] for parallel cylinders. We find a lower bound for the variance of the difference between the two random variables $-\log e_\lambda(0, \partial\Lambda_{L_n})$ and $-\log e_\lambda(0, \partial\Lambda'_{L_n})$ (see Lemma 2.2) as well as an upper bound (see Lemma 2.1). We slightly modify our goals $\partial\Lambda_{L_n}$ and $\partial\Lambda'_{L_n}$. Look at the line segment with endpoints $z_1 + (z_1 - z_2)$ and $z_2 + 2(z_2 - z_1)$ [resp., $z_3 + (z_3 - z_2)$ and $z_2 + 2(z_2 - z_3)$], let ∂_n (resp., ∂'_n) be the closure of the 1/2-neighborhood of first (resp., second) line segment intersected with Λ_{L_n}

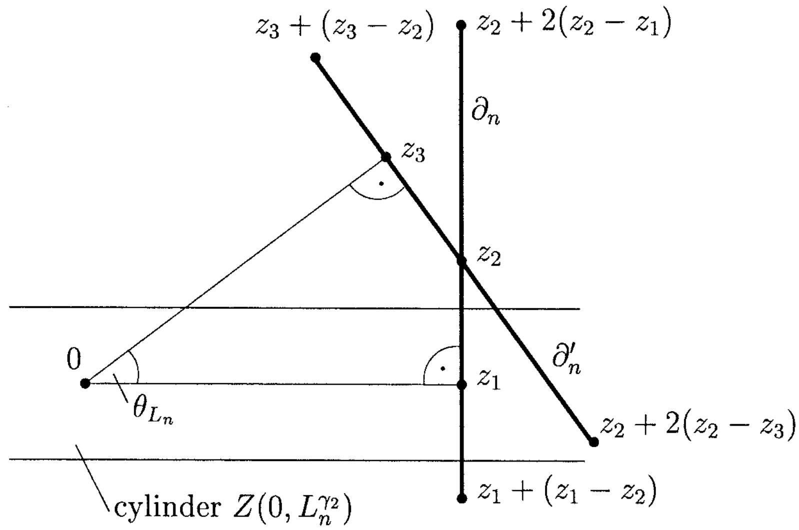

FIG. 4. *The goals ∂_n and ∂'_n.*

(resp., Λ'_{L_n}) (see Figure 4). For $\lambda > 0$, $\omega \in \Omega$ and $n \geq 1$, we define our new random variables as follows:

$$-\log \hat{e}_{\lambda,n}(\omega) = -\log e_\lambda(0, \partial_n, \omega) \tag{2.8}$$

and

$$-\log \hat{e}'_{\lambda,n}(\omega) = -\log e_\lambda(0, \partial'_n, \omega), \tag{2.9}$$

where

$$e_\lambda(0, \partial_n, \omega) = E_0\left[\exp\left\{-\int_0^{H(\partial_n)} (\lambda + V)(Z_s, \omega)\, ds\right\}, H(\partial_n) < \infty\right], \tag{2.10}$$

with $H(\partial_n) = \inf\{s \geq 0; Z_s \in \partial_n\}$, $e_\lambda(0, \partial'_n, \omega)$ is defined analogously with respect to ∂'_n. The path measure starting in 0 conditioned to reach ∂_n (resp., ∂'_n) in finite time with respect to the Poissonian potential ($\omega \in \Omega$) will be denoted by $\hat{P}_0^{\partial_n} = \hat{P}_0^{\partial_n}(\omega)$ (resp., by $\hat{P}_0^{\partial'_n}$). Define

$$\begin{aligned} A_{\partial_n}(L_n, \gamma_2) = \{w \in C(\mathbb{R}_+, \mathbb{R}^d);\, &w(0) = 0 \text{ and } Z_s(w) \in Z(\hat{x}(\theta), L_n^{\gamma_2}) \\ &\text{for all } s \leq H(\partial_n)\}. \end{aligned} \tag{2.11}$$

For all $w \in C(\mathbb{R}_+, \mathbb{R}^d)$ we have $H(\partial_n) \geq H(\partial\Lambda_{L_n})$, whereas if $w \in A_{\partial_n}(L_n, \gamma_2)$, we have a strict equality, $H(\partial_n) = H(\partial\Lambda_{L_n})$. Therefore, using (2.7) we see that

$$\mathbb{E}\big[\hat{P}_0^{\partial_n}[A_{\partial_n}(L_n, \gamma_2)]\big] \to 1 \quad \text{for } n \to \infty. \tag{2.12}$$

Next we state the two main lemmas of this proof.

LEMMA 2.1. *There exists a constant $c_1 \in (0, \infty)$ such that for all large n,*

$$\mathbb{V}\mathrm{ar}\big(-\log \hat{e}_{\lambda,n} + \log \hat{e}'_{\lambda,n}\big) \le c_1 (L_n^{2\gamma_1 - 1})^2. \tag{2.13}$$

PROOF. Of course, ∂_n is within distance $32L_n^{2\gamma_1-1}$ from ∂'_n for large n and also ∂'_n is within distance $32L_n^{2\gamma_1-1}$ from ∂_n for large n. Therefore, using the strong Markov property, we see that for any $\omega \in \Omega$,

$$\big|-\log \hat{e}_{\lambda,n} + \log \hat{e}'_{\lambda,n}\big| \le \sup_{z \in \partial_n} -\log e_\lambda(z, \partial'_n, \omega) + \sup_{z \in \partial'_n} -\log e_\lambda(z, \partial_n, \omega). \tag{2.14}$$

Now using a tubular estimate for Brownian motion [see (1.35) in [9]] and the fact that ∂_n is within distance $32L_n^{2\gamma_1-1}$ of ∂'_n and vice versa, we see that there exists $c_2 \in (0, \infty)$ such that

$$\big|-\log \hat{e}_{\lambda,n} + \log \hat{e}'_{\lambda,n}\big| \le c_2 L_n^{2\gamma_1-1} \quad \text{for all large } n. \tag{2.15}$$

Therefore (2.13) holds. □

LEMMA 2.2. *There exists a constant $c_3 \in (0, \infty)$ such that for all large n,*

$$\mathbb{V}\mathrm{ar}\big(-\log \hat{e}_{\lambda,n} + \log \hat{e}'_{\lambda,n}\big) \ge c_3 L_n^{1-\gamma_1}. \tag{2.16}$$

Lemmas 2.1 and 2.2 lead to the conclusion that $4\gamma_1 - 2 \ge 1 - \gamma_1$, which contradicts $\gamma_1 < 3/5$. This completes the proof of Theorem 0.2. □

So it remains to prove Lemma 2.2. We remark, as mentioned before, that this proof is analogous to the proof of formula (4.3) of [10].

PROOF OF LEMMA 2.2. By $a = a(W) > 0$ we denote the smallest possible $a \in \mathbb{R}_+$ such that $W(\cdot) = 0$ on $\bar{B}(0, a)^c$, where $\bar{B}(0, a)$ is the closed Euclidean ball with center $0 \in \mathbb{R}^2$ and radius $a > 0$. We then start by introducing a paving of the plane $\mathbb{R}^2$. Let $(q_k)_{k \in \mathbb{N}}$ be a deterministic ordering of the points in $\mathbb{Z}^2$. For $k \in \mathbb{N}$, we consider the cubes of size l and center q_k,

$$C_k = \big\{z \in \mathbb{R}^2; -l/2 \le z^i - l q_k^i < l/2, \text{ for } i = 1, 2\big\},$$

where $l = l(d, \nu, a) \in (d(4 + 8a), \infty)$ is fixed but sufficiently large, such that (1.26) of [9] holds. The closed a-neighborhood of C_k is denoted by $\tilde{C}_k$. The entrance time of the motion into the closed cube $\bar{C}_k$ will be denoted by $H_k = H(\bar{C}_k)$ and analogously $\tilde{H}_k$ will be the entrance time of the motion into the closed cube $\tilde{C}_k$. If $U \subset \mathbb{R}^2$ is a subset, we define $\mathscr{F}(U)$ to be the σ-algebra generated by all $\omega(A)$, where $A \in \mathscr{B}(\mathbb{R}^2)$ and $A \subseteq U$. Then we introduce a filtration $(\mathscr{F}_k)_k$ on our probability space $(\Omega, \mathscr{F}, \mathbb{P})$

$$\mathscr{F}_k = \mathscr{F}\left(\bigcup_{i=1}^{k} C_i\right) \quad \text{for } k \ge 1 \text{ and } \mathscr{F}_0 = \{\varnothing, \mathscr{F}\},$$

$$\mathscr{G}_k = \mathscr{F}(C_k) \quad \text{for } k \ge 1.$$

Of course we have that for $k \geq 1$, $\mathscr{F}_k = \mathscr{G}_1 \vee \mathscr{G}_2 \vee \cdots \vee \mathscr{G}_k$, where $\mathscr{G}_j \vee \mathscr{G}_i$ is the smallest σ-algebra containing $\mathscr{G}_i$ and $\mathscr{G}_j$.

Using the filtration, we see [as in [10] formulas (3.3) and (4.4)] that the following martingale estimate holds:

$$\mathbb{V}\mathrm{ar}\big(-\log \hat{e}_{\lambda,n} + \log \hat{e}'_{\lambda,n}\big) \geq \sum_{k\geq 1} \mathbb{V}\mathrm{ar}\big(\mathbb{E}[-\log \hat{e}_{\lambda,n} + \log \hat{e}'_{\lambda,n} \,|\, \mathscr{G}_k]\big). \tag{2.17}$$

Next, for $k \in \mathbb{N}$, we introduce (as in [10]) the events $D_{0,k}$, $D_{1,k}$, D_k^0, D_k^1: If $\omega \in \Omega$ is a cloud configuration, we write $\hat{\omega}_k$ for the restriction of ω to C_k^c and we write ω_k for the restriction of ω to C_k, so we have the following decomposition: $\omega = (\omega_k, \hat{\omega}_k) \in \Omega$. Now we define the events, $D_{0,k} = \{\omega_k; \omega_k(\bar{C}_k) = 0\}$ and $D_k^0 = \{\omega \in \Omega; \omega_k \in D_{0,k}\}$. These are the events that no point of the cloud falls into the closed cube $\bar{C}_k$. The disjoint events on $\bar{C}_k$ will be $D_{1,k} = \{\omega_k;\ \omega_k(\bar{B}(lq_k, 1)) \geq 1\}$ and $D_k^1 = \{\omega \in \Omega; \omega_k \in D_{1,k}\}$. These are the events that we have at least one point of the cloud in the center of C_k (i.e., in the closed ball with radius 1 around lq_k). Of course, D_k^0 and D_k^1 are disjoint, $\mathscr{G}_k$-measurable and $p = \mathbb{P}[D_k^0] > 0$, $q = \mathbb{P}[D_k^1] > 0$. Using Lemma 3 of [7], we find that

$$\mathbb{V}\mathrm{ar}\big(\mathbb{E}[-\log \hat{e}_{\lambda,n} + \log \hat{e}'_{\lambda,n} \,|\, \mathscr{G}_k]\big) \geq \frac{pq}{p+q}(x_1 - x_0)^2, \tag{2.18}$$

with x_δ the numbers

$$x_\delta = \mathbb{E}\left[\frac{\mathbb{E}[-\log \hat{e}_{\lambda,n} + \log \hat{e}'_{\lambda,n} \,|\, \mathscr{G}_k] 1_{D_k^\delta}}{\mathbb{P}[D_k^\delta]}\right] \quad \text{for } \delta = 0, 1. \tag{2.19}$$

We define the random variable $\Psi_{k,n}(\hat{\omega}_k)$ on C_k^c as follows: for $k \in \mathbb{N}$, $\omega \in \Omega$, $\sigma_k^0 \in D_k^0$ (notice that D_k^0 contains only one element),

$$\Psi_{k,n}(\hat{\omega}_k) = \inf_{\omega_k \in D_{1,k}} -\log\left(\frac{\hat{e}_{\lambda,n}(\omega_k, \hat{\omega}_k)}{\hat{e}'_{\lambda,n}(\omega_k, \hat{\omega}_k)}\right) + \log\left(\frac{\hat{e}_{\lambda,n}(\sigma_k^0, \hat{\omega}_k)}{\hat{e}'_{\lambda,n}(\sigma_k^0, \hat{\omega}_k)}\right). \tag{2.20}$$

We remark that $\Psi_{k,n}$ is measurable. This can be seen by using an approximation of the cloud configurations by cloud configurations with rational coordinates. As in (3.4) of [10], we see that

$$|x_1 - x_0| \geq \mathbb{E}[\Psi_{k,n}]_+ = \max\{\mathbb{E}[\Psi_{k,n}], 0\}. \tag{2.21}$$

Next we define the finite set of all the labels $k \in \mathbb{N}$, such that the boxes C_k intersect the truncated cylinder $Z(\hat{x}(0), L_n^{\gamma_2}) \cap \Lambda_{L_n}^c \cap \Lambda_{L_n/2}$,

$$\mathscr{E}_n = \big\{k \in \mathbb{N}; C_k \cap \big(Z(\hat{x}(0), L_n^{\gamma_2}) \cap \Lambda_{L_n}^c \cap \Lambda_{L_n/2}\big) \neq \varnothing\big\}. \tag{2.22}$$

Thus the number of points $k \in \mathbb{N}$ which lie in $\mathscr{E}_n$ is bounded from above by $c_4 L_n^{1+\gamma_2}$. Therefore we find, using (2.17), (2.18), (2.21) and the Cauchy–Schwarz inequality,

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}\big(-\log \hat{e}_{\lambda,n} + \log \hat{e}'_{\lambda,n}\big) &\geq \frac{pq}{p+q} \sum_{k\in\mathcal{E}_n} \big(\mathbb{E}[\Psi_{k,n}]_+\big)^2 \\
&\geq \frac{pq}{p+q} |\mathcal{E}_n|^{-1} \bigg(\sum_{k\in\mathcal{E}_n} \mathbb{E}[\Psi_{k,n}]_+\bigg)^2 \\
&\geq \frac{pq}{c_4(p+q)} L_n^{1-\gamma_2} \bigg(\frac{1}{L_n} \sum_{k\in\mathcal{E}_n} \mathbb{E}[\Psi_{k,n}]\bigg)_+^2 \\
&\geq \frac{pq}{c_4(p+q)} L_n^{1-\gamma_1} \bigg(\frac{1}{L_n} \sum_{k\in\mathcal{E}_n} \mathbb{E}[\Psi_{k,n}]\bigg)_+^2,
\end{aligned}
\tag{2.23}
$$

where in the last step we have used that $\gamma_2 < \gamma_1$. To prove the lower bound (2.16) we have to verify

$$
\liminf_{n\to\infty} \frac{1}{L_n} \sum_{k\in\mathcal{E}_n} \mathbb{E}[\Psi_{k,n}] > 0. \tag{2.24}
$$

Now

$$
\Psi_{k,n}(\hat{\omega}_k) \geq \inf_{\omega_k\in D_{1,k}} -\log\bigg(\frac{\hat{e}_{\lambda,n}(\omega_k,\hat{\omega}_k)}{\hat{e}_{\lambda,n}(\sigma_k^0,\hat{\omega}_k)}\bigg) - \sup_{\omega_k\in D_{1,k}} \log\bigg(\frac{\hat{e}'_{\lambda,n}(\sigma_k^0,\hat{\omega}_k)}{\hat{e}'_{\lambda,n}(\omega_k,\hat{\omega}_k)}\bigg). \tag{2.25}
$$

Therefore it suffices to show that

$$
\liminf_{n\to\infty} \frac{1}{L_n} \sum_{k\in\mathcal{E}_n} \mathbb{E}\bigg[\inf_{\omega_k\in D_{1,k}} \log\bigg(\frac{\hat{e}_{\lambda_n}(\sigma_k^0,\hat{\omega}_k)}{\hat{e}_{\lambda_n}(\omega_k,\hat{\omega}_k)}\bigg)\bigg] > 0 \tag{2.26}
$$

and

$$
\lim_{n\to\infty} \frac{1}{L_n} \sum_{k\in\mathcal{E}_n} \mathbb{E}\bigg[\sup_{\omega_k\in D_{1,k}} \log\bigg(\frac{\hat{e}'_{\lambda,n}(\sigma_k^0,\hat{\omega}_k)}{\hat{e}'_{\lambda,n}(\omega_k,\hat{\omega}_k)}\bigg)\bigg] = 0. \tag{2.27}
$$

The proofs of (2.26) and (2.27) are exactly the same as the proofs of (4.7) and (4.8) in [10]. Therefore we will only give the structure of the rest of the proof. Using Lemmas 3.1, 3.2 and 3.3 of [10], Harnack's inequality and the fact that $\lambda > 0$, the claims (2.26) and (2.27) can be reduced to the following two statements:

$$
\liminf_{n\to\infty} \frac{1}{L_n} \sum_{k\in\mathcal{E}_n\setminus N(\partial_n)} \mathbb{E}\big[\hat{P}_0^{\partial_n}[H_k \leq H(\partial_n)]\big] > 0 \tag{2.28}
$$

and

$$
\lim_{n\to\infty} \frac{1}{L_n} \sum_{k\in\mathcal{E}_n} \mathbb{E}\big[\hat{P}_0^{\partial'_n}[\tilde{H}_k \leq H(\partial'_n)]\big] = 0, \tag{2.29}
$$

where $N(\partial_n)$ denotes the following set: choose R minimal such that $\tilde{C}_k \subset B(lq_k, R)$ and $\tilde{C}_k$ is a neighboring box of the goal ∂_n, if $B(lq_k, R+2) \cap \partial_n \neq \varnothing$.

$$N(\partial_n) = \{k \in \mathbb{N}; \tilde{C}_k \text{ is a neighboring box of } \partial_n\}. \tag{2.30}$$

PROOF OF (2.28). Using the properties of $A_{\partial_n}(L_n, \gamma_2)$, we find

$$\begin{aligned} &\frac{1}{L_n} \sum_{k \in \mathscr{E}_n \setminus N(\partial_n)} \mathbb{E}\big[\hat{P}_0^{\partial_n}[H_k \le H(\partial_n)]\big] \\ &\quad \ge \frac{1}{L_n} \mathbb{E}\Bigg[\hat{E}_0^{\partial_n}\Bigg[\sum_{k \in \mathscr{E}_n \setminus N(\partial_n)} 1_{\{H_k \le H(\partial_n)\}} 1_{A_{\partial_n}(L_n, \gamma_2)}\Bigg]\Bigg] \\ &\quad \ge \frac{1}{L_n} \frac{c_5 L_n}{2} \mathbb{E}\big[\hat{P}_0^{\partial_n}[A_{\partial_n}(L_n, \gamma_2)]\big], \end{aligned} \tag{2.31}$$

which stays strictly positive as n tends to infinity.

PROOF OF (2.29). For all large n the following is true:

$$\bigcup_{k \in \mathscr{E}_n} \tilde{C}_k \cap Z(\hat{x}(\theta_{L_n}), L_n^{\gamma_2}) = \varnothing. \tag{2.32}$$

Thus, if $|\mathscr{A}_n|$ is the number of visited boxes C_k before reaching the goal ∂_n', we have for all large n (using the Cauchy–Schwarz inequality),

$$\begin{aligned} &\frac{1}{L_n} \sum_{k \in \mathscr{E}_n} \mathbb{E}\big[\hat{P}_0^{\partial_n'}[\tilde{H}_k \le H(\partial_n')]\big] \\ &\quad \le \frac{3^2}{L_n} \mathbb{E}\big[\hat{E}_0^{\partial_n'}[|\mathscr{A}_n| 1_{A^c_{\partial_n'}(L_n, \gamma_2)}]\big] \\ &\quad \le 3^2 \mathbb{E}\big[\hat{E}_0^{\partial_n'}[(|\mathscr{A}_n|/L_n)^2]\big]^{1/2} \mathbb{E}\big[\hat{P}_0^{\partial_n'}[A^c_{\partial_n'}(L_n, \gamma_2)]\big]^{1/2}. \end{aligned} \tag{2.33}$$

The last term on the right-hand side of (2.33) tends to zero as n goes to infinity, whereas the first term stays bounded (see (1.31) of [9]). This completes the proof of Lemma 2.2. □

Acknowledgment. I am very grateful to Professor A.-S. Sznitman for many helpful discussions during the writing of this paper.

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