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DIFFUSION PROCESSES AND HEAT KERNELS ON METRIC SPACES

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Dedicated to my teacher Professor Heinz Bauer on the occasion of his 70th birthday

We present a general method to construct m-symmetric diffusion processes $(X_t, \mathbf{P}_x)$ on any given locally compact metric space (X, d) equipped with a Radon measure m. These processes are associated with local regular Dirichlet forms which are obtained as Γ-limits of approximating nonlocal Dirichlet forms. This general method works without any restrictions on (X, d, m) and yields processes which are well defined for quasi every starting point.

The second main topic of this paper is to formulate and exploit the so-called Measure Contraction Property. This is a condition on the original data (X, d, m) which can be regarded as a generalization of curvature bounds on the metric space (X, d). It is a bound for distortions of the measure m under contractions of the state space X along suitable geodesics (or quasi geodesics) w.r.t. the metric d. In the case of Riemannian manifolds, this condition is always satisfied. Several other examples will be discussed, including uniformly elliptic operators, operators with weights, certain subelliptic operators, manifolds with boundaries or corners and glueing together of manifolds.

The Measure Contraction Property implies (upper and lower) Gaussian estimates for the heat kernel and a Harnack inequality for the associated harmonic functions. Therefore, the above-mentioned diffusion processes are strong Feller processes and are well defined for every starting point.

1. Introduction.

1.A. *The idea.* How to construct a diffusion process (e.g., some kind of Brownian motion) on a metric space (X, d)? It is well known that one can do a lot of geometry on X just using the metric structure d. However, in order to do stochastics or analysis on X, one additionally has to fix a *speed* or *reference measure* m on X.

Then the idea is quite easy. Think of Brownian motion on $\mathbb{R}^n$ or on an n-dimensional Riemannian manifold (X, g). It is the unique strong Feller process which is associated to the Dirichlet form

$$E(u, u) = \frac{1}{2}\int_X |\nabla u|^2(x)\, m(dx) \tag{1.1}$$

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with core $C_0^1(X)$. Here the reference measure m is just the Riemannian volume measure.

In the general case, we will try to construct a Dirichlet form E with core $C_0^{\mathrm{Lip}}(X)$ which is analogous to the form (1.1). For this purpose, we have to look for a replacement for the square of the gradient of a function $u \in C_0^1(X)$. Recall that on any Riemannian manifold,

$$|\nabla u|^2(x) = n \lim_{r\to 0} \frac{1}{m(B_r(x))} \int_{B_r^*(x)} \left[\frac{u(z)-u(x)}{d(z,x)}\right]^2 m(dz) \tag{1.2}$$

for $u \in C_0^1(X)$. This leads to the following definition in the general case:

$$\begin{aligned} E^r(u,u) = \frac{1}{2}\int_X N(x)\int_{B_r^*(x)} &\left[\frac{u(z)-u(x)}{d(z,x)}\right]^2 \\ &\times \frac{m(dz)}{\sqrt{m(B_r(z))}}\,\frac{m(dx)}{\sqrt{m(B_r(x))}} \end{aligned} \tag{1.3}$$

for $u \in C_0^{\mathrm{Lip}}(X)$ and

$$E = \lim_{r\to 0} E^r. \tag{1.4}$$

The function N can be any normalization function. It plays the role of the dimension, that is, $N(x)$ is the *local dimension* at $x \in X$.

1.B. *The general approach.* A crucial observation is that the pointwise limit of these forms E^r for $r \to 0$, in general, does not yield a reasonable object. However, there is an appropriate notion of variational convergence, called Γ-convergence. A sequence $(E^{r_n})_n$ is called Γ-convergent if

$$\lim_{\alpha\to 0}\liminf_{n\to\infty}\inf_{\substack{v\in L^2\\ \|u-v\|\le\alpha}} E^{r_n}(v,v) = \lim_{\alpha\to 0}\limsup_{n\to\infty}\inf_{\substack{v\in L^2\\ \|u-v\|\le\alpha}} E^{r_n}(v,v)$$

for all $u \in L^2(X,m)$. The point is, that *without any assumption* on (X,d,m) there always exist sequences $(r_n)_n$ (with $\lim_{n\to\infty} r_n = 0$) such that the sequences $(E^{r_n})_n$ are Γ-convergent. Each such Γ-limit E defines a Dirichlet form on $L^2(X,m)$. Assuming that the state space X is locally compact, this Dirichlet form is strongly local and regular with core $C_0^{\mathrm{Lip}}(X)$. Therefore, for each of these limit forms there exists an m-symmetric diffusion process on X. This is a Hunt process with continuous paths. Its lifetime in X can be finite, but there is no killing inside of X. This diffusion process is defined uniquely for quasi every starting point $x \in X$.

In general, the family $\{E^r, r > 0\}$ may have several Γ-limits for $r \to 0$. Concerning the above constructions, there arise several important questions.

1. When is $\Gamma\text{-}\liminf_{r\to 0} E^r = \Gamma\text{-}\limsup_{r\to 0} E^r$, or in other words, when does the Γ-limit $E^0 = \Gamma\text{-}\lim_{r\to 0} E^r$ exist?
2. When does the pointwise limit $\lim_{r\to 0} E^r(u,u)$ exist [for sufficiently many $u \in L^2(X,m)$] and when does it coincide with $E^0(u,u)$?
3. When does E^0 define a diffusion process uniquely for *every starting point* $x \in X$ and when is this process a strong Feller process?

1.C. *The measure contraction property.* An affirmative answer to the above questions can be given if (X, d, m) satisfies the so called Measure Contraction Property (MCP). Let us describe this property in the easiest case.

Assume for simplicity that (X, d) is a geodesic space. This means that for any two points $x, y \in X$ there exists an arc $\Phi(x, y)$: $[0,1] \to X$, $t \mapsto \Phi_t(x, y)$ of length $d(x, y)$ with $\Phi_0(x, y) = x$ and $\Phi_1(x, y) = y$. Moreover, assume that Φ: $X^2 \to X^{[0,1]}$ is measurable [i.e., geodesics $\Phi(x, y)$ joining x and y can be chosen in such a way that they depend in a measurable way on x and y]. For fixed $x \in X$ and $t \in [0,1]$, the map $\Phi_t(x, \cdot)$: $X \to X$, $y \mapsto \Phi_t(x, y)$ is a contraction of the state space towards the center x in the sense that $d(x, \Phi_t(x, y)) = td(x, y)$ for all $y \in X$. In particular, for each $r \to 0$, the ball $B_r(x)$ will be mapped onto the ball $B_{tr}(x)$. However, if a ball $A = B_r(y)$ is not centered at x, then its image $\Phi_t(x, A)$ is in general no longer a ball. It may be distorted drastically. In the extreme case, it could be an arc of length $2tr$. The Measure Contraction Property is a very weak control for such distortions. It states that on any compact set $Y \subset X$,

$$\frac{m(A)}{m(B_r(x))} \le C \frac{m(\Phi_t(x, A))}{m(B_{tr}(x))} \tag{1.5}$$

for all $A \subset B_r(x) \subset Y$. The LHS of (1.5) measures the proportion of A in $B_r(x)$, whereas the RHS measures the proportion of the image of A in the image of $B_r(x)$.

Actually, we will consider two versions (a weak and a strong one) of this MCP and often it is only required to hold locally on $X \setminus Z$ where Z is an "exceptional" set of measure 0.

1.D. *The examples.* If (X, g) is a smooth Riemannian manifold and if d and m are the Riemannian distance and the Riemannian volume, respectively, then (X, d, m) always satisfies the Measure Contraction Property. This is a consequence of the Bishop volume comparison theorem which implies that if the Ricci curvature on $B_r(x)$ is bounded from below by $-(n-1)\kappa$ then

$$\frac{m(A)}{rS_\kappa(r)} \le \frac{m(\Phi_t(x, A))}{trS_\kappa(tr)} \tag{1.6}$$

for all $A \subset B_r(x)$ with $S_\kappa(r)$ being the area of the sphere of radius r in the space of constant sectional curvature $-\kappa$ and of the fact that (by the Bishop and Bishop–Günther volume comparison theorems) $m(B_r(x))/(rS_\kappa(r)) \to 1/n$ (locally uniformly in x) for $r \to 0$.

Further examples for metric measured spaces (X, d, m) with the MCP are given by manifolds with boundaries or corners and by glueing together of manifolds (where the components could have different dimensions). The MCP also holds true if d is defined by an uniformly elliptic operator on $X = \mathbb{R}^n$ and even if d is derived from certain degenerate elliptic operators like the Grushin operator.

1.E. *The results.* The Measure Contraction Property (say, for simplicity, in its strong version) for the metric measured space (X, d, m) entails several important properties for the possible limits of the approximating Dirichlet forms E^r.

1. For each $u \in C_0^{\mathrm{Lip}}(X)$ the Γ-limit $E^0(u, u) = \Gamma\text{-}\lim_{r \to 0} E^r(u, u)$ as well as the pointwise limit $\lim_{r \to 0} E^r(u, u)$ *exist* and *coincide*;
2. The closure (E, F) of $(E^0, C_0^{\mathrm{Lip}}(X))$ is a strongly local, *regular Dirichlet form*;
3. The associated diffusion process $(X_t, \mathbf{P}_x)$ is a *strong Feller process* (which is defined uniquely for *every* starting point $x \in X$);
4. The corresponding heat kernel is *Hölder continuous* and admits upper and lower *Gaussian estimates*;
5. The intrinsic metric associated with (E, F) on X is locally *equivalent* to the original metric d.

All these results hold true with any normalization function $N \in L^\infty_{\mathrm{loc}}(X, m)$ satisfying $N \geq 1$. An additional condition on (X, d, m) provides a canonical candidate for N. With this choice, the intrinsic metric really coincides with d.

1.F. *The proceeding.* In Section 2, we summarize some basic facts on length spaces, Dirichlet forms (including convergence questions) and diffusion processes.

The goal of Section 3 is to present a general recipe for constructing Dirichlet forms, diffusion processes and heat kernels on metric spaces. Given *any* metric measured space (X, d, m) (with X being locally compact and m being a Radon measure with full support on X), we will define *regular*, *strongly local Dirichlet forms* (E, F) on $L^2(X, m)$ with core $C_0^{\mathrm{Lip}}(X)$. These local forms will be obtained as Γ-limits of the nonlocal forms E^r. The point here is that no “quantitative” assumptions are imposed.

In Section 4, we formulate and discuss the MCP. Among other things, we prove that it always implies the volume doubling property. Much space is given to investigating in detail the main examples (which we have already mentioned).

The strength of the MCP is demonstrated in Sections 5 and 6. In Section 5, we prove the existence and coincidence of the Γ-limit $\Gamma\text{-}\lim_{r \to 0} E^r(u, u)$ and of the pointwise limit $\lim_{r \to 0} E^r(u, u)$. The approach has to take into account that we admit an exceptional set in the formulation of the MCP which brings in its wake the need of subtle localization. But surprisingly enough, the resulting form does not depend on the exhausting sequence (used for the localization) nor on the exceptional set.

Another crucial fact is that the MCP implies a Poincaré inequality. Moreover, it implies that the intrinsic metric is locally equivalent to the original one. These facts are proved in Section 6. They are the ingredients which are used in Section 7 to prove Harnack inequality, Hölder continuity, Feller property and Gaussian estimates.

For simplicity, in Sections 5–7, we restrict ourselves to the case $N \equiv 1$. The general case is treated in Section 8. Moreover, here we introduce functions $\underline{N}$ and $\overline{N}$ which play the role of (lower and upper, respectively) local dimensions. If they coincide, then their common value is the "natural" candidate for the choice of the normalization function N.

2. Preliminaries.

2.A. *The basic set-up.* Throughout this paper we fix a metric measured space (X, d, m) consisting of a *state space* X, a *metric* d on X and a *reference measure* m on X. We always assume the following:

> (X, d) is a locally compact separable metric space and m is a Radon measure on X with $m(U) > 0$ for each nonempty open set $U \subset X$.

In terms of (X, d), we define for each $r > 0$ and $y \in X$ the balls $B_r(y) = \{x \in X\colon d(y, x) < r\}$ of radius r and center y and the sets $B_r^*(y) = B_r(y) \setminus \{y\}$. Similarly, we define for each $r > 0$ and $Y \subset X$ the sets $B_r(Y) = \{x \in X\colon d(x, Y) < r\}$ and $B_r^*(Y) = B_r(Y) \setminus Y$.

We do not require that (X, d) be complete. In particular, balls $B_r(x)$ are not necessarily relatively compact. For each $r > 0$, we define a Radon measure m_r on X by

$$m_r(dx) = \frac{1}{\sqrt{m(B_r(x))}}\, m(dx).$$

The set of (real-valued) Lipschitz continuous functions on X with compact supports will be denoted by $C_0^{\mathrm{Lip}}(X)$.

2.B. *Length spaces and geodesic spaces.* An *arc* in X is a continuous map $\gamma\colon [a, b] \to X$ where $[a, b]$ denotes any compact interval in $\mathbb{R}$. The *length* $L_d(\gamma)$ of an arc $\gamma\colon [a, b] \to X$ is defined as

$$L_d(\gamma) = \sup\left\{ \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1}))\colon n \in \mathbb{N},\ a \le t_0 < t_1 < \cdots < t_n \le b \right\}. \tag{2.1}$$

A *unit speed geodesic arc* is an arc $\gamma\colon [a, b] \to X$ which is locally an isometry, that is, for any $c \in]a, b[$ there exists $\varepsilon > 0$ such that $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [c - \varepsilon, c + \varepsilon]$. A *geodesic arc* is a constant speed reparametrization of a unit speed geodesic arc. Replacing the parameter interval $[a, b]$ by $]a, b[$ (or $[a, \infty[$ or $]-\infty, \infty[$) one obtains the notion of *geodesic curve* (*geodesic ray* and *geodesic line*, resp.). We say that an arc $\gamma\colon [a, b] \to X$ *joins* x and y if $\gamma(a) = x$ and $\gamma(b) = y$. Obviously, the length of any arc joining x and y dominates the distance between x and y, that is, $L_d(\gamma) \geq d(x, y)$ with equality if and only if γ is a *minimal geodesic arc*. Let us define

$$\overline{d}(x, y) = \inf\{L_d(\gamma) \colon \gamma \text{ is an arc in } X \text{ joining } x \text{ and } y\}. \tag{2.2}$$

Then $\overline{d}$ is a pseudometric on X which dominates d, that is, $\overline{d} \geq d$. It can happen that $\overline{d}$ induces a topology on X which is strictly coarser than d; in particular, it can happen that $\overline{d}(x, y) = \infty$ for all $x \neq y$.

DEFINITION 2.1. The metric space (X, d) is called a *length space* (or an *inner metric space*) if $d = \overline{d}$. That is, the distance between any two points $x, y \in X$ is the infimum of the length of arcs joining them. It is called *geodesic space* if any two points $x, y \in X$ are joined by an arc γ of length $d(x, y) = L_d(\gamma)$. (This arc is necessarily a reparametrization of a geodesic but it is not necessarily unique.)

Every length space (X, d) which is locally compact and complete (as a metric space) is a geodesic space [Chavel (1993), Exercise 1.10].

EXAMPLES 2.2. (i) Every Riemannian manifold X equipped with the Riemannian distance d is a length space. It is even a geodesic space if the manifold is complete.

(ii) If (X, d) is a length space and Y an open subset of X, then there are two canonical metrics on Y derived from the metric d on X. The first one is just the restriction $d|_Y$ of d onto Y defined by $d|_Y(x, y) = d(x, y)$ for all $x, y \in Y$. The second one is the metric d_Y defined by $d_Y(x, y) = \inf\{L_d(\gamma) \colon \gamma$ is an arc in Y joining x and $y\}$ for $x, y \in Y$. Obviously, d_Y is the length metric derived from $d|_Y$. These metrics coincide if and only if Y is *convex*.

(iii) Let $X = \mathbb{R}^n$ be the Euclidean space, let $\alpha \in]0, 1[$ and let d be the metric $d(x, y) = |x - y|^\alpha$ which induces the Euclidean topology on X. Here the pseudometric $\overline{d}$ is entirely degenerate; namely, $\overline{d}(x, y) = \infty$ for all $x \neq y$. The space (X, d) is no length space.

(iv) Let $X = \mathbb{R}^n$ be the Euclidean space and put $d(x, y) = \inf\{|x - y|, 1\}$. Then $\overline{d}(x, y) = |x - y|$. Hence, (X, d) is no length space. Nevertheless, for all $x, y \in X$ which are close together [i.e., for which $d(x, y) < 1$] there exists a geodesic connecting them. Indeed, there is exactly one such geodesic.

(v) Let $X = \mathbb{R}^n$ be the Euclidean space and put $d(x, y) = \sup\{|x_i - y_i| \colon i = 1, \dots, n\}$. Then (X, d) is a geodesic space. But for "most" points $x, y \in X$, the geodesic connecting them is not unique. Namely, let $x, y \in X$ with $x - y$ not lying on the diagonal of X in the sense that $|x_i - y_i| < d(x, y)$ for some $i \in \{1, \dots, n\}$. Then every Lipschitz function $f \colon [0, 1] \to \mathbb{R}$ with $f(0) = f(1) = 0$ and Lipschitz constant $\operatorname{dil} f \leq d(x, y) - |x_i - y_i|$ defines a geodesic $\gamma \colon [0, 1] \to \mathbb{R}^n$, $t \mapsto x + t(y - x) + f(t)e_i$ connecting x and y.

For the theory of length spaces and for further examples we recommend Rinow (1961), Gromov (1981) and Ballmann (1995).

2.C. *Preliminaries on Dirichlet forms.* A *Dirichlet form* on the (real) Hilbert space $L^2(X,m)$ is a pair (E,F) consisting of a dense linear subspace $F\subset L^2(X,m)$ and a closed symmetric form E on $L^2(X,m)$, defined on $D(E)=F$, which has the Markovian property

$$E\left(u^\sharp,u^\sharp\right)\le E(u,u)\quad \text{for all } u\in F,$$

where $u^\sharp=(u\wedge 1)\vee 0$. It is called *regular* if F "contains sufficiently many continuous functions" in the sense that $F\cap C_0(X)$ is dense in F (with graph norm $[E(u,u)+\int u^2\,dm]^{1/2}$) as well as in $C_0(X)$ (with uniform norm $\sup_X|u|$). For the highly developed theory of regular Dirichlet forms we refer to the textbook by Fukushima, Oshima and Takeda (1994). We point out that there is an important generalization of that theory, namely, the so-called theory of *quasiregular* Dirichlet forms; see Bouleau and Hirsch (1991) and Ma and Röckner (1992).

In the sequel, we freely use the fact that there is a canonical one-to-one correspondence between *symmetric forms, quadratic forms* and *extended quadratic forms* on $L^2(X,m)$. We recall that evaluating a symmetric form $(E,D(E))$ on the diagonal yields a quadratic form $q\colon D(E)\to[0,\infty[$, $u\mapsto E(u,u)$ with domain $D(q):=D(E)$. Conversely, by polarization, any quadratic form $(q,D(q))$ defines a symmetric form $E\colon D(q)\times D(q)\to]-\infty,\infty[$, $(u,v)\mapsto \frac14[q(u+v)-q(u-v)]$ with domain $D(E):=D(q)$. Moreover, any quadratic form $(q,D(q))$ can be extended to an extended quadratic form Q as follows:

$$Q\colon L^2(X,m)\to[0,\infty],\qquad u\mapsto\begin{cases} q(u), & \text{for } u\in D(q),\\ \infty, & \text{for } u\in L^2(X,m)\setminus D(q).\end{cases}$$

Note that the set $\{u\in L^2(X,m)\colon Q(u)<\infty\}$ coincides with $D(q)$. Finally, by restricting an arbitrary extended quadratic form Q onto the set $D(Q):=\{u\in L^2(X,m)\colon Q(u)<\infty\}$ one obtains a quadratic form $(Q,D(Q))$. Most often, we use the same notation for a symmetric form, its associated quadratic form and extended quadratic form.

A crucial observation is that a symmetric form $(E,D(E))$ on $L^2(X,m)$ is *closed* if and only if the associated extended quadratic form is *lower semicontinuous on* $L^2(X,m)$. Therefore, given any symmetric form $(E,D(E))$ on $L^2(X,m)$, there is a canonical way to obtain a closed symmetric form from it. Namely, let Q denote the extended quadratic form (defined on the whole $L^2(X,m)$ with values in $[0,\infty]$) associated with $(E,D(E))$. Then

$$\underline{Q}(u)=\liminf_{\substack{v\in L^2\\ v\to u}} Q(v)$$

defines an extended quadratic form on $L^2(X,m)$. Obviously, this functional is lower semicontinuous on $L^2(X,m)$ and is dominated by Q in the sense that $\underline{Q}(u)\le Q(u)$ for each $u\in L^2(X,m)$. Actually, $\underline{Q}$ is the biggest lower semicontinuous functional on $L^2(X,m)$, which is dominated by Q. It is called the *relaxation* of Q and the associated symmetric form $(\underline{E},D(\underline{E}))$ is called the relaxation of $(E,D(E))$. Note that the symmetric form $(E,D(E))$ is *closed* if and only if it coincides with its relaxation and that it is *closable* if and only if $E = \underline{E}$ on $D(E)$. It always holds that $D(E) \subset D(\underline{E})$ and that $E(u,u) \geq \underline{E}(u,u)$ for $u \in D(E)$. See also Mosco (1994).

2.D. *A brief introduction to* Γ*-convergence.* Now we turn to convergence questions for symmetric and/or quadratic forms. Here it is much more convenient to formulate everything in terms of extended quadratic forms. We will give a brief survey on Γ-convergence, which is a certain variational convergence. It was introduced by De Giorgi. For more details we recommend the monograph by Dal Maso (1993). We assume that we are given a family $\{E^r, r > 0\}$ of extended quadratic functionals on $L^2(X, m)$.

DEFINITION 2.3. Let $(r_n)_n$ be a sequence with values in $]0,\infty[$. For any $u \in L^2(X, m)$, we define

$$\Gamma\text{-}\limsup_{n\to\infty} E^{r_n}(u,u) = \lim_{\alpha\to 0}\ \limsup_{n\to\infty}\ \inf_{\substack{v\in L^2\\ \|u-v\|\le\alpha}} E^{r_n}(v,v)$$

and

$$\Gamma\text{-}\liminf_{n\to\infty} E^{r_n}(u,u) = \lim_{\alpha\to 0}\ \liminf_{n\to\infty}\ \inf_{\substack{v\in L^2\\ \|u-v\|\le\alpha}} E^{r_n}(v,v).$$

Here and in the sequel $\|\cdot\|$ denotes the norm in $L^2 = L^2(X, m)$. Note that the $\lim_{\alpha\to 0}$ is actually a $\sup_{\alpha>0}$. We say that the sequence $(E^{r_n})_n$ is Γ*-convergent* if $\Gamma\text{-}\limsup_{n\to\infty} E^{r_n}(u,u) = \Gamma\text{-}\liminf_{n\to\infty} E^{r_n}(u,u)$ for each $u \in L^2(X, m)$. In this case, we write $\Gamma\text{-}\lim_{n\to\infty} E^{r_n}(u,u)$ for the common value of $\Gamma\text{-}\limsup_{n\to\infty} E^{r_n}(u,u)$ and $\Gamma\text{-}\liminf_{n\to\infty} E^{r_n}(u,u)$. The functional $\Gamma\text{-}\lim_{n\to\infty} E^{r_n}$ on $L^2(X, m)$ is then called Γ*-limit* of the (quadratic) functionals E^{r_n}, $n \in \mathbb{N}$.

Similarly, we define

$$\overline{E}^0(u,u) = \Gamma\text{-}\limsup_{r\to 0} E^r(u,u) = \lim_{\alpha\to 0}\ \limsup_{r\to 0}\ \inf_{\substack{v\in L^2\\ \|u-v\|\le\alpha}} E^r(v,v)$$

and

$$\underline{E}^0(u,u) = \Gamma\text{-}\liminf_{r\to 0} E^r(u,u) = \lim_{\alpha\to 0}\ \liminf_{r\to 0}\ \inf_{\substack{v\in L^2\\ \|u-v\|\le\alpha}} E^r(v,v)$$

and we say that the family $(E^r)_{r>0}$ is Γ-convergent for $r \to 0$ if the functionals $\Gamma\text{-}\limsup_{r\to 0} E^r$ and $\Gamma\text{-}\liminf_{r\to 0} E^r$ coincide on $L^2(X, m)$. If we choose $E^{r_n} = E$ for all $n \in \mathbb{N}$ and some fixed quadratic functional E on $L^2(X, m)$, then obviously $\Gamma\text{-}\lim_{r\to 0} E^r$ exists and coincides with the relaxation $\underline{E}$ of E (which might be different from E itself).

LEMMA 2.4. *Let $(r_n)_n$ be any sequence and let E^0 be any functional on $L^2(X, m)$ (with values in $[0,\infty]$). Then $E^0 = \Gamma\text{-}\lim_{n\to\infty} E^{r_n}$ if and only if the following two conditions are satisfied*:

(i) $\forall\ u \in L^2(X, m)$, $\forall (u_n)_{n\in N} \subset L^2(X, m)$ *with* $\|u_n - u\| \to 0$:

$$E^0(u,u) \le \liminf_{n\to\infty} E^{r_n}(u_n, u_n);$$

(ii) $\forall\, u \in L^2(X, m)$: $\exists (u_n)_{n\in N} \subset L^2(X, m)$ *with* $\|u_n - u\| \to 0$ *and*

$$E^0(u, u) = \lim_{n\to\infty} E^{r_n}(u_n, u_n).$$

For the proof, see Dal Maso (1993), Proposition 8.1.

LEMMA 2.5. (i) *For every sequence* $(r_n)_n$ *there exists a subsequence* $(r'_n)_n$ *such that the* Γ-*limit of the sequence* $(E^{r'_n})_n$ *exists.*

(ii) *There exist sequences* $(r_n)_n$ *and* $(r'_n)_n$ *such that* $\Gamma\text{-}\limsup_{r\to 0} E^r = \Gamma\text{-}\limsup_{n\to\infty} E^{r_n}$ *and* $\Gamma\text{-}\liminf_{r\to 0} E^r = \Gamma\text{-}\liminf_{n\to\infty} E^{r'_n}$.

(iii) *For every* $u \in L^2(X, m)$ *there exist sequences* $(r'_n)_n$ *and* $(r''_n)_n$ *such that* $\Gamma\text{-}\limsup_{r\to 0} E^r(u, u) = \Gamma\text{-}\lim_{n\to\infty} E^{r'_n}(u, u)$ *and* $\Gamma\text{-}\liminf_{r\to 0} E^r(u, u) = \Gamma\text{-}\lim_{n\to\infty} E^{r''_n}(u, u)$.

For the proof of (i), see Dal Maso (1993), Theorem 8.5; for (ii) and (iii), see Sturm (1997).

PROPOSITION 2.6. *Let* $(r_n)_n$ *be any sequence such that the* Γ-*limit* $E^0 := \Gamma\text{-}\lim_{n\to\infty} E^{r_n}$ *exists and put* $D(E^0) = \{u \in L^2(X, m): E^0(u, u) < \infty\}$.

(i) $(E^0, D(E^0))$ *is always a* (*not necessarily densely defined*) *closed symmetric form on* $L^2(X, m)$.

(ii) *If the* E^{r_n} *have the Markovian property then so has* E^0.

(iii) *Assume that*

$$\liminf_{n\to\infty} E^{r_n}(u, u) < \infty \tag{2.3}$$

for all $u \in F^0$ *where* F^0 *is some dense subset of* $L^2(X, m)$. *Then* (E^0, F^0) *is closable and its closure* (E, F) *is a densely defined symmetric form on* $L^2(X, m)$ *with core* F^0.

(iv) *Under the assumptions of* (ii) *and* (iii), *the symmetric form* (E, F) *is a Dirichlet form. If in addition, the set* F^0 *from* (iii) *is dense in* $C_0(X)$ *then the Dirichlet form* (E, F) *is regular.*

For the proof, see Mosco (1994), Corollary 2.8.

2.E. *The diffusion processes.* Given any *closed symmetric* form (E, F) on the Hilbert space $L^2(X, m)$, there exists a unique positive self-adjoint operator $(A, D(A))$ on $L^2(X, m)$ with the properties $F = D(A^{1/2})$ and

$$E(u, v) = (u, Av)$$

for all $u \in F$ and $v \in D(A)$. In terms of this operator $(A, D(A))$ we can define a strongly continuous contraction semigroup $(e^{-At})_{t>0}$ on $L^2(X, m)$. This semigroup is positivity preserving and extends to a contraction semigroup on each $L^p(X, m)$, $p \in [1, \infty]$, provided (E, F) is a *Dirichlet form*. If (E, F) is a *regular* Dirichlet form then there exists an m-symmetric Markov process $(X_t, \mathbf{P}_x)_{t\geq 0,\, x\in X}$ whose transition semigroup [extended to the space

$L^2(X, m)$] coincides with the semigroup $(e^{-At})_{t>0}$. Even more, such a Markov process $(X_t, \mathbf{P}_x)$ can be chosen "properly associated to (E, F)," which means that for each $u \in F \cap C_0(X)$ the function $x \mapsto \mathbf{E}_x[u(X_t), t < \zeta]$ is a quasicontinuous version of $e^{-At}u$. By this property, the process is determined *uniquely for quasi every starting point* $x \in X$.

This Markov process $(X_t, \mathbf{P}_x)$ is even a diffusion process (i.e., its paths $t \to X_t$ are continuous maps from $[0, \infty[$ to the one-point-compactification $X \cup \{\Delta\}$ of X) if and only if the Dirichlet form is *strongly local* in the sense that $E(u, v) = 0$ whenever $u \in F$ is constant on a neighborhood of the support of $v \in F$. See Fukushima, Oshima and Takeda (1994) for further details.

3. Local Dirichlet forms as Γ-limits of nonlocal approximations. The goal of this chapter is to present a general recipe for constructing Dirichlet forms, diffusion processes and heat kernels on metric spaces. Given *any* metric measured space (X, d, m) which satisfies the general assumptions from Section 2.A, we will define regular, *strongly local Dirichlet forms* (E, F) on $L^2(X, m)$ with core $C_0^{\mathrm{Lip}}(X)$. These *local* Dirichlet forms are restrictions of Dirichlet forms (E^0, F^0), which will be obtained as Γ-limits of *nonlocal* Dirichlet forms (E^r, F^r) on $L^2(X, m)$.

3.A. *The nonlocal approximations.* From now on, fix a function $N \in L^\infty_{\mathrm{loc}}(X, m)$ with $N \geq 0$ on X. For each $r > 0$ and $u \in L^2(X, m)$, define

$$Q^r(u, u) = \int_X N(x) \int_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x, y)}\right]^2 \times \frac{1}{m(B_r(x)) + m(B_r(y))}\, m(dy)\, m(dx)$$

and

$$E^r(u, u) = \frac{1}{2} \int_X N(x) \int_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x, y)}\right]^2 m_r(dy)\, m_r(dx). \tag{3.1}$$

Recall that $m_r(dy) = m(dy)/\sqrt{m(B_r(y))}$.

LEMMA 3.1. (i) $(E^r, D(E^r))$ *as well as* $(Q^r, D(Q^r))$ *are Dirichlet forms on* $L^2(X, m)$.

(ii) *For any* $u \in C_0^{\mathrm{Lip}}(X)$ (*with compact support* K *and Lipschitz constant* L),

$$Q^r(u, u) \leq E^r(u, u) \leq \frac{L^2}{2} m(B_r(K)) \operatorname*{ess\text{-}sup}_{B_r(K)} N. \tag{3.2}$$

PROOF. (i) Fix $r > 0$ and choose an exhaustion of X by relatively compact sets $Y_n \subset X$, $n \in \mathbb{N}$. The form $(E^r, D(E^r))$ is the increasing limit for

$n \to \infty$ of the forms $(E^{r,n}, L^2(X, m))$ where

$$E^{r,n}(u,u) = \frac{1}{2}\int_{Y_n}\int_{Y_n \cap B_r(x)} [u(x) - u(y)]^2 \times \left[\frac{N(x)}{\sqrt{m(B_r(y))\,m(B_r(x))}\,d^2(x,y)} \wedge n\right] m(dy)\,m(dx)$$

for $u \in L^2(X, m)$ and $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, the quadratic form $u \mapsto E^{r,n}(u,u)$ is continuous (!) on $L^2(X, m)$ since

$$\begin{aligned} E^{r,n}(u,u) &\le \frac{n}{2}\int_{Y_n}\int_{Y_n \cap B_r(x)} \left[2u^2(x) + 2u^2(y)\right] m(dy)\,m(dx) \\ &\le 2nm(Y_n)\|u\|_2^2. \end{aligned}$$

Hence, $(E^{r,n}, L^2(X, m))$ is a closed symmetric form on $L^2(X, m)$. For $n \to \infty$, this carries over to the increasing limit [see Dal Maso (1993)]. That is, (E^r, F^r) with $F^r = \{u \in L^2(x, m)\colon \sup_{n \in N} E^{r,n}(u,u) < \infty\} = \{u \in L^2(X, m)\colon E^r(u,u) < \infty\} = D(E^r)$ is a closed symmetric form on $L^2(X, m)$. One easily checks that $(E^r, D(E^r))$ has the Markovian property. The same arguments apply to $(Q^r, D(Q^r))$.

(ii) The first inequality in (3.2) is an immediate consequence of the inequality between the arithmetic and the geometric mean of $m(B_r(x))$ and $m(B_r(y))$. Moreover, for any u as in the claim

$$\begin{aligned} 2 \cdot E^r(u,u) &\le \operatorname*{ess-sup}_{B_r(K)} N \int_{B_r(K)}\int_{B_r(K)\cap B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)}\right]^2 m_r(dy)\,m_r(dx) \\ &\le \operatorname*{ess-sup}_{B_r(K)} N \int_{B_r(K)}\int_{B_r(K)\cap B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)}\right]^2 \\ &\qquad \times \frac{1}{m(B_r(x))}\,m(dy)\,m(dx) \\ &\le \operatorname*{ess-sup}_{B_r(K)} NL^2\,m(B_r(K)). \qquad \square \end{aligned}$$

REMARKS 3.2. (i) The closure $(\hat{E}^r, \hat{F}^r)$ of $(E^r, C_0^{\mathrm{Lip}}(X))$ is a regular Dirichlet form on $L^2(X, m)$ with core $C_0^{\mathrm{Lip}}(X)$. See Proposition 2.6.

(ii) If $N > 0$ m-a.e. and if there exist points $z_1, z_2 \in X$ with $0 < d(z_1, z_2) < r$ then the form $(\hat{E}^r, \hat{F}^r)$ is nonlocal. See Sturm (1997).

(iii) Assume for simplicity that $m(B_r(x))$ and $N(x)$ do not depend on $x \in X$ and put

$$\alpha = \frac{N(x)}{m(B_r(x))}\int_{B_r^*(x)} \frac{m(dy)}{d^2(x,y)}.$$

[Note that $\alpha = (n-2)/r^2$ for (X, d) being the Euclidean space $(\mathbb{R}^n, |\cdot|)$, $n \geq 3$, and m being the Lebesgue measure on it.] Then the Markov process $(X_t, \mathbf{P}_x)$ properly associated with $(\hat{E}^r, \hat{F}^r)$ can be constructed as follows.

(a) Let $(Z_n, \mathbf{Q}_x)_{n \in N_0, x \in X}$ with path space Ω_1 (e.g., $\{\omega: \mathbb{N}_0 \to X\}$) be a Markov chain on X which $\mathbf{Q}_x$-a.s. starts at x and then at the first step jumps into the ball $B_r(x)$ with

$$\mathbf{Q}_x(Z_1 \in A) = \int_{A \cap B_r^*(x)} \frac{m(dy)}{d^2(x,y)} \Big/ \int_{B_r^*(x)} \frac{m(dy)}{d^2(x,y)}$$

for all measurable $A \subset X$ and all $x \in X$.

(b) Let $(n_t, \mathbf{P})_{t \geq 0}$ with path space Ω_2 (e.g., {right cont. $\omega: \mathbb{R} \to \mathbb{N}_0$}) be a Poisson process with parameter α. That is, an $\mathbb{N}_0$-valued Markov process starting at 0 with independent increments $n_t - n_s$, which are Poisson distributed with parameter $\alpha(t-s)$.

(c) Then $(Z_{n_t}, \mathbf{Q}_x \otimes \mathbf{P})_{t \geq 0, x \in X}$ with path space $\Omega = \Omega_1 \times \Omega_2$ is a Markov process properly associated to $(\hat{E}^r, \hat{F}^r)$; see, for example, Ethier and Kurtz (1986), page 163.

Analogous results hold true for the closure of $(Q^r, C_0^{\mathrm{Lip}}(X))$.

3.B. *The limit forms.* Now we come to the main result of this section.

THEOREM 3.3. *Let $(r_n)_n$ be any sequence with $r_n \xrightarrow{n\to\infty} 0$ for which $E^0 = \Gamma\text{-}\lim_{n\to\infty} E^{r_n}$ exists and let $(r'_n)_n$ be any sequence with $r'_n \xrightarrow{n\to\infty} 0$ for which $Q^0 = \Gamma\text{-}\lim_{n\to\infty} Q^{r'_n}$ exists.*

(i) *Then $(E^0, C_0^{\mathrm{Lip}}(X)$ as well as $(Q^0, C_0^{\mathrm{Lip}}(X))$ are closable symmetric forms on $L^2(X, m)$.*

(ii) *Their closures $(E, D(E))$ and $(Q, D(Q))$, respectively, are regular Dirichlet forms on $L^2(X, m)$ with core $C_0^{\mathrm{Lip}}(X)$.*

(iii) *The form $(Q, D(Q))$ is always strongly local. The form $(E, D(E))$ is strongly local provided*

$$\limsup_{r\to 0} \sup_{\substack{x, y \in Y \\ d(x,y) < r}} \frac{m(B_r(x))}{m(B_r(y))} < \infty \tag{3.3}$$

for each compact set $Y \subset X$.

PROOF. (i) Let $D(E^0) = \{u \in L^2(X, m): E^0(u,u) < \infty\}$. Then by Dal Maso (1993), Proposition 6.8, Theorem 11.10 and Proposition 12.16, $(E^0, D(E^0))$ is a closed symmetric form on $L^2(X, m)$. Moreover, $C_0^{\mathrm{Lip}}(X) \subset D(E^0)$ since for $u \in C_0^{\mathrm{Lip}}(X)$ (with support $K \subset X$ and Lipschitz constant L),

$$\begin{aligned} E^0(u,u) &= \Gamma\text{-}\lim_{n\to\infty} E^{r_n}(u,u) \leq \liminf_{n\to\infty} E^{r_n}(u,u) \\ &\leq \frac{L^2}{2} m(K) \operatorname*{ess\text{-}sup}_{B_\delta(K)} N < \infty. \end{aligned}$$

Hence, $(E^0, C_0^{\mathrm{Lip}}(X))$ is a closable symmetric form on $L^2(X, m)$.

(ii) By Lemma 3.1 each of the approximating forms E^{r_n} has the Markovian property. Hence, by Mosco (1994), Corollary 2.8, also the Γ-limit has the Markovian property. This property, of course, carries over from $(E^0, D(E^0))$ to $(E^0, C_0^{\mathrm{Lip}}(X))$ and $(E, D(E))$. Thus $(E, D(E))$ is a Dirichlet form. The same arguments apply to $(Q, D(Q))$.

By construction, $(E, D(E))$ and $(Q, D(Q))$ are regular with core $C_0^{\mathrm{Lip}}(X)$.

In order to see the strong locality, let $u, v \in C_0^{\mathrm{Lip}}(X)$ with u being constant on a neighborhood of $A_1 = \mathrm{supp}[v]$, say $u \equiv \beta$ on $B_{3R}(A_1)$. Put $A_0 = \mathrm{supp}[u]$. Then there exist closed sets $X_0 = X \setminus B_R(A_0)$ and $X_1 = \overline{B}_{2R}(A_1) \setminus B_R(A_1)$ with $u \equiv 0$ on a neighborhood of X_0 and $u \equiv \beta$ as well as $v \equiv 0$ on a neighborhood of X_1. Put $Y = X \setminus (X_0 \cup X_1)$. According to Lemma 3.4 there exists a sequence $(f_n)_n \subset C_0^{\mathrm{Lip}}(Y)$ with $f_n \to 0$ in $L^2(X, m)$ and with

$$E(u+v, u+v) = \lim_{n\to\infty} E^{r_n}(u+v+f_n, u+v+f_n).$$

Now put $Y_0 = B_R(A_0) \setminus \overline{B}_{2R}(A_1)$ and $Y_1 = B_R(A_1)$, such that $Y = Y_0 \cup Y_1$. Then, obviously, $f_n = g_n + h_n$ with $g_n \in C_0^{\mathrm{Lip}}(Y_0)$ and $h_n \in C_0^{\mathrm{Lip}}(Y_1)$ and, for fixed $R > 0$ and sufficiently small $r > 0$,

$$\begin{aligned} &E^r(u+v+g_n+h_n, u+v+g_n+h_n) \\ &\qquad = E^r(u-v+g_n-h_n, u-v+g_n-h_n). \end{aligned}$$

Hence,

$$\begin{aligned} E(u+v, u+v) &= \lim_{n\to\infty} E^{r_n}(u+v+g_n+h_n, u+v+g_n+h_n) \\ &= \lim_{n\to\infty} E^{r_n}(u-v+g_n-h_n, u-v+g_n-h_n) \\ &\geq E(u-v, u-v). \end{aligned}$$

In the same way, we can prove $E(u-v, u-v) \geq E(u+v, u+v)$. That is, $E(u+v, u+v) = E(u-v, u-v)$. Since $E(u,v) = \frac{1}{4}[E(u+v, u+v) - E(u-v, u-v)]$, this implies $E(u,v) = 0$. This proves the strong locality of $(E, D(E))$. The same arguments apply to $(Q, D(Q))$. $\square$

LEMMA 3.4. *Let $(Q, D(Q))$ and $(E(D(E))$ be as in Theorem 3.3 and fix closed sets $X_0, X_1 \subset X$ as well as real numbers $\beta_0 = 0$, $\beta_1 \neq 0$. Then for all $u \in C_0^{\mathrm{Lip}}(X)$ which are identical to β_0 on a neighborhood of X_0 and identical to β_1 on a neighborhood of X_1,*

$$Q(u,u) = \lim_{\alpha\to 0} \limsup_{n\to\infty} \inf_{\substack{w \in C_0^{\mathrm{Lip}}(Y) \\ \|w\| \leq \alpha}} Q^{r'_n}(u+w, u+w)$$

where $Y = X \setminus (X_0 \cup X_1)$. If (3.3) is satisfied then also

$$E(u,u) = \lim_{\alpha\to 0} \limsup_{n\to\infty} \inf_{\substack{w \in C_0^{\mathrm{Lip}}(Y) \\ \|w\| \leq \alpha}} E^{r_n}(u+w, u+w).$$

The emphasis here is on the fact that the inf is over $w \in C_0^{\mathrm{Lip}}(Y)$ [and not over all $w \in C_0^{\mathrm{Lip}}(X)$]. Recall that $\|\cdot\|$ denotes the norm in $L^2(X, m)$.

PROOF. Fix $X_1, X_2 \subset X$ as well as β_0, β_1 and $u \in C_0^{\mathrm{Lip}}(X)$ as above. Assume without restriction that $Y = X \setminus (X_0 \cup X_1)$ is relatively compact. Choose $R > 0$ such that $u \equiv \beta_i$ on $B_{2R}(X_i)$ for $i = 0, 1$ and choose a function $\psi \in C_0^{\mathrm{Lip}}(X)$ with $\psi \equiv 1$ on $X \setminus B_R(X_0 \cup X_1)$, $\psi \equiv 0$ on $X_0 \cup X_1$ and $0 \le \psi \le 1$ on X. Note that $C_1 := \limsup_{n\to\infty} E^{r_n}(u, u) < \infty$ [cf. (3.2)]. Now obviously

$$\begin{aligned} E(u,u) &= \lim_{\alpha\to 0} \limsup_{n\to\infty} \inf_{\substack{w\in C_0^{\mathrm{Lip}}(X)\\ \|w\|\le 1}} E^{r_n}(u+\alpha w, u+\alpha w) \\ &\le \lim_{\alpha\to 0} \limsup_{n\to\infty} \inf_{\substack{w\in C_0^{\mathrm{Lip}}(Y)\\ \|w\|\le 1}} E^{r_n}(u+\alpha w, u+\alpha w) \\ &\le \lim_{\alpha\to 0} \limsup_{n\to\infty} \inf_{\substack{w\in C_0^{\mathrm{Lip}}(X)\\ \|w\|\le 1}} E^{r_n}(u+\alpha \tilde{w}, u+\alpha \tilde{w}) \end{aligned}$$

where for each $w \in C_0^{\mathrm{Lip}}(X)$ with $\|w\| \le 1$, we define $\tilde{w} = w \cdot \psi$. Note that then $\tilde{w} \in C_0^{\mathrm{Lip}}(Y)$ with $\|\tilde{w}\| \le 1$. Obviously,

$$\begin{aligned} |\tilde{w}(x) - \tilde{w}(y)|^2 &= \frac{1}{4}|(w(x)+w(y))(\psi(x)-\psi(y)) \\ &\quad + (w(x)-w(y))(\psi(x)+\psi(y))|^2 \\ &\le \frac{1+\alpha}{4\alpha}|w(x)+w(y)|^2|\psi(x)-\psi(y)|^2 \\ &\quad + \frac{1+\alpha}{4}|w(x)-w(y)|^2|\psi(x)+\psi(y)|^2 \\ &\le \frac{1+\alpha}{2\alpha}(w^2(x)+w^2(y))|\psi(x)-\psi(y)|^2 \\ &\quad + (1+\alpha)|w(x)-w(y)|^2 \\ &\le \frac{1+\alpha}{2\alpha}C_2^2(w^2(x)+w^2(y))d^2(x,y) \\ &\quad + (1+\alpha)|w(x)-w(y)|^2 \end{aligned}$$

for each $\alpha > 0$ with $C_2 = \operatorname{dil}\psi = \sup_{x\ne y}|\psi(x)-\psi(y)|/d(x,y) < \infty$. Hence,

$$\begin{aligned} E^r(\tilde{w},\tilde{w}) \le \frac{1+\alpha}{8\alpha}C_2^2 \int_Y\int_{B_r(x)} &\left[w^2(x)+w^2(y)\right] \\ &\times \frac{N(x)+N(y)}{\sqrt{m(B_r(x))\cdot m(B_r(y))}}\, m(dy)\, m(dx) \\ + \frac{1+\alpha}{4}\int_Y\int_{B_r(x)} &\left[\frac{w(x)-w(y)}{d(x,y)}\right]^2 \end{aligned}$$

$$\times \frac{N(x)+N(y)}{\sqrt{m(B_r(x))\cdot m(B_r(y))}}\, m(dy)\, m(dx)$$

$$\leq \frac{1+\alpha}{\alpha} C_3 + (1+\alpha) E^r(w,w)$$

for each $w \in C_0^{\mathrm{Lip}}(X)$ with $\|w\| \leq 1$, each $r \in]0, R[$ and each $\alpha > 0$ with

$$C_3 = C_2^2 \operatorname*{ess\text{-}sup}_{x\in B_r(Y)} N(x) \sup_{r<R} \sup_{\substack{x,\, y\in B_R(Y)\\ d(x,y)<r}} \sqrt{\frac{m(B_r(x))}{m(B_r(y))}} < \infty.$$

This implies

$$\begin{aligned} E^{r_n}(u+\alpha\tilde{w}, u+\alpha\tilde{w}) &= E^{r_n}(u,u) + 2\alpha E^{r_n}(u,\tilde{w}) + \alpha^2 E^{r_n}(\tilde{w},\tilde{w}) \\ &\leq E^{r_n}(u,u) + 2\alpha E^{r_n}(u,w) \\ &\quad + \alpha^2(1+\alpha)E^{r_n}(w,w) + \alpha(1+\alpha)C_3 \\ &= \frac{1}{1+\alpha} E^{r_n}(u+\alpha(1+\alpha)w, u+\alpha(1+\alpha)w) \\ &\quad + \frac{\alpha}{(1+\alpha)} E^{r_n}(u,u) + \alpha(1+\alpha)C_3 \end{aligned}$$

for all $\alpha > 0$, $w \in C_0^{\mathrm{Lip}}(X)$ with $\|w\| \leq 1$ and $n \in \mathbb{N}$ with $r_n < R$. Therefore,

$$\begin{aligned} E(u,u) &\leq \lim_{\alpha\to 0} \limsup_{n\to\infty} \inf_{\substack{w\in C_0^{\mathrm{Lip}}(Y)\\ \|w\|\leq 1}} E^{r_n}(u+\alpha w, u+\alpha w) \\ &\leq \lim_{\alpha\to 0} \limsup_{n\to\infty} \inf_{\substack{w\in C_0^{\mathrm{Lip}}(X)\\ \|w\|\leq 1}} \frac{1}{1+\alpha} E^{r_n}(u+\alpha(1+\alpha)w, u+\alpha(1+\alpha)w) \\ &\quad + \frac{\alpha}{(1+\alpha)} C_1 + \alpha(1+\alpha)C_3 \\ &= E(u,u). \end{aligned}$$

Similar (actually easier) arguments apply to $(Q, D(Q))$ [for details, see Sturm (1997)]. □

REMARKS 3.5. (i) Assume that $(r_n)_n = (r_n')_n$. Then under (3.3) the limit forms $(E, D(E))$ and $(Q, D(Q))$ are equivalent. They coincide provided

$$\limsup_{r\to 0} \sup_{\substack{x,\, y\in Y\\ d(x,y)<r}} \frac{m(B_r(x))}{m(B_r(y))} = 1 \tag{3.4}$$

for each compact set $Y \subset X$.

(ii) Condition (3.3) obviously follows from the doubling property

$$\limsup_{r\to 0} \sup_{x\in Y} \frac{m(B_{2r}(x))}{m(B_r(x))} < \infty$$

for each compact set $Y \subset X$ [see also (4.8)]. Actually, (3.3) is much weaker.

In the sequel we restrict ourselves to the form $(E, D(E))$ and we assume for simplicity that (3.3) is satisfied.

COROLLARY 3.6. *For each limit point E (in the sense of Γ-convergence) of the family $(E^r)_{r>0, r\to 0}$ there exists an m-symmetric diffusion process $(X_t, \mathbf{P}_x)$ on X with the properties*

$$\mathbf{E}_x[u(X_t)] = e^{-At}u(x) \tag{3.5}$$

for m-a.e. $x \in X$ and all $u \in C_0^{\mathrm{Lip}}(X, m)$ and

$$E(u, u) = \lim_{t\to 0} \frac{1}{2t}\int \mathbf{E}_x\big[(u(X_t) - u(x))\big]^2 m(dx) \tag{3.6}$$

for all $u \in C_0^{\mathrm{Lip}}(X, m)$. By each of these properties, the process is determined uniquely for quasi every starting point.

For the proof, see Theorem 3.3 together with Fukushima, Oshima and Takeda (1994).

DEFINITION 3.7. If $\underline{E}^0$ (or $\overline{E}^0$) is a limit point (in the sense of Γ-convergence) of the family $(E^r)_{r>0, r\to 0}$ then the diffusion process associated with the Dirichlet form $\underline{E}$ (or $\overline{E}$, resp.) is called the *relaxed diffusion* (or the *excited diffusion*, resp.) on (X, d, m) with normalization function N. If $\underline{E}$ and $\overline{E}$ coincide, then the associated diffusion process is called the *canonical diffusion* on (X, d, m) with normalization function N.

EXAMPLE 3.8. Let $X = \mathbb{R}^n$ be the Euclidean space equipped with the metric $d(x, y) = |x - y|^\alpha$ for some $\alpha \in]0, 1[$ and let m be the Lebesgue measure on $\mathbb{R}^n$. Then

$$\underline{E}(u, u) = \overline{E}(u, u) = 0$$

for all $u \in \underline{F} = \overline{F} = L^2(X, m)$. The properly associated diffusion process $(X_t, \mathbf{P}_x)$ is just the process which always stays at the starting place. That is,

$$\mathbf{P}_x(X_t = x, \forall\, t \geq 0) = 1.$$

REMARKS 3.9. (i) Let X be an open subset of $\mathbb{R}^n$ and let m be the Lebesgue measure on X. Then according to the second Beurling–Deny formula [cf. Fukushima, Oshima and Takeda (1994)] each limit point E of the family $(E^r)_{r>0}$ for $r \to 0$ can be represented on $C_0^\infty(X)$ uniquely as

$$E(u,v) = \frac{1}{2}\sum_{i,j=1}^n \int_X \frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\, d\nu_{ij}$$

with some uniquely determined Radon measures ν_{ij}, $i, j = 1, \dots, n$ on X.

(ii) If in addition d is the Euclidean metric on X and $N \in C(X)$, then these measures ν_{ij} are absolutely continuous with a density

$$a_{ij}(x) = \frac{N(x)}{n}\delta_{ij}.$$

See Section 8.

(iii) We emphasize that the difference between two limit points E' and E'' of the family $(E^r)_{r>0}$ for $r \to 0$ never is caused by different "boundary conditions." Actually, both correspond to the same "Dirichlet boundary condition" on X in the sense that both have the same form core $C_0^{\mathrm{Lip}}(X)$. If different limit points E' and E'' exist, then they really have different "diffusion coefficients" $(\nu_{ij})_{i,j}$.

4. The Measure Contraction Property.

4.A. *The definition.* Throughout the sequel, the basic assumptions on (X, d, m) from Section 2.A are still in force. Recall that $m_r(dy) = \left(1/\sqrt{m(B_r(y))}\right)m(dy)$.

DEFINITION 4.1. We say that the metric measured space (X, d, m) satisfies the *weak Measure Contraction Property* (MCP) with *exceptional set* iff there is a closed set $Z \subset X$ with $m(Z) = 0$ such that for each compact set $Y \subset X \setminus Z$ there exist numbers $R > 0$, $\Theta < \infty$ and $\vartheta < \infty$ and m^2-measurable maps Φ_t: $X \times X \to X$ (for all $t \in [0,1]$) with the following properties.

(i) For m-a.e. $x, y \in Y$ with $d(x,y) < R$ and all $s, t \in [0,1]$,

$$\begin{gathered}\Phi_0(x,y) = x, \qquad \Phi_t(x,y) = \Phi_{1-t}(y,x),\\ \Phi_s(x, \Phi_t(x,y)) = \Phi_{st}(x,y),\end{gathered} \tag{4.1}$$

$$d(\Phi_s(x,y), \Phi_t(x,y)) \le \vartheta |s-t| d(x,y); \tag{4.2}$$

(ii) For all $r < R$, m-a.e. $x \in Y$, all m-measurable $A \subset B_r(x) \cap Y$ and all $t \in [0,1]$,

$$\frac{m_r(A)}{\sqrt{m(B_r(x))}} \le \Theta \frac{m_{rt}(\Phi_t(x,A))}{\sqrt{m(B_{rt}(x))}}. \tag{4.3}$$

Of course, the constants R, Θ, ϑ are not unique. Let $R(Y)$ be the supremum of such R. *Don't worry about the* ϑ. In most cases, one can choose $\vartheta = 1$.

We say that (X, d, m) satisfies the *strong* Measure Contraction Property with exceptional set Z iff for each compact $Y \subset X \setminus Z$ the constants Θ and ϑ can be chosen arbitrarily close to 1 and if for every $\Theta' > 1$ there exists some

$\vartheta' > 1$ such that for m-a.e. $x \in Y$ and all $r < R$ with $B_r(x) \subset Y$,

$$m\big(B_{r\vartheta'}(x)\big) \le \Theta' m\big(B_r(x)\big). \tag{4.4}$$

In this case, without restriction we will always assume that $\Theta = \Theta'$ and $\vartheta < \vartheta'$.

If the above conditions are satisfied with $Z = \emptyset$ then we say that (X, d, m) satisfies the (*weak or strong*, resp.) *Measure Contraction Property without exceptional set*.

REMARKS 4.2 [Concerning (4.1) and (4.2)]. (i) Assume that (4.1) and (4.2) hold true for all $Y \subset X$ with $R = \infty$ and $\vartheta = 1$. Then (X, d) is a length space. For m^2-a.e. $(x, y) \in X^2$, the map $\Phi(x, y): [0,1] \to X$, $t \mapsto \Phi_t(x, y)$ is a geodesic connecting x and y. In general, this map is a *quasi geodesic*.

(ii) If (X, d) is a geodesic space then for all $(x, y) \in X^2$ there exists a geodesic $\gamma = \gamma(x, y): [0,1] \to X$ joining x and y. Conditions (4.1) and (4.2) (with $\vartheta = 1$) are obviously satisfied if this geodesic $\gamma(x, y)$ can be chosen as a *measurable* function of x and y.

(iii) Assumption (4.1) and (4.2) (with $\vartheta = 1$) are always satisfied if (X, d) is a geodesic space with curvature in the sense of Alexandrov being locally bounded from above. In this case, on each compact set $Y \subset X$ there is a strictly positive *injectivity radius* $R(Y)$ in the sense that for each $(x, y) \in Y \times Y$ with $d(x, y) < R(Y)$ there exists exactly one geodesic connecting x and y and this geodesic depends *continuously* on x and y [Ghys and de la Harpe (1990), Chapter 10].

REMARKS 4.3 [Concerning (4.3)]. (i) Assume that for all $r < R$ and for m-a.e. $x \in Y$ and $y \in X$ with $d(x, y) < \vartheta r$,

$$\frac{1}{\alpha} \le \frac{m\big(B_r(y)\big)}{m\big(B_r(x)\big)} \le \alpha. \tag{4.5}$$

Then (4.3) implies

$$\frac{m(A)}{m\big(B_r(x)\big)} \le \Theta^* \frac{m\big(\Phi_t(x, A)\big)}{m\big(B_{rt}(x)\big)} \tag{4.6}$$

with $\Theta^* = \alpha \cdot \Theta$ and, conversely, (4.6) implies (4.3) with $\Theta = \alpha \cdot \Theta^*$. In the case $\vartheta = 1$, property (4.6) has an obvious geometric–measure theoretic meaning: it is a control for distortions of the volume of sets under the "contraction" $\Phi_t(x, \cdot)$: $X \to X$ of the state space. The LHS of (4.6) measures the proportion of A in $B_r(x)$ whereas the RHS measures the proportion of the image of A in the image of $B_r(x)$.

(ii) For $t \in [0,1]$ let Φ_t: $X^2 \to X$ be a map with the properties (4.1) and (4.2) from Definition 4.1 and assume for simplicity that $\vartheta = 1$. For $s, t \in [0,1]$ define

$$\Phi_{s,t}: X^2 \to X^2, \qquad (x, y) \mapsto \big(\Phi_s(x, y), \Phi_t(x, y)\big).$$

[By the way, note that (4.2) states that $d \circ \Phi_{s,t} \le \vartheta|s-t|d$ a.e. on $\{(x,y) \in X^2: d(x,y) < R\}$.] Consider the measure $m_r^2(dx\,dy) = m_r(dx) \times m_r(dy)$ on X^2. Then (4.3) implies

$$m_r^2(A) \le \Theta^* m^2_{|s-t|r}(\Phi_{s,t}(A)) \tag{4.7}$$

for all m^2-measurable, symmetric $A \subset \{(x,y) \in Y^2: d(x,y) < r\}$ and all $s,t \in [0,1]$ with $\Theta^* = \Theta^2$. (The proof of this fact is implicitly contained in the proof of Lemma 5.2 below.) Conversely, if (4.7) holds true with $s = 0$ for all m^2-measurable $A \subset \{(x,y) \in Y^2: d(x,y) < r\}$ and $t \in [0,1]$, then it implies (4.3) with $\Theta = \Theta^*$. Property (4.7) is exactly that required in the sequel.

REMARK 4.4 [Concerning (4.4)]. Property (4.4) as well as (4.5) follow from (4.3); see Corollary 4.6 below. The point, however, is that $\Theta \to 1$ in (4.3) does not imply $\Theta' \to 1$ in (4.4) nor $\alpha \to 1$ in (4.5). On the other hand, if $\vartheta = 1$, then we can modify our approach in order to dispense entirely with (4.4).

PROPOSITION 4.5. *The property* (4.3) *of the weak MCP with exceptional set Z implies the volume doubling property on $X \setminus Z$. That is, for each compact set Y there exist constants M and $R > 0$ such that*

$$m(B_{2r}(x)) \le M m(B_r(x)) \tag{4.8}$$

for all $r \in]0, R[$ and m-a.e. $x \in Y$. The same holds true with (4.6) *in the place of* (4.3).

The number M is called *doubling constant*.

PROOF. Without restriction $Y \ne \emptyset$. Fix a compact set $Y' \subset X \setminus Z$ and some $R \in]0, R(Y')[$ with $B_{2R}(Y) \subset Y'$. Let R, Θ and ϑ be the constants from the MCP for the compact set Y' and put $R_0 = R/(3\vartheta)$. Apply (4.3) with $t = 3r/R$ to the sets $A = B_{R_0}(x)$ and $B_R(x)$ in order to obtain

$$\begin{aligned}
\frac{1}{\Theta}\frac{m(B_{R_0}(x))}{m(Y')} &\le \frac{1}{\Theta}\frac{1}{\sqrt{m(B_R(x))}}\int_{B_{R_0}(x)}\frac{m(dy)}{\sqrt{m(B_R(y))}}\\
&= \frac{1}{\Theta}\frac{m_R(B_{R_0}(x))}{\sqrt{m(B_R(x))}} \le \frac{m_{Rt}(\Phi_t(x, B_{R_0}(x))}{\sqrt{m(B_{Rt}(x))}} \le \frac{m_{Rt}(B_{R_0 t\vartheta}(x))}{\sqrt{m(B_{Rt}(x))}}\\
&= \frac{1}{\sqrt{m(B_{3r}(x))}}\int_{B_r(x)}\frac{m(dy)}{\sqrt{m(B_{3r}(y))}} \le \frac{m(B_r(x))}{m(B_{2r}(x))}.
\end{aligned}$$

Finally, note that

$$\inf_{x \in Y} m(B_{R_0}(x)) \ge \inf_{j=1,\dots,k} m(B_{R_0/2}(x_j)) > 0$$

[since Y is covered by finitely many nonempty $B_{R_0/2}(x_j)$, $j = 1, \dots, k$]. The proof of (4.6) $\Rightarrow$ (4.8) is essentially the same. $\square$

COROLLARY 4.6. *Conditions* (4.3) *and* (4.6) *are equivalent. Each of them implies* (4.4) [*as well as* (4.5)].

4.B. *Example*: *Riemannian manifolds*. Let (X, g) be an n-dimensional Riemannian manifold. Let d be the Riemannian distance and m be the Riemannian volume on X. Obviously, (X, d, m) satisfies the basic assumptions from Section 2.A.

PROPOSITION 4.7. *(X, d, m) satisfies the strong MCP without exceptional set.*

PROOF. Fix a compact set $Y \subset X$, let $R > 0$ be a lower bound for the injectivity radius on Y and choose $\kappa \geq 0$ such that $-(n-1)\kappa$ is a lower bound for the Ricci curvature on $B_R(Y)$.

For $x, y \in Y$ with $d(x, y) < R$, let $\Phi(x, y)$: $[0, 1] \to X$ be the *unique* minimal geodesic connecting x and y (i.e., with $\Phi_0(x, y) = x$ and $\Phi_1(x, y) = y$). For all other $x, y \in X$ and all $t \in [0, 1]$, put $\Phi_t(x, y) = z$ with some fixed $z \in X$. This choice of geodesics depends in a measurable way on x and y and obviously satisfies (4.1) and (4.2).

If $\kappa > 0$, let $S_\kappa(r) = c_n[\sinh(r\sqrt{\kappa})/\sqrt{\kappa}]^{n-1}$ be the area of the sphere of radius $r > 0$ in the space of constant sectional curvature $-\kappa$ and put $S_0(r) = c_n \cdot r^{n-1}$. Here c_n denotes the area of the unit sphere in $\mathbb{R}^n$. The Bishop volume comparison theorem implies that

$$\begin{aligned} m\big(\Phi_t(x, A)\big) &= t\int_0^R \int 1_A(\exp \rho\xi) \det A(t\rho, \xi)\, d\mu_x(\xi)\, d\rho \\ &\geq t\int_0^R \int 1_A(\exp \rho\xi) \det A(\rho, \xi) \frac{S_\kappa(t\rho)}{S_\kappa(\rho)}\, d\mu_x(\xi)\, d\rho \\ &\geq t\frac{S_\kappa(tr)}{S_\kappa(r)} m(A) \end{aligned}$$

for all $r < R$, $x \in Y$, $A \subset B_r(x)$ and all $t \in [0, 1]$ [Chavel (1993); Theorem 3.10]. That is,

$$\frac{m(A)}{rS_\kappa(r)} \leq \frac{m\big(\Phi_t(x, A)\big)}{trS_\kappa(tr)}. \tag{4.9}$$

The Bishop and Bishop–Günther volume comparison theorems imply that

$$\frac{rS_\kappa(r)/n}{m\big(B_r(x)\big)} \to 1 \qquad \text{for } r \to 0$$

uniformly in $x \in Y$. This proves (4.3) and (4.4). □

4.C. *Example*: *uniformly elliptic matrices*. Let $a = (a_{ij}(x))_{i,j=1,\dots,n}$ be a uniformly elliptic, symmetric matrix on $\mathbb{R}^n$, $n \geq 1$, with bounded measurable coefficients. That is, $x \mapsto a_{ij}(x)$ is a bounded, (Lebesgue) measurable function on $\mathbb{R}^n$ [for each $(i,j) \in \{1,\dots,n\}^2$] and $a(x) = (a_{ij}(x))_{i,j=1,\dots,n}$ is a symmetric $n \times n$-matrix (for each $x \in \mathbb{R}^n$) satisfying

$$\lambda^{-1}|\xi|^2 \leq \xi a(x)\xi \leq \lambda|\xi|^2 \tag{4.10}$$

for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^n$ with some constant $\lambda > 0$. Moreover, let b be a (Lebesgue) measurable function on $\mathbb{R}^n$ with $\Lambda^{-1} \leq b(x) \leq \Lambda$ for every $x \in \mathbb{R}^n$ with some constant $\Lambda > 0$. The function b defines a measure m_b on $\mathbb{R}^n$ by $m_b(dx) = b(x)\,dx$ and the matrix a defines a metric d_a on $\mathbb{R}^n$ by

$$d_a(x,y) = \inf\left\{L_a(\gamma) \colon \gamma \in C^1([0,1] \to \mathbb{R}^n), \gamma(0) = x, \gamma(1) = y\right\}, \tag{4.11}$$

where

$$L_a(\gamma) = \int_0^1 \sqrt{\dot\gamma_t a^{-1}(\gamma_t)\dot\gamma_t}\, dt. \tag{4.12}$$

(Here $\gamma_t := \gamma(t)$ and $\dot\gamma_t := (d/dt)\gamma_t$.) Obviously, d_a is a length metric on $\mathbb{R}^n$ comparable with the Euclidean metric according to

$$\lambda^{-1/2}|x-y| \leq d_a(x,y) \leq \lambda^{1/2}|x-y| \tag{4.13}$$

for all $x, y \in \mathbb{R}^n$. Therefore, obviously, the basic assumptions from Section 2.A are satisfied for the triplet $(\mathbb{R}^n, d_a, m_b)$.

PROPOSITION 4.8. *The metric measured space $(\mathbb{R}^n, d_a, m_b)$ satisfies the weak MCP without exceptional set.*

PROOF. In order to see this, choose $\Phi(x,y)$ to be the Euclidean geodesic connecting x and y, that is, $\Phi_t(x,y) = x + t(y-x)$. Then obviously (4.1), (4.3) (with any $\Theta \geq \Lambda^4$) and (4.4) (with any $\Theta > \Lambda^2\vartheta^n$) are satisfied. Moreover, (4.2) (with $\vartheta = \lambda$) follows from (4.13) according to

$$\begin{aligned} d_a(\Phi_s(x,y), \Phi_t(x,y)) &= d_a(x + s(y-x), x + t(y-x)) \\ &\leq \lambda^{1/2}|s-t|\,|y-x| \leq \lambda|s-t|\,d_a(x,y). \qquad \square \end{aligned}$$

PROPOSITION 4.9. *Let $(\mathbb{R}^n, d_a, m_b)$ as before and assume in addition now that $x \mapsto b(x)$ as well as $x \mapsto a_{ij}(x)$ are continuous functions on $\mathbb{R}^n$ (for each $(i,j) \in \{1,\dots,n\}^2$). Then $(\mathbb{R}^n, d_a, m_b)$ satisfies the strong MCP without exceptional set.*

The proof will be decomposed into several steps.

LEMMA 4.10. *For every compact set $K \subset \mathbb{R}^n$ and every $\vartheta > 1$ there exists a uniformly elliptic, symmetric matrix $\tilde a$ on $\mathbb{R}^n$ with C^∞-coefficients such that*

$$\vartheta^{-1/2} d_{\tilde a}(x,y) \leq d_a(x,y) \leq \vartheta^{1/2} d_{\tilde a}(x,y) \tag{4.14}$$

for all $x, y \in K$.

PROOF. Let us write $|\xi| = (\sum_{i=1}^n \xi_i^2)^{1/2}$ for a vector $\xi \in \mathbb{R}^n$ and $|a(x)| = (\sum_{i,j=1}^n a_{ij}(x)^2)^{1/2}$ for the matrix $a(x) \in \mathbb{R}^{n\times n}$. Obviously, $\lambda^{-1} \le |a(x)| \le n \cdot \max_{i,j}|a_{ij}(x)| \le n \cdot \lambda$ according to (4.10).

Let $\psi \in C^\infty(\mathbb{R}^n)$ with $\psi \ge 0$, $\operatorname{supp}[\psi] \subset B_1(0)$ and $\int \psi(x)\,dx = 1$ and define

$$a_{ij}^{(k)}(x) = \int_{B_1(0)} a_{ij}\left(x + \frac{y}{k}\right)\psi(y)\,dy.$$

Then one easily checks that $a^{(k)} = (a_{ij}^{(k)})_{i,j=1,\dots,n}$ is a uniformly elliptic matrix with

$$\lambda^{-1}|\xi|^2 \le \xi a^{(k)}(x)\xi \le \lambda|\xi|^2 \tag{4.15}$$

for each $k \in \mathbb{N}$. For all $i, j = 1, \dots, n$ the functions $a_{ij}^{(k)}$ converge for $k \to \infty$ locally uniformly to a_{ij}. More precisely, on each compact set $K' \subset \mathbb{R}^n$,

$$|a_{ij}^{(k)}(x) - a_{ij}(x)| \le f\left(\frac{1}{k}\right)$$

for all $x \in K'$, $k \in \mathbb{N}$ and $i, j = 1, \dots, n$ with $f\colon r \mapsto f(r) = \max\{|a_{ij}(y) - a_{ij}(y')|\colon y, y' \in B_1(K('), \; |y - y'| < r, \; i, j = 1, \dots, n\}$ being the modulus of continuity of a on $B_1(K')$. Therefore,

$$\begin{aligned}
&\left|\xi a^{(k)^{-1}}(x)\xi - \xi a^{-1}(x)\xi\right| \\
&\quad = \left|\xi a^{(k)^{-1}}(x)\big(a^{(k)}(x) - a(x)\big)a^{-1}(x)\xi\right| \\
&\quad \le |\xi|^2 \left|a^{(k)}(x)^{-1}\right| \left|a^{(k)}(x) - a(x)\right| \left|a(x)^{-1}\right| \\
&\quad \le |\xi|^2 \lambda^2 n^3 f(1/k) \\
&\quad \le |\xi a^{(k)^{-1}}(x)\xi| \lambda^3 n^3 f(1/k).
\end{aligned}$$

Hence,

$$|\xi a^{-1}(x)\xi| \le \big[1 + \lambda^3 n^3 f(1/k)\big] |\xi a^{(k)^{-1}}(x)\xi| \le \vartheta |\xi a^{(k)^{-1}}(x)\xi|$$

if k is chosen sufficiently large. Similarly,

$$|\xi a^{(k)^{-1}}(x)\xi| \le \vartheta |\xi a^{-1}(x)\xi|.$$

That is, with $\tilde a = a^{(k)}$,

$$\vartheta^{-1}|\xi \cdot \tilde a^{-1}(x)\xi| \le |\xi a^{-1}(x)\xi| \le \vartheta|\xi \tilde a^{-1}(x)\xi| \tag{4.16}$$

for all $x \in K'$ and $\xi \in \mathbb{R}^n$. Inequalities (4.16) immediately imply

$$\vartheta^{-1/2} L_{\tilde a}(\gamma) \le L_a(\gamma) \le \vartheta^{1/2} L_{\tilde a}(\gamma) \tag{4.17}$$

for all arcs $\gamma \in C^1([0,1] \to K')$. (Note that the arc may not leave K'.)

Now fix a compact set $K \subset \mathbb{R}^n$, choose an Euclidean ball $B_r^{\mathrm{id}}(z) = \{x\colon |x - z| < r\}$ with $K \subset B_r^{\mathrm{id}}(z)$ and put $K' = \overline{B}_{(1+2\lambda)r}^{\mathrm{id}}(z)$. Then for an appropriate choice of $\tilde a$, (4.16) holds true. Let $x, y \in K$ and consider an arc $\gamma \in C^1([0,1] \to X)$ connecting x and y and having length $L_a(\gamma) \le \frac{3}{2} d_a(x, y)$. If $\gamma_t \notin K'$ for some $t \in [0,1]$ then there would exist $t_1 < t_2$ with $\gamma_{t_1}, \gamma_{t_2} \in \partial K'$

and $\gamma_t \in K'$ for all $t \in [0, t_1] \cup [t_2, 1]$. This would imply

$$L_a(\gamma) \geq d_a(x, \gamma_{t_1}) + d_a(\gamma_{t_2}, y) \geq \lambda^{-1/2}\big(|x - \gamma_{t_1}| + |\gamma_{t_2} - y|\big)$$
$$\geq \lambda^{-1/2} 4\lambda r \geq 2\, d_a(x, y) \geq \tfrac{4}{3} L_a(\gamma),$$

which is a contradiction. Therefore, all such arcs stay in K'. Hence, the estimate (4.17) can be applied in order to estimate

$$\begin{aligned} d_a(x, y) &= \inf\big\{L_a(\gamma)\colon \gamma \in C^1([0,1] \to X)\big\} \\ &= \inf\big\{L_a(\gamma)\colon \gamma \in C^1([0,1] \to K')\big\} \\ &\geq \vartheta^{-1/2} \inf\big\{L_{\tilde a}(\gamma)\colon \gamma \in C^1([0,1] \to K')\big\} \\ &\geq \vartheta^{-1/2} \inf\big\{L_{\tilde a}(\gamma)\colon \gamma \in C^1([0,1] \to X)\big\} = \vartheta^{-1/2} d_{\tilde a}(x, y). \end{aligned}$$

Interchanging the roles of a and $\tilde a$ yields

$$d_{\tilde a}(x, y) \geq \vartheta^{-1/2} d_a(x, y).$$

This proves the claim. □

LEMMA 4.11. *Let $\tilde d = d_{\tilde a}$ with $\tilde a$ from Lemma 4.10 and put $\tilde m(dx) = (\det \tilde a(x))^{-1/2}\, dx$ and $m(dx) = m_b(dx) = b(x)\, dx$ with b from Proposition 4.9. Then $(\mathbb{R}^n, \tilde d, \tilde m)$ as well as $(\mathbb{R}^n, \tilde d, m)$ satisfies the strong MCP without exceptional set.*

PROOF. Let $g(x) = \tilde a^{-1}(x)$ be the inverse of the matrix $\tilde a(x)$. Then $(\mathbb{R}^n, g)$ is a smooth Riemannian manifold with Riemannian distance $\tilde d = d_{\tilde a}$ and Riemannian volume $\tilde m(dx) = (\det \tilde a(x))^{-1/2}\, dx$. Hence, $(\mathbb{R}^n, \tilde d, \tilde m)$ satisfies the strong MCP without exceptional set according to Proposition 4.7.

In order to prove the MCP for the triplet $(\mathbb{R}^n, \tilde d, m)$, we choose $t \mapsto \Phi_t$ to be minimal geodesics w.r.t. the metric d. This proves (4.1) and (4.2) with $\vartheta = 1$. It remains to prove (4.3) and (4.4). Let $\psi(x) = \log b(x) + \frac{1}{2} \log\det \tilde a(x)$. Note that $m(dx) = e^{\psi(x)} \tilde m(dx)$. Moreover, note that ψ is continuous. In particular, for any compact $K \subset \mathbb{R}^n$ and any $\varepsilon > 0$ there exists $R > 0$ such that $|\psi(x) - \psi(y)| \leq \varepsilon$ for all $x, y \in K$ with $\tilde d(x, y) < R$. Therefore, for all $r < R$, a.e. $x \in \mathbb{R}^n$ and all $A \subset \tilde B_r(x) = \{y\colon \tilde d(x, y) < r\} \subset K$,

$$\begin{aligned} \frac{m_r(A)}{\sqrt{m\big(\tilde B_r(x)\big)}} &\leq e^{2\varepsilon} \frac{\tilde m_r(A)}{\sqrt{\tilde m\big(\tilde B_r(x)\big)}} \leq e^{2\varepsilon} \tilde\Theta \frac{\tilde m_{rt}(\Phi_t(x, A))}{\sqrt{\tilde m\big(\tilde B_{rt}(x)\big)}} \\ &\leq e^{4\varepsilon} \tilde\Theta \frac{m_{rt}(\Phi_t(x, A))}{\sqrt{m\big(\tilde B_{rt}(x)\big)}} = \Theta \frac{m_{rt}(\Phi_t(x, A))}{\sqrt{m\big(\tilde B_{rt}(x)\big)}} \end{aligned}$$

with $\Theta = e^{4\varepsilon}\tilde\Theta$ and $\tilde\Theta$ being the constant from the MCP for $(\mathbb{R}^n, \tilde d, \tilde m)$. Similarly,

$$\frac{m\big(\tilde B_{r\vartheta'}(x)\big)}{m\big(\tilde B_r(x)\big)} \leq e^{2\varepsilon} \frac{\tilde m\big(\tilde B_{r\vartheta'}(x)\big)}{\tilde m\big(\tilde B_r(x)\big)} \leq e^{2\varepsilon}\tilde\Theta \leq \Theta.$$

According to the first part of the proof, $(\mathbb{R}^n, \tilde{d}, \tilde{m})$ satisfies the strong MCP. Hence, $\tilde{\Theta}$ and in turn also Θ can be chosen arbitrarily close to 1 (if R is chosen sufficiently small). This proves the claim. □

PROOF OF PROPOSITION 4.9. Fix a compact set $K \subset \mathbb{R}^n$ and numbers $\vartheta > 1$ and $\Theta > 1$. Let $K' = \overline{B}^{\mathrm{id}}_{(1+2\lambda)r}(z)$ where $B^{\mathrm{id}}_r(z)$ is some Euclidean ball containing K. According to Lemma 4.10 there exists a smooth matrix $\tilde{a}$ on $\mathbb{R}^n$ with

$$\vartheta^{-1/2} d_{\tilde{a}}(x, y) \le d_a(x, y) \le \vartheta^{1/2} d_{\tilde{a}}(x, y)$$

for all $x, y \in K'$. Choosing $t \mapsto \Phi_t$ to be minimal geodesics w.r.t. the metric $d_{\tilde{a}}$ immediately yields (4.1). Moreover, (4.3) and (4.4) were proved in Lemma 4.11. Finally, note that $x, y \in K$ implies $\Phi_t(x, y) \in K'$ for all $t \in [0,1]$. Hence,

$$\begin{aligned} d_a\big(\Phi_s(x, y), \Phi_t(x, y)\big) &\le \vartheta^{1/2} d_{\tilde{a}}\big(\Phi_s(x, y), \Phi_t(x, y)\big) \\ &= \vartheta^{1/2} |s - t| d_{\tilde{a}}(x, y) \le \vartheta |s - t| d_a(x, y) \end{aligned}$$

for all $x, y \in K$ and $s, t \in [0,1]$. That is, (4.2) holds true. □

4.D. *Example*: *degenerate elliptic matrices.* As in the previous section, let $X = \mathbb{R}^n$ and let $d = d_a$ be given by a matrix $a = (a_{ij}(x))_{i,j=1,\dots,n}$ according to (4.11) and (4.12). However, now we do not assume that this matrix is uniformly (or strictly) elliptic. We are interested in degenerate elliptic matrices. We do not treat the general case but treat some particular cases. We restrict ourselves to $n = 2$ and matrices of the form

$$a(x) = \begin{pmatrix} 1 & 0 \\ 0 & \varphi^2(x_1) \end{pmatrix}$$

with some function φ on $\mathbb{R}^1$. We always choose $m(dx) = dx$.

PROPOSITION 4.12. *Let $\varphi \in C(\mathbb{R})$ with $\{\varphi = 0\}$ discrete in $\mathbb{R}^1$. Then the metric measured space $(\mathbb{R}^2, d_a, m)$ satisfies the strong MCP with exceptional set.*

PROOF. Without restriction, φ is bounded and nonnegative. Let $Z_1 = \{x_1 \in \mathbb{R}: \varphi(x_1) = 0\}$ and $Z = Z_1 \times \mathbb{R}$. Then Z is closed (w.r.t. the Euclidean topology) and $m(Z) = 0$. Of course, on $\mathbb{R}^2 \setminus Z$ the matrix a is nondegenerate. Hence, on $\mathbb{R}^2 \setminus Z$ the topology induced by $d = d_a$ coincides with the Euclidean topology. In particular, Z is closed w.r.t. the topology induced by d.

Fix a compact set $K \subset \mathbb{R}^2 \setminus Z$ and let $\alpha^K = \inf\{\varphi(x_1): x = (x_1, x_2) \in K\}$. Put $\varphi^K = \varphi \vee \alpha^K$ on $\mathbb{R}$ and $K' = \{x_1: \varphi(x_1) \ge \alpha^K\} \times \mathbb{R} \supset K$. Define a matrix a^K by $a^K_{11}(x) = 1$, $a^K_{12}(x) = a^K_{21}(x) = 0$ and $a^K_{22}(x) = \varphi^K(x_1)^2$. Obviously, this matrix is uniformly elliptic and has bounded continuous coefficients. It satisfies $a^K(x) \ge a(x)$ for all $x \in \mathbb{R}^2$ with equality if $x \in K'$. The associated metric $d^K = d_{a^K}$ satisfies $d^K(x, y) \le d(x, y)$ for all $x, y \in \mathbb{R}^2$ with equality if the d^K-geodesic arc connecting x and y does not leave K'.

Choose $t \mapsto \Phi_t$ to be geodesics in the metric $d^K = d_{a^K}$. Then obviously (4.1) is satisfied.

Observe that each component K'_j of K' is of the form $K'_j = [s_j, t_j] \times \mathbb{R}$ which is a convex set in the metric d^K. Therefore, if $x, y \in K$ lie in the same component of K' then the d^K-geodesic $t \mapsto \Phi_t(x, y)$ connecting them will stay in K' for all $t \in [0, 1]$. Choose $R > 0$ sufficiently small such that all $x, y \in K$ with $d^K(x, y) < R$ lie in the same component of K'. Then all $x, y \in K$ with $d(x, y) < R$ lie in the same component of K' and

$$\begin{aligned} d\big(\Phi_s(x, y), \Phi_t(x, y)\big) &= d^K\big(\Phi_s(x, y), \Phi_t(x, y)\big) \\ &= |s - t| d^K(x, y) = |s - t| d(x, y). \end{aligned}$$

This is (4.2) with $\vartheta = 1$. Moreover, it follows that the ball $B_r(x)$ in the metric d and the ball $B_r^K(x)$ in the metric d^K coincide for each $x \in K$ and each $r < R$ with $B_r(x) \subset K$. Therefore, it suffices to verify (4.3) and (4.4) with d replaced by d^K. But in this case, (4.3) and (4.4) hold true (with Θ arbitrarily close to 1) according to Proposition 4.9. □

REMARKS 4.13. (i) If $\varphi(x_1) = x_1^k$ for some $k \in \mathbb{N}$ then the degenerate elliptic operator A associated with the diffusion matrix a is called *Grushin operator*. It is a Hörmander type operator, that is, it is the sum of squares of vector fields (namely, of $X_1 = (\partial/\partial x_1)$ and $X_2 = x_1^k(\partial/\partial x_2)$).

(ii) If $\varphi(x_1) = \exp(-1/x_1^2)$ then the associated operator is highly degenerate; it is *not subelliptic.*

4.E. *Example*: *manifolds with corners.*

A POSITIVELY CURVED CORNER. Let the two-dimensional set X be given by

$$X = \partial(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+) \subset \mathbb{R}^3$$

(i.e., X can be obtained by glueing together three copies of $\mathbb{R}_+^2$) and let d be the length metric on X derived from the Euclidean metric on $\mathbb{R}^n$; that is,

$$d(x, y) = \inf\big\{ L(\gamma) \colon \gamma \in C\big([0, 1] \to X\big), \gamma(0) = x, \gamma(1) = y \big\} \tag{4.18}$$

with $L(\gamma)$ being the Euclidean length of an arc γ in $\mathbb{R}^3$ [see (2.1)]. Then obviously (X, d) is a geodesic space with curvature (in the sense of Alexandrov) being $+\infty$ at the origin $(0, 0, 0)$ and curvature being 0 elsewhere. Choose m to be the two-dimensional Lebesgue measure on X.

PROPOSITION 4.14. *(X, d, m) satisfies the strong MCP with exceptional set and the weak MCP without exceptional set.*

PROOF. Obviously, (X, d, m) satisfies the strong MCP with exceptional set $Z = \{(0, 0, 0)\}$ and $\vartheta = \Theta = 1$ (since outside of the origin everything is locally the same as for the Euclidean triplet $(\mathbb{R}^2, |\cdot|, dx))$. In order to prove the weak MCP *without* exceptional set, let $t \mapsto \Phi_t$ be geodesics w.r.t. d. Then (a straightforward generalization of) the Bishop–Gromov comparison theorem implies

$$\frac{m(A)}{r^2\pi} \leq \frac{m(\Phi_t(x, A))}{(rt)^2\pi}$$

for all $x \in X$, $r > 0$ and $A \subset B_r(x)$. Moreover, obviously

$$\frac{3}{4} \leq \frac{m(B_r(x))}{r^2\pi} \leq 1$$

for all $x \in X$ and $r > 0$. This yields the claim with $\Theta = 4/3$. $\square$

This example can be generalized in a straightforward way. For $\alpha \in]0, 2\pi[$, let the two-dimensional set $X = X_\alpha$ be obtained by glueing together two copies of the sector

$$X_\alpha^0 = \{x = (r \cdot \cos \varphi, r \cdot \sin \varphi) \in \mathbb{R}^2 \colon 0 \leq \varphi \leq \alpha\}$$

along their boundaries. For instance, $X_{\pi/2}$ can be identified with the set X from above, X_π can be identified with $\mathbb{R}^2$ and $X_{3\pi/2}$ can be identified with

$$X = \partial\left(\mathbb{R}^2_+ \times \mathbb{R} \cup \mathbb{R}^2 \times \mathbb{R}_-\right) \subset \mathbb{R}^3$$

(which is obtained by glueing together five copies of $\mathbb{R}^2_+$). On each of these sets X_α there exists a "natural" distance d and a "natural" volume measure m. Then for any $\alpha \in]0, \pi[$, the assertions of Proposition 4.14 also hold true for the triplets (X_α, d, m).

A NEGATIVELY CURVED CORNER. The (strong) MCP with exceptional set actually is also true for the metric measured spaces (X_α, d, m) with $\alpha \in]\pi, 2\pi[$ in which case the geodesic spaces (X_α, d) have curvature $-\infty$ at the origin. However, the MCP without exceptional set no longer holds true. Let us explain this effect in a more simple example.

Let X be obtained by glueing two copies of $\mathbb{R}^2$ at the origin and let the metric d and the measure m on X be derived from the corresponding Euclidean quantities on $\mathbb{R}^2$. More explicitly, let $X = \{o\} \cup X_- \cup X_+$ with $X_\pm = (\mathbb{R}^2 \setminus \{(0,0)\} \times \{\pm 1\}$ and $o = (0, 0, +1) = (0, 0, -1)$. Then for $\xi, \eta \in \mathbb{R}^2$ and $\sigma, \tau \in \{\pm 1\}$,

$$d((\xi, \sigma), (\eta, \tau)) = \begin{cases} |\xi - \eta|, & \sigma\tau = +1, \\ |\xi| + |\eta|, & \sigma\tau = -1. \end{cases}$$

Of course, again (X, d, m) satisfies the (strong) MCP with exceptional set.

PROPOSITION 4.15. *(X, d, m) does not satisfy the (weak) MCP without exceptional set.*

Proof. Assume that the (weak) MCP holds true without exceptional set. Let $B_r(x)$ be a ball in X which contains the origin $o \in X$ but which is not centered at the origin, that is, $0 < d(x, o) < r$. Without restriction, $x \in X_-$, say $x = (\xi, -1)$. Let $W = \{\Phi_t(x, o): t \in [0,1]\}$ be the graph of the quasi geodesic $\Phi(x, o)$ connecting x with the origin o and let $A = B_r(x) \cap X_+ = \{(\eta, +1): \eta \in \mathbb{R}^2, |\eta| < r - |\xi|\}$. We claim that $\Phi_t(x, A) \subset W$ for all $t \le |\xi|/\vartheta r$. Namely, for each $y \in A$, the map $t \mapsto \Phi_t(x, y)$ is a continuous arc connecting $x \in X_-$ and $y \in X_+$. Hence, for some $t_y \in]0,1[$ we must have $\Phi_{t_y}(x, y) = o$. From (4.2) it follows that $t_y \ge |\xi|/\vartheta r$. But according to (4.1), the graph of $\Phi(x, y)|_{[0, t_y]}$ coincides with the graph of $\Phi(x, \Phi_{t_y}(x, y))|_{[0,1]} = \Phi(x, o)|_{[0,1]} = W$. This proves the above claim. Therefore, $m(\Phi_t(x, A)) \le m(W) = 0$ for all $t \le |\xi|/(\vartheta r)$ whereas of course $m(A) = (r - |\xi|)^2 \pi > 0$. This yields a contradiction to (4.3) [since $1 \le m(B_s(y))/(s^2\pi) \le 2$ for all $s > 0$ and all $y \in X$]. □

4.F. *Example*: *manifolds with boundaries.*

Dirichlet or absorbing boundary. Let $(\tilde{X}, \tilde{g})$ be a Riemannian manifold with Riemannian metric $\tilde{d}$ and Riemannian volume $\tilde{m}$ and let X be an open subset of $\tilde{X}$. Choose $m = \tilde{m}|_X$. There are two "natural" ways to define a metric d on X derived from the metric $\tilde{d}$ on $\tilde{X}$, namely, either $d = \tilde{d}|_X$ or $d = \tilde{d}_X$; see Example 2.2(ii). Recall that $\tilde{d}_X$ is the length metric derived from $\tilde{d}|_X$ (and that both coincide if and only if X is a convex subset of $\tilde{X}$). With d being either of these two metrics we always obtain that (X, d, m) *satisfies the strong MCP without exceptional set.*

Neumann or reflecting boundary. Let $(\tilde{X}, \tilde{g})$, $\tilde{d}$ and $\tilde{m}$ as before and now let X be a closed, convex subset of $\tilde{X}$. Choose $m = \tilde{m}|_X$ and $d = \tilde{d}|_X = \tilde{d}_X$. Then (X, d, m) *satisfies the weak MCP without exceptional set and the strong MCP with exceptional set* ∂X. Here the convexity is crucial.

4.G. *Example*: *glueing together of manifolds.*

The general principle. Let (X, d) be a locally compact length space and let m be a Radon measure on X. Assume that there exist Riemannian manifolds (Y_i, g_i), $i \in I$ (without boundary) with the properties that $d|_{Y_i}$ is the Riemannian distance on Y_i and $m|_{Y_i}$ is the Riemannian volume measure on Y_i (for each $i \in I$) and that $m(X \setminus \bigcup_{i \in I} Y_i) = 0$. Then (X, d, m) satisfies the *strong MCP with exceptional set*. Note that the Y_i may have different dimensions.

A line and a wedge. Let $Y_1 = \{(x_1, x_2) \in \mathbb{R}^2: x_1 < 0, x_2 = 0\}$, $Y_2 = \{(x_1, x_2) \in \mathbb{R}^2: |x_2| < x_1\}$, $Z = \{(0,0)\}$ and $X = Y_1 \cup Y_2 \cup Z$. Furthermore, let d be the length metric on X obtained from the Euclidean metric on $\mathbb{R}^2$ and let m be the sum of the one-dimensional Lebesgue measure on Y_1 and of the two-dimensional Lebesgue measure on Y_2. Then (X, d, m) satisfies the strong

MCP with exceptional set Z. From the probabilistic point of view, however, this example is of minor interest, since the associated diffusion process will never reach Z (and thus Y_1) if it starts in Y_2.

Alternative constructions. In order to replace the Dirichlet boundary condition on $\partial Y_2 \setminus Z$ by a Neumann boundary condition, let us choose $X' = \overline{X} = \overline{Y}_1 \cup \overline{Y}_2$ and extend d and m (as before) on X'. Then (X', d, m) satisfies the weak MCP with exceptional set Z and the strong MCP with exceptional set ∂Y_2. Instead of being killed at the boundary $\partial Y_2 \setminus Z$, the process will be reflected there. But still it will never reach Z if it starts in Y_2.

Another construction. If we want to ensure that the diffusion process will reach the origin if it starts inside of the wedge, we have to produce a drift inside of the wedge towards the origin. This can be done by modifying the reference measure m. Let us now choose

$$m'(dx) = \begin{cases} \lambda^1(dx_1), & x_1 \leq 0, \\ \dfrac{1}{|x|}\lambda^2(dx), & x_1 > 0. \end{cases}$$

Then as before (X', d, m') satisfies the weak MCP with exceptional set Z and the strong MCP with exceptional set ∂Y_2.

Three half planes. Let X be the union of three half planes Y_1, Y_2, Y_3 being copies of $\mathbb{R}_+ \times \mathbb{R}$ glued together at the "y-axis" $\{0\} \times \mathbb{R}$. (More concretely, let $X = (\mathbb{R}_+ \times \mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R} \times \mathbb{R}) \subset \mathbb{R}^3$.) Then as before the (strong) MCP holds with exceptional set but obviously also the weak MCP holds *without exceptional set*, namely, with $\vartheta = 1$ and $\Theta = 3/2$. Note that (X, d) is a geodesic space with curvature $-\infty$ on the singular line (and curvature 0 elsewhere).

5. The Measure Contraction Property and pointwise convergence of the approximating forms. Throughout this section, we require that (X, d, m) satisfies the weak MCP with an exceptional set which will be fixed and denoted by Z.

5.A. *Convergence of the approximating forms on compact sets.* We fix a relatively compact set Y with $\overline{Y} \subset X \setminus Z$, numbers R, Θ, ϑ and a map Φ satisfying the conditions from Definition 4.1. We put

$$Q(Y) = \big\{(x, y) \in X \times X\colon d\big(\Phi_s(x, y), \Phi_t(x, y)\big) < d\big(\Phi_u(x, y), X \setminus Y\big) \\ \text{for all } s, t, u \in [0, 1]\big\}.$$

Note that $(x, y) \in Q(Y)$ together with $d(x, y) < R$ implies $(x, \Phi_t(x, y)) \in Q(Y)$ for all $t \in [0, 1]$. [If Y is convex and $\vartheta = 1$, then the latter property also holds for $Y \times Y$ in the place of $Q(Y)$. The reason for using $Q(Y)$ is to avoid convexity assumptions on Y which might be very restrictive.] Moreover, note that for $(x, y) \in X \times X$ with $d(x, y) < R$, the map $t \mapsto \Phi_t(x, y)$ is continuous. Hence, the set $\{(x, y) \in Q(Y)\colon d(x, y) < R\}$ is measurable.

For $u \in L^2(X, m)$ define

$$E_Y^r(u, u) = \frac{1}{2}\int_Y \int_{Y \cap B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x, y)}\right]^2 m_r(dy)\, m_r(dx)$$

and

$$E_{Y,\Phi}^r(u, u) = \frac{1}{2}\iint_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x, y)}\right]^2 1_{Q(Y)}(x, y)\, m_r(dy)\, m_r(dx).$$

LEMMA 5.1. *For each $r > 0$ and Y as above, $(E_Y^r, D(E_Y^r))$ and $(E_{Y,\Phi}^r, D(E_{Y,\Phi}^r))$ are Dirichlet forms on $L^2(X, m)$. For each $u \in C_0^{\mathrm{Lip}}(X)$ (with compact support K and Lipschitz constant L),*

$$E_Y^r(u, u) \le \frac{L^2}{2} m\big(B_r(K)\big). \tag{5.1}$$

The proof is trivial. See also the proof of Lemma 3.1.

LEMMA 5.2 (Subpartitioning lemma). *Let $u \in L^2(X, m)$ and $r < R$. For any $n \in \mathbb{N}$ and any partition $0 = t_0 < t_1 < \cdots < t_n = 1$,*

$$E_{Y,\Phi}^{r/\vartheta^2}(u, u) \le C \sum_{k=1}^{n} (t_k - t_{k-1}) E_{Y,\Phi}^{r(t_k - t_{k-1})}(u, u) \tag{5.2}$$

with $C = \Theta^4 \vartheta^2$. In particular, for any $n \in \mathbb{N}$,

$$E_{Y,\Phi}^{r/\vartheta^2}(u, u) \le C E_{Y,\Phi}^{r/n}(u, u) \tag{5.3}$$

and for any $t \in [0, 1]$,

$$E_{Y,\Phi}^{r/\vartheta^2}(u, u) \le Ct E_{Y,\Phi}^{tr}(u, u) + C(1 - t) E_{Y,\Phi}^{(1-t)r}(u, u). \tag{5.4}$$

PROOF. It obviously suffices to prove (5.2). Let us fix u, r and the partition $\{t_k, k = 0, \dots, n\}$. Put $w_r(x, y) = 1_{Q(Y)}(x, y) 1_{B_r^*(x)}(y)$ and note that $w_r(x, y) = w_r(y, x)$ as well as

$$w_r(x, y) \le w_{\vartheta t r}\big(x, \Phi_t(x, y)\big) \tag{5.5}$$

for all $t \in [0, 1]$. Therefore,

$$\begin{aligned}
E_{Y,\Phi}^r(u, u) &= \frac{1}{2}\iint \left[\frac{u(x) - u(y)}{d(x, y)}\right]^2 w_r(x, y)\, m_r(dy)\, m_r(dx) \\
&= \frac{1}{2}\iint \left[\sum_{k=1}^{n} \frac{u\big(\Phi_{t_{k-1}}(x, y)\big) - u\big(\Phi_{t_k}(x, y)\big)}{d(x, y)}\right]^2 \\
&\qquad \times w_r(x, y)\, m_r(dy)\, m_r(dx) \\
&\le \frac{1}{2}\iint \sum_{k=1}^{n} \frac{1}{t_k - t_{k-1}} \left[\frac{u\big(\Phi_{t_{k-1}}(x, y)\big) - u\big(\Phi_{t_k}(x, y)\big)}{d(x, y)}\right]^2 \\
&\qquad \times w_r(x, y)\, m_r(dy)\, m_r(dx)
\end{aligned}$$

according to the elementary algebraic inequality $[\sum_{k=1}^n a_k]^2 \le \sum_{k=1}^n a_k^2/(t_k - t_{k-1})$.

Using (4.2), then (4.1) and finally (5.5) we get

$$\cdots \leq \frac{\vartheta^2}{2} \iint \sum_{k=1}^{n} (t_k - t_{k-1}) \left[\frac{u\big(\Phi_{t_{k-1}}(x,y)\big) - u\big(\Phi_{t_k}(x,y)\big)}{d\big(\Phi_{t_{k-1}}(x,y), \Phi_{t_k}(x,y)\big)} \right]^2$$
$$\times w_r(x,y)\, m_r(dy)\, m_r(dx)$$
$$= \frac{\vartheta^2}{2} \sum_{k=1}^{n} (t_k - t_{k-1}) \iint \left[\frac{u\Big(\Phi_{t_{k-1}/t_k}\big(x, \Phi_{t_k}(x,y)\big)\Big) - u\big(\Phi_{t_k}(x,y)\big)}{d\Big(\Phi_{t_{k-1}/t_k}\big(x, \Phi_{t_k}(x,y)\big), \Phi_{t_k}(x,y)\Big)} \right]^2$$
$$\times w_r(x,y)\, m_r(dy)\, m_r(dx)$$
$$\leq \frac{\vartheta^2}{2} \sum_{k=1}^{n} (t_k - t_{k-1}) \iint \left[\frac{u\Big(\Phi_{t_{k-1}/t_k}\big(x, \Phi_{t_k}(x,y)\big)\Big) - u\big(\Phi_{t_k}(x,y)\big)}{d\Big(\Phi_{t_{k-1}/t_k}\big(x, \Phi_{t_k}(x,y)\big), \Phi_{t_k}(x,y)\Big)} \right]^2$$
$$\times w_{\vartheta t_k r}\big(x, \Phi_{t_k}(x,y)\big)\, m_r(dy)\, m_r(dx).$$

Now we are in position to apply (4.3) which yields

$$\cdots \leq \frac{\Theta\vartheta^2}{2} \sum_{k=1}^{n} (t_k - t_{k-1}) \iint \left[\frac{u\big(\Phi_{t_{k-1}/t_k}(x,y')\big) - u(y')}{d\big(\Phi_{t_{k-1}/t_k}(x,y'), y'\big)} \right]^2$$
$$\times w_{\vartheta t_k r}(x, y')\, m_{t_k r}(dy')\, m_{t_k r}(dx)$$
$$\leq \frac{\Theta^2\vartheta^2}{2} \sum_{k=1}^{n} (t_k - t_{k-1}) \iint \left[\frac{u\big(\Phi_{t_{k-1}/t_k}(x,y')\big) - u(y')}{d\big(\Phi_{t_{k-1}/t_k}(x,y'), y'\big)} \right]^2$$
$$\times w_{\vartheta t_k r}(y', x)\, m_{\vartheta t_k r}(dx)\, m_{\vartheta t_k r}(dy'),$$

the last inequality being a consequence of (4.4). By (4.1) and (5.5) it follows

$$\cdots \leq \frac{\Theta^2\vartheta^2}{2} \sum_{k=1}^{n} (t_k - t_{k-1}) \iint \left[\frac{u\big(\Phi_{1-t_{k-1}/t_k}(y',x)\big) - u(y')}{d\big(\Phi_{1-t_{k-1}/t_k}(y',x), y'\big)} \right]^2$$
$$\times w_{\vartheta^2(t_k - t_{k-1})r}\big(y', \Phi_{1-t_{k-1}/t_k}(y',x)\big)$$
$$\times m_{\vartheta t_k r}(dx)\, m_{\vartheta t_k r}(dy')$$

which again allows applying (4.3). Hence, together with (4.4)

$$\cdots \leq \frac{\Theta^4\vartheta^2}{2} \sum_{k=1}^{n} (t_k - t_{k-1}) \iint \left[\frac{u(x') - u(y')}{d(x', y')} \right]^2$$
$$\times w_{\vartheta^2(t_k - t_{k-1})r}(y', x')\, m_{\vartheta^2(t_k - t_{k-1})r}(dx')$$
$$\times m_{\vartheta^2(t_k - t_{k-1})r}(dy')$$
$$= \Theta^4\vartheta^2 \sum_{k=1}^{n} (t_k - t_{k-1}) E_{Y,\Phi}^{\vartheta^2(t_k - t_{k-1})r}(u,u). \qquad \square$$

The basic idea for this subpartitioning lemma is well known. It is essentially the same argument which implies that the "approximate length" $\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i))$ of an arc $\gamma: [0,1] \to X$ *increases* if the partition $\{t_0, \dots, t_n\}$ is replaced by any subpartition. This leads to the definition (2.1) for the length $L_d(\gamma)$ of γ. The same idea was used by Korevaar and Schoen (1993) and Jost (1994) to treat energy integrals for maps $f: M \to N$ where M is either the $\mathbb{R}^n$ or an n-dimensional Riemannian manifold (with bounded Ricci curvature) and N some "singular space" (i.e., geodesic space with nonpositive curvature). In the present paper, however, the difficulties are not caused by singular target spaces N (which in our case is just $\mathbb{R}$) but by singular domains M (which in our case is the metric space X).

LEMMA 5.3. *There exist constants $C < \infty$ (with C arbitrarily close to* 1 *under the strong MCP), $R_0 > 0$ and $\delta > 0$ such that for all $u \in L^2(X, m)$ and all $R < R_0$ and $r < \delta R$,*

$$E^R_{Y,\Phi}(u,u) \le C E^r_Y(u,u). \tag{5.6}$$

PROOF. Fix $R > 0$, put $R' = \vartheta^2 R$ and observe that for any $r > 0$ there exists $k \in \mathbb{N}$ with $R'/k \le r < R'/(k-1)$. With this k one obtains

$$\begin{aligned} E^r_Y(u,u) &\ge \frac{1}{2}\iint_{B^*_r(x)} \left[\frac{u(x)-u(y)}{d(x,y)}\right]^2 1_{Q(Y)}(x,y)\, m_r(dy)\, m_r(dx) \\ &\ge \frac{1}{2}\iint_{B^*_{R'/k}(x)} \left[\frac{u(x)-u(y)}{d(x,y)}\right]^2 \\ &\qquad\qquad \times 1_{Q(Y)}(x,y)\, m_{R'/k-1}(dy)\, m_{R'/k-1}(dx). \end{aligned}$$

Now assume that $r < (1 - 1/\vartheta')\vartheta^2 R$. Then $1/k < (1 - 1/\vartheta')$ and thus $k/(k-1) < \vartheta'$. Hence, we may apply (4.4) to deduce

$$\begin{aligned} \cdots &\ge \frac{1}{2\Theta}\iint_{B^*_{R'/k(x)}} \left[\frac{u(x)-u(y)}{d(x,y)}\right]^2 1_{Q(Y)}(x,y)\, m_{R'/k}(dy)\, m_{R'/k}(dx) \\ &= \frac{1}{\Theta} E^{R'/k}_{Y,\Phi}(u,u) \ge \frac{1}{C} E^R_{Y,\Phi}(u,u) \end{aligned}$$

according to (5.3).

Let us turn to the crucial question of Γ-convergence of E^r_Y for $r \to 0$. Note that always $\Gamma\text{-}\liminf \cdots \le \liminf \cdots$ as well as $\Gamma\text{-}\limsup \cdots \le \limsup \cdots$ and of course $\Gamma\text{-}\liminf \cdots \le \Gamma\text{-}\limsup \cdots$.

LEMMA 5.4. *There exist constants $C < \infty$ and $R_0 > 0$ (with C arbitrarily close to* 1 *under the strong MCP) such that for all $u \in L^2(X, m)$ and all $R < R_0$,*

$$E^R_{Y,\Phi}(u,u) \le C \cdot \Gamma\text{-}\liminf_{r \to 0} E^r_Y(u,u). \tag{5.7}$$

Moreover,

$$\limsup_{r\to 0} E^r_Y(u,u) \le C\cdot \Gamma\text{-}\liminf_{r\to 0} E^r_Y(u,u). \tag{5.8}$$

PROOF. In order to see (5.7), write

$$\begin{aligned}\Gamma\text{-}\liminf_{r\to 0} E^r_Y(u,u) &= \lim_{\alpha\to 0}\liminf_{r\to 0}\inf_{\|v-u\|\le\alpha} E^r_Y(v,v)\\ &\ge C^{-1}\liminf_{v\to u} E^R_Y(v,v) = C^{-1}E^R_Y(v,v)\end{aligned}$$

since E^R_Y is lower semicontinuous on $L^2(X,m)$. This proves (5.7).

In order to see (5.8), fix $u\in C_0^{\mathrm{Lip}}(X)$ (with Lipschitz constant L). Let $Y^r = \{x\in X: d(x, X\setminus Y > 2\vartheta r\}\subset Y$ and note that $1_{Y^r}(x)1_{Y^r}(y)\le 1_{Q(Y)}(x,y)$ for x, y with $d(x,y)<r$. Then a straightforward calculation [cf. proof of Theorem 5.6(ii)] shows

$$\begin{aligned}0 &\le E^r_Y(u,u) - E^r_{Y,\Phi}(u,u)\\ &\le \frac{1}{2}\int_Y\int_{Y\cap B^*_r(x)}\left[\frac{u(x)-u(y)}{d(x,y)}\right]^2 m_r(dy)\,m_r(dx)\\ &\quad -\frac{1}{2}\int_{Y^r}\int_{Y^r\cap B^*_r(x)}\left[\frac{u(x)-u(y)}{d(x,y)}\right]^2 m_r(dy)\,m_r(dx)\\ &\le \frac{L^2}{2}\left[\int_Y\int_{Y\cap B^*_r(x)} m_r(dy)\,m_r(dx)\right.\\ &\qquad\left. -\int_{Y^r}\int_{Y^r\cap B^*_r(x)} m_r(dy)\,m_r(dx)\right]\\ &\le L^2 m\big(Y\setminus Y^{2r}\big)\to 0\end{aligned}$$

for $r\to 0$. That is, $\limsup_{r\to 0}E^r_Y(u,u) = \limsup_{r\to 0}E^r_{Y,\Phi}(u,u)$ which together with (5.7) yields the claim. □

COROLLARY 5.5. (i) *Assume that the strong MCP (with exceptional set) holds. Fix any sequence $(r_n)_n$ with $\lim r_n = 0$ for which $E^0_Y := \Gamma\text{-}\lim_{n\to\infty}E^{r_n}_Y$ exists. Then for each $u\in C_0^{\mathrm{Lip}}(X)$, the limit*

$$\lim_{r\to 0} E^r_Y(u,u) \tag{5.9}$$

exists, is finite and coincides with $E^0_Y(u,u)$.

(ii) *$(E^0_Y, C_0^{\mathrm{Lip}}(X))$ is closable and its closure (E_Y, F_Y) is a regular, strongly local Dirichlet form on $L^2(X,m)$ with core $C_0^{\mathrm{Lip}}(X)$. It is independent of the choice of $(r_n)_n$.*

PROOF. (i) is obvious from Lemma 5.4. (ii) By Mosco [(1994), Theorem 2.8], the Γ-limit $(E^0_Y, D(E^0_Y))$ from (i) is a (not necessarily densely defined) closed Markovian symmetric form. Moreover, $C_0^{\mathrm{Lip}}(X)\subset D(E^0_Y)$. Hence, this form is densely defined. Its restriction $(E^0_Y, C_0^{\mathrm{Lip}}(X))$ is of course closable with closure (E_Y, F_Y) being again a Dirichlet form. Obviously, the latter is regular with core $C_0^{\mathrm{Lip}}(X)$ and it is easily seen that it is strongly local. □

5.B. *Convergence on the whole space.* Let us assume that (X, d, m) satisfies the strong MCP with exceptional set, say $Z \subset X$, and fix an exhaustion $(Y_i)_{i \in N}$ of $X \setminus Z$, that is, an increasing sequence of relatively compact open sets Y_i with $\overline{Y_i} \subset X \setminus Z$ and $\bigcup_{i \in N} Y_i = X \setminus Z$.

THEOREM 5.6. (i) *On $L^2(X, m)$, the limits*

$$\Gamma\text{-}\lim_{i \to \infty} E_{Y_i} \quad and \quad \lim_{i \to \infty} E_{Y_i} \tag{5.10}$$

exist and coincide.

(ii) *For any $u \in C_0^{\mathrm{Lip}}(X)$,*

$$E^0(u, u) = \lim_{r \to 0} \frac{1}{2} \int_X \int_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x, y)} \right]^2 m_r(dy)\, m_r(dx) < \infty. \tag{5.11}$$

In particular, $(E^0, C_0^{\mathrm{Lip}}(X))$ is independent of the exceptional set Z and of the exhaustion $(Y_i)_i$ of $X \setminus Z$.

(iii) *$(E^0, C_0^{\mathrm{Lip}}(X))$ is closable and its closure (E, F) is a regular, strongly local Dirichlet form with core $C_0^{\mathrm{Lip}}(X)$.*

PROOF. (i) The convergence assertions are obvious since the sequence $(E_{Y_i})_i$ is increasing, which in turn follows from the fact that, for each $r > 0$, the sequence $(E_{Y_i}^r)_i$ is increasing.

(ii) The finiteness assertion follows from the estimate (5.1) which holds true for all $u \in C_0^{\mathrm{Lip}}(X)$, uniformly in r and i. Fix $u \in C_0^{\mathrm{Lip}}(X)$ (with Lipschitz constant L and compact support K) and $i \in \mathbb{N}$. Denote the $\lim_{r \to 0} \cdots$ in (5.11) by $\tilde{E}(u, u)$. Then with $v_r(x) = 1/m(B_r(x))$,

$$\begin{aligned}
0 &\le \tilde{E}(u, u) - E_{Y_i}^0(u, u) \\
&= \frac{1}{2} \lim_{r \to 0} \Bigg[\int_X \int_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x, y)} \right]^2 m_r(dy)\, m_r(dx) \\
&\qquad\qquad - \int_{Y_i} \int_{Y_i \cap B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x, y)} \right]^2 m_r(dy)\, m_r(dx) \Bigg] \\
&\le \frac{L^2}{2} \lim_{r \to 0} \Bigg[\int_{B_r(K)} \int_{B_r(K) \cap B_r^*(x)} m_r(dy)\, m_r(dx) \\
&\qquad\qquad - \int_{B_r(K) \cap Y_i} \int_{B_r(K) \cap Y_i \cap B_r^*(x)} m_r(dy)\, m_r(dx) \Bigg] \\
&\le \frac{L^2}{2} \lim_{r \to 0} \Bigg[\int_{B_r(K)} \int_{B_r(K) \cap B_r^*(x)} v_r(x)\, m(dy)\, m(dx) \\
&\qquad\qquad - \int_{B_r(K) \cap Y_i} \int_{B_r(K) \cap Y_i \cap B_r^*(x)} v_r(x)\, m(dy)\, m(dx) \Bigg]
\end{aligned}$$

$$= \frac{L^2}{2} \lim_{r\to 0} \left[\int_{B_r(K)\setminus Y_i} \int_{B_r(K)\cap B_r^*(x)} v_r(x)\, m(dy)\, m(dx) \right.$$

$$\left. + \int_{B_r(K)\cap Y_i} \int_{B_r(K)\setminus Y_i \cap B_r^*(x)} v_r(x)\, m(dy)\, m(dx) \right]$$

$$\leq \frac{L^2}{2} \lim_{r\to 0} m(B_r(K)\setminus Y_i) = \frac{L^2}{2} m(K\setminus Y_i).$$

In the limit $i \to \infty$, the latter goes to 0 since $(Y_i)_i$ is an exhaustion of $X \setminus Z$ and $m(Z) = 0$.

(iii) Same argument as in the proof of Corollary 5.5(ii). □

COROLLARY 5.7. (i) *For any* $u, v \in C_0^{\mathrm{Lip}}(X)$,

$$E^0(u,v) = \lim_{r\to 0} \frac{1}{2} \int_X \int_{B_r^*(x)} \frac{[u(x)-u(y)][v(x)-v(y)]}{d^2(x,y)} \times m_r(dy)\, m_r(dx) < \infty. \tag{5.12}$$

(ii) *For any* $u \in C_0^{\mathrm{Lip}}(X)$

$$E^0(u,u) = \Gamma\text{-}\limsup_{r\to 0} E^r(u,u) = \Gamma\text{-}\liminf_{r\to 0} E^r(u,u). \tag{5.13}$$

That is, $E^0 = \overline{E}^0 = \underline{E}^0$ *on* $C_0^{\mathrm{Lip}}(X)$ *and thus*

$$E = \overline{E} = \underline{E} \quad \text{on } L^2(X,m). \tag{5.14}$$

PROOF. (i) follows from (5.11) by polarization.

(ii) For any $u \in C_0^{\mathrm{Lip}}(X)$ and any exhaustion $(Y_i)_i$ of $X \setminus Z$,

$$\overline{E}^0(u,u) \leq \limsup_{r\to 0} E^r(u,u) = E^0(u,u) = \sup_{i\in N} E^0_{Y_i}(u,u)$$

according to (5.11) and (5.10). Moreover, for any $i \in \mathbb{N}$ by (5.8),

$$E^0_{Y_i}(u,u) = \Gamma\text{-}\liminf_{r\to 0} E^r_{Y_i}(u,u) \leq \Gamma\text{-}\liminf_{r\to 0} E^r(u,u)$$

and thus

$$\sup_{i\in N} E^0_{Y_i}(u,u) \leq \Gamma\text{-}\liminf_{r\to 0} E^r(u,u).$$

This proves (5.13). Then (5.14) is an obvious consequence. □

6. Energy measure, Poincaré inequality and intrinsic metric. Throughout this section, we assume that (X, d, m) satisfies the strong MCP with exceptional set. The exceptional set for the *weak* MCP (which might be much smaller than that for the strong MCP) will be denoted by Z.

6.A. *The energy measure of the limit form.* We recall that for every regular Dirichlet form (E, F) on $L^2(X, m)$ and every $u \in F$ there exists a uniquely determined Radon measure $\mu_{\langle u \rangle}$ on X which satisfies

$$\int_X \varphi(x)\mu_{\langle u \rangle}(dx) = 2E(u, \varphi u) - E(u^2, \varphi) \quad \forall \varphi \in C_0(X).$$

From now on, let (E, F) be the regular, strongly local Dirichlet form defined in Theorem 5.6.

THEOREM 6.1. *For any $u \in C_0^{\mathrm{Lip}}(X)$, the energy measure $\mu_{\langle u \rangle}$ w.r.t. the Dirichlet form (E, F) is given by the formula*

$$\int_X \varphi(x)\mu_{\langle u \rangle}(dx) = \lim_{r \to 0} \int_X \varphi(x) \int_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x, y)} \right]^2 m_r(dy)\, m_r(dx) \tag{6.1}$$

for $\varphi \in C_0(X)$. Actually, formula (6.1) *holds true whenever $\varphi \in C_b(Y)$ for some open $Y \subset X$ with $m(X \setminus Y) = 0$.*

PROOF. Assume that $\varphi \in C_b(Y)$ for some open $Y \subset X$ with $m(X \setminus Y) = 0$. Without restriction, we may assume that $X \setminus Y$ is the exceptional set for the MCP. (Otherwise replace Y by $Y \setminus Z$.) Let $(Y_i)_i$ be an exhaustion of Y. Then for each $i \in \mathbb{N}$, the function φ can be uniformly approximated on Y_i by functions $\varphi_{ik} \in C_0^{\mathrm{Lip}}(X)$, $k \in \mathbb{N}$. Note that for any i, $k \in \mathbb{N}$ the function φ_{ik} lies in the domain F_{Y_i} of the Dirichlet form E_{Y_i}. Therefore, by the definition of the energy measure $\mu_{\langle u \rangle}^{(i)}$ w.r.t. the Dirichlet form (E_{Y_i}, F_{Y_i}) from (5.5),

$$\begin{aligned}
\int_X \varphi_{ik}(x)\mu_{\langle u \rangle}^{(i)}(dx) &= 2E_{Y_i}(u, \varphi_{ik} u) - E_{Y_i}(u^2, \varphi_{ik}) \\
&= \lim_{r \to 0} \int_{Y_i} \int_{Y_i \cap B_r^*(x)} \left[\frac{\varphi_{ik}(x) + \varphi_{ik}(y)}{2} \right] \left[\frac{u(x) - u(y)}{d(x, y)} \right]^2 \\
&\qquad \times m_r(dy)\, m_r(dx) \\
&= \lim_{r \to 0} \int_{Y_i} \varphi_{ik}(x) \int_{Y_i \cap B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x, y)} \right]^2 \\
&\qquad \times m_r(dy)\, m_r(dx).
\end{aligned}$$

These equalities remain valid in the limit $k \to \infty$ (with φ in the place of φ_{ik}) since φ_{ik} converges uniformly on Y_i to φ. That is,

$$\int_X \varphi(x)\mu_{\langle u \rangle}^{(i)}(dx) = \lim_{r \to 0} \int_{Y_i} \varphi(x) \int_{Y_i \cap B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x, y)} \right]^2 \times m_r(dy)\, m_r(dx) \tag{6.2}$$

for all $i \in \mathbb{N}$. We claim that for $i \to \infty$ the RHS of (6.2) converges monotonically to

$$\lim_{r\to 0} \int_X \varphi(x) \int_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)}\right]^2 m_r(dy)\, m_r(dx).$$

The proof of this claim is the same as that of Theorem 5.6(ii). Moreover, according to (5.10) in the limit $i \to \infty$, the LHS of (6.2) converges monotonically to

$$\int_X \varphi(x)\, \mu_{\langle u\rangle}(dx).$$

This proves the assertion of the theorem. □

For the following result, we recall our assumptions which are quite sophisticated: we require the strong MCP with some exceptional set and the weak MCP with the exceptional set Z.

COROLLARY 6.2. *For any compact set $Y \subset X \setminus Z$, there exist constants C and R and a map Φ satisfying* (4.1) *and* (4.2) *such that, for all $r < R$ and all $u \in F$,*

$$\int_Y d\mu_{\langle u\rangle} \geq \frac{1}{C} \iint_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)}\right]^2 1_{Q(Y)}(x,y)\, m_r(dy)\, m_r(dx), \tag{6.3}$$

where as usual

$$Q(Y) = \Big\{(x,y) \in X \times X\colon \sup_{s,t} d\big(\Phi_s(x,y), \Phi_t(x,y)\big) < \inf_u d\big(\Phi_u(x,y), X \setminus Y\big)\Big\}.$$

PROOF. By density arguments it suffices to prove the assertion for $u \in C_0^{\mathrm{Lip}}(X)$. For $\varepsilon > 0$ let $Y^\varepsilon = \{x \in X\colon d(x, X \setminus Y) > \varepsilon\}$ and choose $\varphi_\varepsilon \in C_0^{\mathrm{Lip}}(X)$ with $\varphi_\varepsilon \equiv 1$ on Y^ε and $\varphi_\varepsilon \equiv 0$ on $X \setminus Y$. Then (6.1) (which is a consequence of the *strong* MCP with *some* exceptional set) implies

$$\begin{aligned}\int_Y d\mu_{\langle u\rangle} &\geq \limsup_{r\to 0} \int_{Y^\varepsilon} \int_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)}\right]^2 m_r(dy)\, m_r(dx)\\ &\geq \limsup_{r\to 0} \iint_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)}\right]^2 1_{Q(Y^\varepsilon)}(x,y)\, m_r(dy)\, m_r(dx).\end{aligned}$$

Now for any $R < R(Y)$ the *weak* MCP with the exceptional set $Z \subset X \setminus Y$ implies

$$\begin{aligned}&\liminf_{r\to 0} \iint_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)}\right]^2 1_{Q(Y^\varepsilon)}(x,y)\, m_r(dy)\, m_r(dx)\\ &\quad\geq \frac{1}{C} \iint_{B_R^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)}\right]^2 1_{Q(Y^\varepsilon)}(x,y)\, m_R(dy)\, m_R(dx).\end{aligned}$$

That is,

$$\int_Y d\mu_{\langle u \rangle} \geq \frac{1}{C} \iint_{B_R^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)} \right]^2 1_{Q(Y^\varepsilon)}(x,y)\, m_R(dy)\, m_R(dx)$$

(independently of u and ε). Hence, (by monotone convergence in ε)

$$\int_Y d\mu_{\langle u \rangle} \geq \frac{1}{C} \iint_{B_R^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)} \right]^2 1_{Q(Y)}(x,y)\, m_R(dy)\, m_R(dx)$$

for all $u \in C_0^{\mathrm{Lip}}(X)$. □

6.B. *The Poincaré inequality.* It is a surprising fact that the MCP not only implies the doubling property but automatically also implies a Poincaré inequality. This is more or less a consequence of the lower estimate (5.7) which allows estimating the limit form $E_Y^0 = \Gamma\text{-}\lim_r E_Y^r = \lim_r E_Y^r$ from below by one fixed approximating form $E_{Y,\Phi}^r$. This lower estimate is rewritten in a more appropriate way in Corollary 6.2 which is already something like a Poincaré inequality.

THEOREM 6.3 (Weak Poincaré inequality). *For any compact set $Y \subset X \setminus Z$, there exist constants C, k and $R > 0$ such that for all $u \in F$, all $z \in Y$ and all $r < R$,*

$$\int_{B_{kr}(z)} d\mu_{\langle u \rangle} \geq \frac{1}{Cr^2} \int_{B_r(z)} [u(x) - \overline{u}(x)]^2\, m(dx) \tag{6.4}$$

where $\overline{u}(x) = (1/m(B_r(z))) \int_{B_r(z)} u(y) m(dy)$.

PROOF. Choose $k = 4\vartheta$ and $R > 0$ such that $Y_0 := \overline{B}_{kR}(Y)$ is a compact subset of $X \setminus Z$ with $R(Y_0) > 2R$. We apply Corollary 6.2 with $B_{kr}(z)$ in the place of Y and $2r$ in the place of r in order to obtain

$$\begin{aligned}
\int_{B_{kr}(z)} d\mu_{\langle u \rangle} &\geq \frac{1}{C} \iint_{B_{2r}^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)} \right]^2 1_{Q(B_{kr}(z))}(x,y)\, m_{2r}(dy)\, m_{2r}(dx) \\
&\geq \frac{1}{C} \int_{B_r(z)} \int_{B_r^*(z)} \left[\frac{u(x) - u(y)}{d(x,y)} \right]^2 m_{2r}(dy)\, m_{2r}(dx) \\
&\geq \frac{1}{M^2 Cr^2} \int_{B_r(z)} \frac{1}{m(B_r(z))} \int_{B_r(z)} [u(x) - u(y)]^2\, m(dy)\, m(dx) \\
&\geq \frac{1}{M^2 C^2 r^2} \int_{B_r(z)} \left[u(x) - \frac{1}{m(B_r(z))} \int_{B_r(z)} u(y)\, m(dy) \right]^2 m(dx),
\end{aligned}$$

where M denotes the doubling constant on Y. □

For subelliptic operators, Jerison (1986) has proven that a weak Poincaré inequality (as above) always implies the corresponding strong Poincaré inequality where the integration on the left- and right-hand sides is over the same regions. This result was generalized in Sturm (1996) to general Dirichlet operators.

THEOREM 6.4 (Strong Poincaré inequality). *For any compact set $Y \subset X \setminus Z$, there exists a constant C such that for all $u \in F$, all $z \in X$ and all $r > 0$ with $B_r(z) \subset Y$,*

$$\int_{B_r(z)} d\mu_{\langle u\rangle} \geq \frac{1}{Cr^2} \cdot \int_{B_r(z)} \left[u(x) - \overline{u}(x) \right]^2 m(dx) \tag{6.5}$$

where $\overline{u}(x) = 1/m(B_r(z)) \int_{B_r(z)} u(y) m(dy)$.

PROOF. The result would follow immediately from Sturm (1996), Theorem 2.4, if d would be the intrinsic metric for the Dirichlet form (E, F) (which, in general, will not be true). However, one easily checks that for the arguments in that paper it suffices that d is a length metric on X with the property that $\mu_{\langle d_z\rangle} \leq m$ for all $z \in X$ [with $d_z : x \mapsto d(z, x)$] which will be proved as inequality (6.8) in the course of the proof of the following Proposition 6.6. □

6.C. *The intrinsic metric.* The notion of intrinsic metric was introduced and investigated in Sturm (1994) and (1995b). Here we use a slightly different definition.

DEFINITION 6.5. The *intrinsic metric* ρ on X associated with (E, F) is defined by

$$\rho(x, y) = \sup\left\{ u(x) - u(y) \colon u \in C_0^{\mathrm{Lip}}(X), \mu_{\langle u\rangle} \leq m \right\}. \tag{6.6}$$

A priori, we only know that ρ is a pseudometric on X. It might be degenerate.

PROPOSITION 6.6. *The intrinsic metric ρ associated with the Dirichlet form (E, F) satisfies*

$$\rho \geq d \quad \text{on } X \times X. \tag{6.7}$$

PROOF. For any $x, y, z \in X$ the triangle inequality yields

$$\left| \frac{d(x, z) - d(y, z)}{d(x, y)} \right| \leq 1.$$

Hence, for each $z \in X$, the energy measure of the function $d_z \colon x \mapsto d(x, z)$ satisfies

$$\begin{aligned}
\int_X \varphi(x) \mu_{\langle d_z\rangle}(dx) &= \lim_{r\to 0} \int_X \varphi(x) \int_{B_r^*(x)} \left[\frac{d(x, z) - d(y, z)}{d(x, y)} \right]^2 m_r(dy)\, m_r(dx) \\
&\leq \lim_{r\to 0} \int_X \varphi(x) \int_{B_r^*(x)} m_r(dy)\, m_r(dx) \\
&\leq \lim_{r\to 0} \int_X \int_{B_r^*(x)} \sqrt{\varphi(x)} \sqrt{\varphi(y)}\, m_r(dy)\, m_r(dx)
\end{aligned}$$

for all nonnegative $\varphi \in C_0(X)$. The last inequality follows from

$$\begin{aligned}
&\iint_{B_r^*(x)} \left(\varphi(x) - \sqrt{\varphi(x)\varphi(y)}\right) m_r(dy)\, m_r(dx) \\
&\quad \le \iint_{B_r^*(x)} |\varphi(x) - \varphi(y)|\, m_r(dy)\, m_r(dx) \\
&\quad \le \int_{B_r(K)} \int_{B_r^*(x)} \delta(d(x,y))\, m_r(dy)\, m_r(dx) \\
&\quad \le \delta(r) \int_{B_r(K)} \int_{B_r^*(x)} m_r(dy)\, m_r(dx) \to 0
\end{aligned}$$

for $r \to 0$. Here $r \mapsto \delta(r)$ denotes the modulus of continuity of $\varphi \in C_0(X)$ and $K = \operatorname{supp} \varphi$. That is,

$$\int \varphi(x)\, \mu_{\langle u \rangle}(dx) \le \lim_{r \to 0} \iint_{B_r^*(x)} \sqrt{\frac{\varphi(x)}{m(B_r(x))}} \sqrt{\frac{\varphi(y)}{m(B_r(y))}}\, m(dy)\, m(dx)$$

which by the Cauchy–Schwarz inequality is less than or equal to

$$\lim_{r \to 0} \int \varphi(x)\, m(dx) = \int \varphi(x)\, m(dx)$$

for all nonnegative $\varphi \in C_0(X)$. Hence,

$$\mu_{\langle d_z \rangle} \le m \tag{6.8}$$

and thus

$$\rho(x, y) \ge d_z(x) - d_z(y)$$

for all $x, y, z \in X$. Choosing $z = y$, this proves the claim. □

PROPOSITION 6.7. *For each compact set $Y \subset X \setminus Z$ there exists a constant L and a map Φ satisfying* (4.1) *and* (4.2) *such that the intrinsic metric ρ associated with the Dirichlet form (E, F) satisfies*

$$\rho \le Ld \quad \text{on } Q(Y), \tag{6.9}$$

where as usual

$$\begin{aligned}
Q(Y) = \Big\{ (x, y) \in X \times X \colon & \sup_{s,t} d(\Phi_s(x,y), \Phi_t(x,y)) \\
& < \inf_u d(\Phi_u(x,y), X \setminus Y) \Big\}.
\end{aligned}$$

PROOF. We have to consider such $u \in C_0^{\mathrm{Lip}}(X)$ for which $\mu_{\langle u \rangle} \le m$. According to (6.3), this implies

$$\iint_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)}\right]^2 1_{Q(Y_0)}(x, y)\, m_r(dy)\, m_r(dx) \le Cm(Y_0) \tag{6.10}$$

for any open set $Y_0 \subset Y$ and all $r > 0$ where C is a constant determined by the MCP.

Let $L = \sup\{|u(x) - u(y)|/d(x, y) : (x, y) \in Q(Y), x \neq y\}$ be the "Lipschitz constant of u on $Q(Y)$" and fix $(z_1, z_2) \in Q(Y)$ with

$$|u(z_1) - u(z_2)| \geq \frac{L}{2} d(z_1, z_2) > 0.$$

Such $(z_1, z_2) \in Q(Y)$ can be chosen with arbitrarily small $d(z_1, z_2)$ (just replacing successively either z_1 or z_2 by their midpoint $\Phi_{1/2}(z_1, z_2)$). Let $r = \frac{4}{3} d(z_1, z_2)$. For $x \in B_{r/8}(z_1)$ and $y \in B_{r/8}(z_2)$,

$$\begin{aligned} |u(x) - u(y)| &\geq |u(z_1) - u(z_2)| - |u(z_1) - u(x)| - |u(y) - u(z_2)| \\ &\geq \frac{L}{2}\frac{3}{4} r - 2L\frac{r}{8} \geq \frac{L}{8} r \geq \frac{L}{16} d(x, y). \end{aligned}$$

Therefore, for $x \in B_{r/8}(z_1)$

$$\begin{aligned} &\frac{1}{\sqrt{m(B_r^*(x))}} \int_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x, y)}\right]^2 m_r(dy) \\ &\quad\geq \frac{1}{m(B_{2r}^*(z_2))} \int_{B_{r/8}^*(z_2)} \left[\frac{u(x) - u(y)}{d(x, y)}\right]^2 m(dy) \\ &\quad\geq \frac{L^2}{2^8} \frac{m\big(B_{r/8}(z_2)\big)}{m\big(B_{2r}(z_2)\big)} \geq \frac{L^2}{2^8} M^{-4} \end{aligned}$$

with M being the doubling constant. Together with (6.10) [applied to $Y_0 = B_{2r}(z_1)$] this yields

$$\begin{aligned} Cm\big(B_{2r}(z_1)\big) &\geq \int_{B_r(z_1)} \int_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x, y)}\right]^2 m_r(dy)\, m_r(dx) \\ &\geq \frac{L^2}{2^8} M^{-4} m\big(B_{r/8}(z_1)\big) \\ &\geq \frac{L^2}{2^8} M^{-8} m\big(B_{2r}(z_1)\big). \end{aligned}$$

Hence, we finally obtain

$$L^2 \leq 2^8 C M^8.$$

In other words,

$$|u(x) - u(y)| \leq L d(x, y)$$

for all $(x, y) \in Q(Y)$ with $L = 16\sqrt{C} M^4$. Since this holds for all $u \in C_0(X) \cap F$ we get

$$\rho(x, y) \leq L d(x, y)$$

for all $(x, y) \in Q(Y)$ with L as above. □

THEOREM 6.8. *Assume that the weak MCP holds without exceptional set. Then the intrinsic metric ρ is locally equivalent to the original metric d. In particular, it is nondegenerate and induces on X the same topology as d. Moreover, (X, ρ) is a length space.*

We recall once more that for each result in this section, in particular for Theorem 6.8, we require that (X, d, m) satisfies the strong MCP with some exceptional set.

PROOF OF THEOREM 6.8. Let us prove the first assertion. For $z \in X$, choose $R > 0$ such that $B_{2R}(z)$ is relatively compact. Then Propositions 6.6 and 6.7 [with $Y = B_{2R}(z)$] imply that there is a constant L such that

$$d(x, y) \le \rho(x, y) \le Ld(x, y)$$

for all $x, y \in B_R(z)$ with $d(x, y) \le R$. This proves the first claim. Next we turn to the last assertion. It follows by a slight modification of the proof of Lemma 3 in Sturm (1995b). Let ψ_0 be the function from that paper. In the present case, we have to check that $\psi_0 \in C_0^{\mathrm{Lip}}(X)$ [and not only in $C_0(X) \cap F$]. But according to Proposition 6.7, the function $y \mapsto \rho(x, y)$ lies in $C_0^{\mathrm{Lip}}(X)$, hence, also $\psi_0 \in C_0^{\mathrm{Lip}}(X)$. $\square$

COROLLARY 6.9. *The volume doubling property and the weak* (*and strong*, *resp*.) *Poincaré inequality as stated in Proposition* 4.5 *and Theorem* 6.3 (*and Corollary* 6.4, *resp*.) *remain valid if all balls w.r.t. the metric d are replaced by balls w.r.t. the metric ρ. The constants, however, will change.*

7. Heat kernel estimates and strong Feller property. Again, we assume that (X, d, m) satisfies the strong MCP with exceptional set and we denote the exceptional set for the weak MCP by Z.

7.A. *The parabolic Harnack inequality.* The *Harnack inequality* is a uniform estimate for the growth of local solutions of certain operator equations (usually, partial differential equations). The *elliptic* Harnack inequality deals with local solutions $u\colon x \mapsto u(x)$ of the equation $Au = 0$ on X; the *parabolic* Harnack inequality with local solutions $u\colon (t, x) \mapsto u(t, x)$ of the equation $(A - \partial/\partial t)u = 0$ on $\mathbb{R} \times X$. For the precise notion of "local solution" (which actually means local weak solution) of these equations, we refer to Sturm (1995a).

THEOREM 7.1 (Parabolic Harnack inequality). *For each compact $Y \subset X \setminus Z$, there exists a constant $C_H = C_H(Y)$ such that for all balls $B_{2r}(x) \subset Y$ and all $t \in \mathbb{R}$,*

$$\sup_{(s, y) \in Q^-} u(s, y) \le C_H \inf_{(s, y) \in Q^+} u(s, y) \tag{7.1}$$

whenever u is a nonnegative local solution of the parabolic equation $(A - \partial/\partial t)u = 0$ on $Q =]t - 4r^2, t[\times B_{2r}(x)$. Here $Q^- =]t - 3r^2, t - 2r^2[\times B_r(x)$ and $Q^+ =]t - r^2, t[\times B_r(x)$.

PROOF. The result would follow immediately from Sturm (1996) if d would be the intrinsic metric for the Dirichlet form (E, F). Namely, Theorem 3.5 in that paper states that the parabolic Harnack inequality (7.1) follows from a doubling property [like (4.8)] and a Poincaré inequality [like (6.4)], provided the balls under consideration are balls w.r.t. the intrinsic metric. However, here we have balls w.r.t. the metric d and (as pointed out in the proof of Corollary 6.4), in general, the intrinsic metric does not coincide with d. But actually for the validity of Theorem 3.5 in Sturm (1996) it suffices that d is a length metric with $\mu_{\langle d_z \rangle} \leq m$ for all $z \in Y$. Fortunately, this is the case due to (6.8). □

In order to be precise, one should replace the "inf" and "sup" in (7.1) by "ess inf" and "ess sup". The following Proposition 7.3, however, states that all functions u under consideration can be chosen to be continuous (more precisely: admit a continuous version). Hence, there is no reason to use this cumbersome notation.

REMARK 7.2. The parabolic Harnack inequality obviously implies the *elliptic Harnack inequality*. That is, there exists a constant $C = C(Y)$ such that for all balls $B_{2r}(x) \subset X$,

$$\sup_{y \in B_r(x)} u(y) \leq C \inf_{y \in B_r(x)} u(y) \tag{7.2}$$

whenever u is a nonnegative local solution of the elliptic equation $Au = 0$ on $B_{2r}(x)$. Note that the elliptic Harnack inequality *does not* imply the parabolic one.

It is a well-known fact that the parabolic Harnack inequality is quite a powerful property which has many important consequences. We will state only one of them.

COROLLARY 7.3 (Hölder continuity). *Fro each compact $Y \subset X \setminus Z$ there exist constants $\alpha \in]0, 1[$ and C such that for all balls $B_{2r}(x) \subset Y$ and all $T \in \mathbb{R}$,*

$$|u(s, y) - u(t, z)| \leq C \sup_{Q_2} |u| \left(\frac{|s - t|^{1/2} + |y - z|}{r} \right)^{\alpha} \tag{7.3}$$

whenever u is a local solution of the parabolic equation $(A - \partial/\partial t)u = 0$ on $Q_2 =]T - 4r^2, T[\times B_{2r}(x)$ and (s, y) and (t, z) are points in $Q_1 = [T - r^2, T[\times B_r(x)$.

The proof of Moser (1964) carries over without any essential change.

Of course, the precise assertion of (7.3) is that any local solution of the equation $(A - \partial/\partial t)u = 0$ admits a version which satisfies (7.3). This continuous version is uniquely determined and without restriction in the sequel we always assume that all local solutions of that equation are chosen to be continuous on $X \setminus Z$.

Instead of considering "harmonic functions" $u: X \to \mathbb{R}$, one also could consider "harmonic maps" $f: X \to X'$, defined on the metric measured space (X, d, m) with values in some metric space (X', d'). More precisely, one looks for minimizers f of the energy form (or generalized Dirichlet form)

$$E(f) = \lim_{r \to 0} \frac{1}{2} \int_X \int_{B_r^*(x)} \frac{d'(f(x), f(y))^2}{d(x, y)^2} m_r(dy)\, m_r(dx).$$

For details, we refer to Korevaar and Schoen (1993) and Jost (1994). The aim is to prove existence and regularity of such minimizers. If one is interested in minimal assumptions on the original data, one typically requires that (X', d') is a complete geodesic space with nonpositive curvature (in the sense of Alexandrov) and that (X, d, m) is (derived from) a smooth Riemannian manifold. However, the previous results together with recent results of Jost (1997) indicate that in order to prove Hölder continuity for such minimizers it suffices that (X, d, m) is a metric measured space satisfying the MCP. This problem will be addressed in detail in a subsequent paper.

THEOREM 7.4. *There exists a measurable function* $p\colon]0, \infty[\times X \times X \to [0, \infty], (t, x, y) \mapsto p_t(x, y)$ *with the following properties.*

(i) *For every* $t > 0$, *every* $u \in L^2(X, m)$ *and* m-*a.e.* $x \in X$,

$$e^{-At} u(x) = \int_X p_t(x, y) u(y)\, m(dy); \tag{7.4}$$

(ii) *The function* $(t, x, y) \mapsto p_t(x, y)$ *is locally Hölder continuous on* $]0, \infty[\times (X \setminus Z) \times (X \setminus Z)$ *and identically* 0 *on the complement of that set in* $]0, \infty[\times X \times X$;

(iii) *For all* $s, t > 0$ *and all* $x, y \in X$,

$$p_t(x, y) = p_t(y, x) \tag{7.4a}$$

and

$$p_{s+t}(x, y) = \int_X p_s(s, z)\, p_t(z, y)\, m(dz). \tag{7.4b}$$

By these properties, the function p is determined pointwise uniquely. It is called heat kernel for A.

PROOF. The proof follows the lines of the proof of Proposition 2.3 in Sturm (1995a). Fix $t > 0$ and let $B_r(x_0)$ be any ball with $4r^2 < t$ and $B_{2r}(x_0)$ being a relatively compact subset of $X \setminus Z$. Put $Q_r(t, x_0) =]t - r^2, t + r^2[\times B_r(x_0)$. For $f \in L^1(X, m) \cap L^2(X, m)$ the function $u\colon (s, x) \mapsto e^{-As} f(x)$ is a local solution of the parabolic equation $(A - \partial/\partial t)u = 0$ on $]0, \infty[\times X$. Hence, the subsolution estimate, Theorem 2.1 in Sturm (1995a), together with the contraction properties of e^{-At} on $L^\infty(X, m)$ and $L^1(X, m)$ imply

$$\begin{aligned}
\operatorname*{ess\text{-}sup}_{B_r(x_0)} u(t, \cdot) &\le \operatorname*{ess\text{-}sup}_{Q_r(t, x_0)} u \le C \int_{Q_{2r}(t, x_0)} u(s, x)\, m(dx)\, ds \\
&\le C \int_{t - 4r^2}^{t + r^2} \int_X e^{-As} f(x)\, m(dx)\, ds \le C 8 r^2 \int_X f(x)\, m(dx)
\end{aligned}$$

for all $f \in L^1(X, m) \cap L^2(X, m)$. By monotonicity and continuity of e^{-At} on $L^1(X, m)$, this extends to all $f \in L^1(X, m)$. That is, for any such $B = B_r(x_0)$, the map $T_t^B\colon f \mapsto (e^{-At}f)|_B$ defines a positivity preserving, bounded linear operator

$$T_t^B\colon L^1(X, m) \to L^\infty(B, m)$$

with norm $\leq C8r^2$. From the theorem of Dunford–Pettis it now follows that T_t^B is an integral operator with a density $p_t^B(\cdot, \cdot)$ satisfying $\sup_{x \in B} \sup_{y \in X} p_t^B(x, y) \leq C8r^2$ and

$$e^{-At}f(x) = \int_X p_t^B(x, y) f(y)\, m(dy)$$

for all $f \in L^1(X, m)$ and m-a.e. $x \in B$. Covering the space $X \setminus Z$ by countably many such balls B, we obtain a function $p_t^0(\cdot, \cdot)$ which satisfies

$$e^{-At}f(x) = \int_X p_t^0(x, y) f(y)\, m(dy)$$

for all $f \in L^1(X, m)$ and m-a.e. $x \in X$.

The operator identity $e^{-A(s+t)} = e^{-As} \circ e^{-At}$ implies

$$p_{s+t}^0(x, y) = \int p_s^0(x, z)\, p_t^0(z, y)\, m(dz) \tag{7.4c}$$

for all $s, t \in]0, \infty[$ and m-a.e. $x, y \in X$. The symmetry of e^{-At} [which is a consequence of the symmetry of the Dirichlet form (E, F)] implies that

$$p_t^0(x, y) = p_t^0(y, x) \tag{7.4d}$$

for m-a.e. $x, y \in X$. From these two properties it follows that

$$\int p_t^0(y, z)^2\, m(dy) = \int p_t^0(z, y)\, p_t^0(y, z)\, m(dy) = p_{2t}^0(z, z) < \infty,$$

that is, that $p_t^0(z, \cdot) \in L^2(X, m)$, for all $t > 0$ and m-a.e. $z \in X$. Put $f(y) = p_\varepsilon^0(y, z)$ and consider

$$u(t, x) = e^{-A(t-\varepsilon)} f(y).$$

This is a solution of the equation $(A - \partial/\partial t)u = 0$ on $]\varepsilon, \infty[\times X$. On the other hand, having in mind the choice of f and using (7.4c) we get

$$u(t, x) = \int p_{t-\varepsilon}^0(x, y)\, p_\varepsilon^0(y, z)\, m(dy) = p_t^0(x, z).$$

That is, for m-a.e. $z \in X$ the function

$$(t, x) \mapsto p_t^0(x, z)$$

is a solution of the equation $(A - \partial/\partial t)u = 0$ on $]0, \infty[\times X$. By Corollary 7.3, it admits a version which is Hölder continuous on $]0, \infty[\times (X \setminus Z)$. But according to (7.4d), for m-a.e. $x \in X$ also the function $(t, y) \mapsto p_t^0(x, y)$ admits a version which is Hölder continuous on $]0, \infty[\times (X \setminus Z)$. This finally implies, that the function $(t, x, y) \mapsto p_t^0(x, y)$ admits a version $p_t(x, y)$ which is

Hölder continuous on $]0,\infty[\times(X\setminus Z)\times(X\setminus Z)$. By continuity, properties (7.4c) and (7.4d) carry over from m-a.e. $x, y\in X$ to every $x, y\in X\setminus Z$. Without restriction, one can put $p_t(x, y)=0$ if either $x\in Z$ or $y\in Z$ [since $m(Z)=0$]. With this convention, the function p_t is determined uniquely and obviously satisfies (7.4c) as well as (7.4d) for all $x, y\in X$. □

PROPOSITION 7.5. *Assume that the weak MCP holds without exceptional set. Then the diffusion process $(X_t, \mathbf{P}_x)$ properly associated with (E, F) can be chosen to be a strong Feller process. By this property, it is uniquely determined for every starting point $x\in X$.*

PROOF. It suffices to prove the strong Feller property of the semigroup $(e^{-At})_t$. For each $u\in L^2(X, m)$ the function

$$(t, x)\mapsto e^{-At}u(x)=\int_X p_t(x, y)u(y)\,m(dy)$$

is a local solution of the equation $(A-\partial/\partial t)u=0$ on $]0,\infty[\times X$ [Sturm (1995a), Section 1.4C]. Therefore, by Corollary 7.3 this function (admits a version which) is Hölder continuous on $]0,\infty[\times X$. Approximating $u\in L^\infty(X, m)$ by $u_n\in L^2(X, m)$, $n\in\mathbb{N}$, yields the Hölder continuity of $(t, x)\mapsto e^{-At}u(x)$ for all $u\in L^\infty(X, m)$. In particular, for any $t>0$ the function

$$x\mapsto e^{-At}u(x)$$

is (Hölder) continuous on X. This proves the strong Feller property for the semigroup $(e^{-At})_t$. □

COROLLARY 7.6. *The diffusion process $(X_t, \mathbf{P}_x)$ properly associated with (E, F) can be uniquely determined for every starting point $x\in X\setminus Z$ by the property that*

$$x\mapsto \mathbf{E}_x\big[u(X_t)\big]$$

is continuous in $x\in X\setminus Z$ for every $u\in C_0(X)$.

7.B. *Gaussian estimates for the heat kernel.* In the sequel we derive pointwise estimates on $X\setminus Z$ for the heat kernel $p_t(x, y)$ of the operator A on X. We emphasize that $p_t(x, y)$ is the heat kernel on the whole space X whereas the (weak) MCP is only required on the subset $X\setminus Z$. Using the parabolic Harnack inequality, one easily derives pointwise estimates for $p_t(x,y)$ with $t>0$ and $x, y\in X\setminus Z$.

THEOREM 7.7. *For every compact set $Y\subset X\setminus Z$ and every $\varepsilon>0$ there exists a constant C such that for all points $x, y\in Y$ and all $t>0$,*

$$\begin{aligned}p_t(x, y)\le{}& \frac{C}{\sqrt{m\big(B_{\sqrt{t_0}}(x)\big)\cdot m\big(B_{\sqrt{t_0}}(y)\big)}}\\ &\times\exp\left(-\frac{d^2(x, y)}{(2+\varepsilon)t}\right)\exp(-(1+\varepsilon)\lambda t).\end{aligned}\tag{7.5}$$

Here $t_0 = \inf\{t, d^2(x, X \setminus Y), d^2(y, X \setminus Y)\}$. *Furthermore,* $\lambda = \inf\{E(u,u)/\|u\|^2 \colon u \in F, u \neq 0\} \geq 0$ *denotes the bottom of the spectrum of the selfadjoint operator* A *on* $L^2(X, m)$.

For the proof, see Sturm (1995a), Theorem 2.4. □

REMARKS 7.8. (i) The number λ in the estimate (7.5) can always be replaced by 0. That is, the last term on the RHS of (7.5) can always be dropped.

Let us mention that the bottom of the spectrum of A is zero if X is complete and if the volume of balls grows subexponentially [Sturm (1994), Theorem 5]. For instance, the latter is satisfied if the doubling property holds true uniformly on X (which even implies that the volume grows at most polynomially).

(ii) In addition to the assumptions of Theorem 7.7, assume that there exists an arc γ of length $L_d(\gamma) \leq C_1 d(x,y)$ joining x and y and that the doubling property holds true on the neighborhood $B_R(\gamma) = \{x \in X \colon d(x, \gamma) < R\}$ of γ. Then the usual chaining argument yields

$$m\big(B_{\sqrt{t}}(x)\big) \leq C_2\, m\big(B_{\sqrt{t}}(y)\big) \exp\left(C_2 \frac{d(x,y)}{\sqrt{t}}\right). \tag{7.6}$$

Hence, *for every* $\varepsilon > 0$ *there exists a constant* C' *such that*

$$p_t(x,y) \leq \frac{C'}{m\big(B_{\sqrt{t_0}}(x)\big)} \exp\left(-\frac{d^2(x,y)}{(2+\varepsilon)t}\right) \exp(-(1+\varepsilon)\lambda t) \tag{7.7}$$

for all $x, y \in Y$ *and* $t > 0$ *with* t_0 *as above.*

Estimate similar to (7.5) also hold for the time derivatives $(\partial/\partial t)^k p_t(x,y)$ of the heat kernel. See Sturm (1995a) for details.

THEOREM 7.9. *For each compact set* $Y \subset X \setminus Z$ *there exists a constant* C *such that*

$$p_t(x,y) \geq \frac{1}{C m\big(B_{\sqrt{t} \wedge R}(x)\big)} \exp\left(-C \frac{d^2(x,y)}{2t}\right) \exp\left(-\frac{Ct}{R^2}\right) \tag{7.8}$$

for all $t > 0$ *and all* $x, y \in X$ *which are joined in* Y *by an arc* γ *of length* $d(x,y)$. *Here* $R = d(\gamma, X \setminus Y) = \inf_{0 \leq s \leq 1} d(\gamma_s, X \setminus Y)$.

REMARKS 7.10. (i) Theorem 7.9 together with the symmetry of $p_t(\cdots)$ immediately implies the following symmetrized version of (7.8):

$$\begin{aligned} p_t(x,y) \geq{}& \frac{1}{C\sqrt{m\big(B_{\sqrt{t} \wedge R}(x)\big) m\big(B_{\sqrt{t} \wedge R}(y)\big)}} \\ &\times \exp\left(-C \frac{d^2(x,y)}{2t}\right) \exp\left(-\frac{Ct}{R^2}\right) \end{aligned} \tag{7.9}$$

for t, x, y and R as above.

(ii) If x and y are not joined by a geodesic in Y of length $d(x,y)$ but only by some arc $\gamma: [0,1] \to Y$ of length $L_d(\gamma)$ then the estimate (7.8) remains true with $L_d(\gamma)$ in the place of $d(x,y)$.

7.C. *Estimates for Green functions and first exit times.* Integrating the heat kernel $p_t(x,y)$ against t we obtain the *fundamental solution* or *Green function*

$$G(x,y) = \int_0^\infty p_t(x,y)\,dt$$

for the operator A on X. Theorem 7.4 immediately implies that $G(x,y) = G(y,x)$ for all $x, y \in X$ and that

$$\int_X G(x,y) f(y)\, m(dy) = \mathbf{E}_x\left[\int_0^\infty f(X_t)\,dt\right]$$

for all $f \in L^2(X,m)$ and m-a.e. $x \in X$. Even more interesting is the Green function

$$G^Y(x,y) = G(x,y) - \mathbf{E}_x\left[G\left(X_{\tau(Y)}, y\right); \tau(Y) < \infty\right]$$

for A on a given open set $Y \subset X$. Here $\tau(Y) = \inf\{t > 0\colon X_t \in X \setminus Y\}$. The strong Markov property implies that

$$\int_Y G^Y(x,y) f(y)\, m(dy) = \mathbf{E}_x\left[\int_0^{\tau(Y)} f(X_t)\,dt\right] \tag{7.10}$$

for all $f \in L^2(X,m)$ and m-a.e. $x \in Y$.

COROLLARY 7.11. *For every compact set $Y \subset X \setminus Z$ there exists a constant C such that for all balls $B = B_r(x) \subset Y$ and all $y \in B_{r/2}(x)$,*

$$C^{-1}\int_{d(x,y)}^r \frac{s\,ds}{m\left(B_s(x)\right)} \le G^B(x,y) \le C\int_{d(x,y)}^r \frac{s\,ds}{m\left(B_s(x)\right)}. \tag{7.11}$$

Biroli-Mosco (1995), Theorem 1.3, together with some simple covering arguments gives the proof.

THEOREM 7.12. *For every compact set $Y \subset X \setminus Z$ there exists a constant C such that for all balls $B_r(x) \subset Y$,*

$$C^{-1} r^2 \le \mathbf{E}_x(\tau_r) \le C r^2. \tag{7.12}$$

Here $\tau_r = \inf\{t > 0\colon d(X_t, X_0) > r\}$.

PROOF. According to (7.10),

$$\mathbf{E}_x[\tau_r] = \int_{B_r(x)} G^{B_r(x)}(x,y)\, m(dy).$$

By (7.11), the RHS is comparable with

$$\int_{B_r(x)} \left(\int_{d(x,y)}^{r} \frac{s\,ds}{m(B_s(x))} \right) m(dy) = \int_0^r \left(\int_{B_s(y)} m(dy) \right) \frac{s\,ds}{m(B_s(x))}$$
$$= \int_0^r s\,ds = \frac{1}{2} r^2. \qquad \square$$

8. The local dimension and Dirichlet forms with general normalization function. Throughout this section, we assume that the weak MCP holds without exceptional set and that the strong MCP holds with some exceptional set.

8.A. *Dirichlet forms with general normalization function.* Denote the previously defined Dirichlet form (E, F) from now on by (E^1, F^1). For $u \in C_0^{\mathrm{Lip}}(X)$, let $\mu^1_{\langle u \rangle}$ denote the energy measure of u w.r.t. the Dirichlet form (E^1, F^1).

For any Borel function $N \geq 0$ on X and $u \in C_0^{\mathrm{Lip}}(X)$ define

$$E_0^N(u,u) = \tfrac{1}{2} \int_X N(x)\, \mu^1_{\langle u \rangle}(dx). \tag{8.1}$$

THEOREM 8.1. (i) *Let $N \geq 1$ be any locally bounded Borel function on X. Then $(E_0^N, C_0^{\mathrm{Lip}}(X))$ defines a closable symmetric form on $L^2(X,m)$. Its closure (E^N, F^N) is a regular, strongly local Dirichlet form with energy measure given by*

$$\mu^N_{\langle u \rangle}(dx) = N(x)\, \mu^1_{\langle u \rangle}(dx)$$

for each $u \in C_0^{\mathrm{Lip}}(X)$.

(ii) *Assume in addition that $N \in C(Y)$ for some open set Y with $m(X \setminus Y) = 0$. Then*

$$E^N(u,u) = \lim_{r \to 0} \frac{1}{2} \int_X N(x) \int_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)} \right]^2 m_r(dy)\, m_r(dx) \tag{8.2}$$

for all $u \in C_0^{\mathrm{Lip}}(X)$. In this case, the energy measure of $u \in C_0^{\mathrm{Lip}}(X)$ w.r.t. (E^N, F^N) is given by the formula

$$\int_X \varphi(x)\, \mu^N_{\langle u \rangle}(dx) = \lim_{r \to 0} \int_X \varphi(x)\, N(x) \int_{B_r^*(x)} \left[\frac{u(x) - u(y)}{d(x,y)} \right]^2 \times m_r(dy)\, m_r(dx) \tag{8.3}$$

for all $\varphi \in C_0(X)$.

(iii) *Assume that $N \in L^\infty_{\mathrm{loc}}(X, m)$ and $N \geq 1$. Then the intrinsic metric*

$$\rho^N(x,y) = \sup\left\{ u(x) - u(y) \colon u \in C_0^{\mathrm{Lip}}(X) \cap F,\ \mu^N_{\langle u \rangle} \leq m \right\}$$

associated with the Dirichlet form (E^N, F^N) *is locally equivalent to the original metric* d.

PROOF. (i) E^N is the increasing limit of the $E^{N\wedge k}$, $k \in \mathbb{N}$. Each $E^{N\wedge k}$ is comparable with E^1, hence, closable with closure being a strongly locally regular Dirichlet form with core $C_0^{\mathrm{Lip}}(X)$. For $k \to \infty$ these properties carry over to the increasing limit E^N; (ii) is an immediate consequence of Theorem 6.1; (iii) follows from Theorem 6.8. □

COROLLARY 8.2. *Assume that* $N \in L^\infty_{\mathrm{loc}}(X, m)$ *with* $N \geq 1$. *Then all the regularity assertions stated in Chapter* 7 *for* (*local solutions, heat kernels, Green functions, diffusion processes associated with*) *the Dirichlet form* (E, F) *remain valid for* (*the respective quantities associated with*) *the Dirichlet form* (E^N, F^N).

8.B. *The local dimension.* In the sequel we will introduce functions $\overline{N}$ and $\underline{N}$ on X, which play the role of the (upper and lower, resp.) local dimension of the space X. For this purpose, let us fix throughout this chapter a core $C^* \subset C_0^{\mathrm{Lip}}(X)$ which is dense in $C_0(X)$ as well as in F (w.r.t. the respective norms) and which defines the intrinsic metric in the sense that

$$\rho(x, y) = \sup\left\{u(x) - u(y) \colon u \in C^*, \mu_{\langle u \rangle} \leq m\right\}$$

for all $x, y \in X$. Note that the results below may depend on the choice of C^*. Of course, one can always choose $C^* = C_0^{\mathrm{Lip}}(X)$.

We define the *dilation* of a function $u\colon X \to \mathbb{R}$ at the point $x \in X$ by

$$\mathrm{dil}_x u = \limsup_{z \to x} \mathrm{dil}_z^0 u \quad \text{where } \mathrm{dil}_z^0 u = \lim_{r \to 0} \sup_{y \in B_r^*(z)} \frac{|u(z) - u(y)|}{d(z, y)}.$$

We will compare it with the following L^2-versions:

$$\overline{D}_x u = \operatorname*{ess\text{-}lim\,sup}_{z \to x} \overline{D}_z^0 u \quad \text{and} \quad \underline{D}_x = \operatorname*{ess\text{-}lim\,sup}_{z \to x} \underline{D}_z^0 u,$$

where

$$\overline{D}_z^0 u = \limsup_{r \to 0} \left(\frac{1}{\sqrt{m(B_r(z))}} \int_{B_r^*(z)} \left| \frac{u(z) - u(y)}{d(z, y)} \right|^2 m_r(dy) \right)^{1/2}$$

and

$$\underline{D}_z^0 u = \liminf_{r \to 0} \left(\frac{1}{\sqrt{m(B_r(z))}} \int_{B_r^*(z)} \left| \frac{u(z) - u(y)}{d(z, y)} \right|^2 m_r(dy) \right)^{1/2}.$$

DEFINITION 8.3. The *upper local dimension* at $x \in X$ is

$$\overline{N}(x) = \sup_{\substack{u \in C^* \\ \mathrm{dil}_x u = 1}} [\underline{D}_x u]^{-2} \tag{8.4}$$

and the *lower local dimension* at $x \in X$ is

$$\underline{N}(x) = \left[\overline{D}_x(d_x)\right]^{-2}. \tag{8.5}$$

[Here d_x denotes the function $y \mapsto d(x, y)$.] If $\overline{N}(x)$ and $\underline{N}(x)$ coincide, then the common value is called the *local dimension* (of the space X) at $x \in X$.

REMARKS 8.4. (i) Fix $z \in X$. If

$$\sup_{x, y \in B_r(z)} \frac{m(B_r(x))}{m(B_r^*(y))} \to 1 \tag{8.6}$$

for $r \to 0$ then

$$\overline{D}_z^0 u = \limsup_{r \to 0} \left(\frac{1}{m(B_r(z))} \int_{B_r^*(z)} \left| \frac{u(z) - u(y)}{d(z, y)} \right|^2 m(dy) \right)^{1/2}$$

for all u and therefore the triangle inequality obviously implies

$$\overline{D}_z^0(d_x) \le 1$$

for all $x \in X$. Hence, if (8.6) holds for all $z \in X$ then $\underline{N}(x) \ge 1$ for all $x \in X$.

(ii) If $C^* = C_0^{\mathrm{Lip}}(X)$ (or, more generally, if $d_z \in C^*$ for all $z \in X$) then

$$\underline{N}(x) \le \overline{N}(x)$$

for all $x \in X$ (since $\overline{N}(x) \ge [\underline{D}_x(d_x)]^{-2} \ge [\overline{D}_x(d_x)]^{-2} = \underline{N}(x)$).

LEMMA 8.5. (i) *Assume that the doubling property holds on some open subset $X_0 \subset X$. Then $\underline{N}^{-1}$, $\overline{N} \in L^\infty_{\mathrm{loc}}(X_0, m)$. In particular, the (weak) MCP implies*

$$\underline{N}^{-1}, \overline{N} \in L^\infty_{\mathrm{loc}}(X \setminus Z, m). \tag{8.7}$$

(ii) *If some open subset $X_0 \subset X$ is an n-dimensional Riemannian manifold, then for all $x \in X_0$*

$$\underline{N}(x) = \overline{N}(x) = n. \tag{8.8}$$

PROOF. (i) The triangle inequality implies

$$\left| \frac{d(z, x) - d(x, y)}{d(z, y)} \right|^2 \le 1$$

for all $x, y, z \in X$. Hence, for any compact $Y \subset X_0$,

$$\overline{D}_z^0(dx) \le \limsup_{r\to 0}\left(\frac{1}{\sqrt{m(B_r(z))}}\int_{B_r(z)}\frac{m(dy)}{\sqrt{m(B_r(y))}}\right)^{1/2}$$

$$\le \limsup_{r\to 0}\left(\frac{M^{1/2}}{\sqrt{m(B_r(z))}}\int_{B_r(z)}\frac{m(dy)}{\sqrt{m(B_r(y))}}\right)^{1/2} \le M^{1/4}$$

for all $z \in Y$ with M being the doubling constant. That is,

$$\underline{N}(z) \ge M^{-1/2}.$$

This proves that $1/\underline{N} \in L^\infty_{\mathrm{loc}}(X_0, m)$.

The proof of the assertion $\overline{N} \in L^\infty_{\mathrm{loc}}(X_0, m)$ is essentially the same as that of Proposition 6.7. Fix a compact set $Y \subset X_0$. Given $x \in Y$ and $u \in C^*$, let

$$L_\varepsilon = L_\varepsilon(u, x) = \sup\left\{\left|\frac{u(y)-u(z)}{d(y,z)}\right| : (y,z) \in Q(B_\varepsilon(x)),\ y \ne z\right\}.$$

Then $\mathrm{dil}_x u = \sup_{\varepsilon > 0} L_\varepsilon$. As in the proof of Proposition 6.7, for each $\varepsilon > 0$ one finds $(z_1, z_2) \in Q(B_\varepsilon(x))$ [with arbitrarily small $d(z_1, z_2)$] such that

$$|u(z_1) - u(z_2)| \ge \tfrac{3}{4} L_\varepsilon d(z_1, z_2) > 0.$$

Hence, for some $\delta > 0$ and all $z \in B_\delta(z_2)$,

$$|u(z) - u(z_2)| \ge \tfrac{1}{2} L_\varepsilon d(z, z_2) > 0.$$

Following the argumentation from the proof of Proposition 6.7, this implies

$$\frac{1}{\sqrt{m(B_r(z))}}\int_{B_r^*(z)}\left[\frac{u(z)-u(y)}{d(z,y)}\right]^2 m_r(dy) \ge \frac{L_\varepsilon}{2^8 M^4}$$

for all $r \le \frac{4}{3} d(z, z_2)$. Therefore,

$$\underline{D}_z^0 u \ge \frac{L_\varepsilon}{16 M^2}$$

for all $z \in B_\delta(z_2) \subset B_\varepsilon(x)$ and thus

$$\underline{D}_x u \ge \frac{L_\varepsilon}{16M^2} \to \frac{\mathrm{dil}_x u}{16 M^2}$$

for $\varepsilon \to 0$. That is, $\overline{N}(x) \le (16M^2)^2$.

(ii) Fix $x \in X_0$. By local regularity, we may choose C^* such that $u|_{X_0} \in C^\infty(X_0)$ for all $u \in C^*$. That is,

$$\overline{N}(x) = \sup_{\substack{u \in C^\infty(X_0)\\ \mathrm{dil}_x u = 1}} [\underline{D}_x u]^{-2}.$$

Moreover, without restriction we may assume $X_0 \subset \mathbb{R}^n$. For $u \in C^\infty(X_0)$ and all y near x,

$$u(y) = u(x) + \nabla u(x)(y-x) + o\big(|y-x|^2\big).$$

Now $\operatorname{dil}_x u = 1$ implies $|\nabla u(x)| = 1$ and without restriction we may assume that, for example, $\nabla u(x) = (1, 0, \dots, 0)$. Then

$$u(y) = u(x) + (y_1 - x_1) + o\big(|y-x|^2\big).$$

Hence,

$$\underline{D}_z^0 u = \liminf_{r\to 0} \left(\frac{1}{m(B_r(z))} \int_{B_r^*(z)} \frac{|z_1 - y_1|^2}{|z-y|^2}\, dy \right)^{1/2}.$$

But for each $r > 0$ and $z \in \mathbb{R}^n$,

$$\frac{1}{m(B_r(z))} \int_{B_r^*(z)} \frac{|z_1 - y_1|^2}{|z_1-y_1|^2 + \cdots + |z_n - y_n|^2}\, dy = \frac{1}{n}.$$

Therefore, $\underline{D}_x u = 1/\sqrt{n}$ and $\overline{N}(x) = n$ for all $x \in X_0$.

Similarly, one obtains $\overline{D}_z^0 u = 1/\sqrt{n}$ for all u which are C^∞ in a neighborhood of z. In particular, $\overline{D}_z^0(d_x) = 1/\sqrt{n}$ for all $z \neq x$. Hence, $\overline{D}_x(d_x) = 1/\sqrt{n}$ and thus $\underline{N}(x) = n$. □

Proposition 8.6. *Let N be a bounded Borel function on X with $N \in C(Y)$ for some open set Y with $m(X \setminus Y) = 0$.*

(i) *If* $\operatorname{ess-lim\,inf}_{z\to x} N(z) \geq \overline{N}(x)$ *for all $x \in X$ then*

$$\rho^N \leq d \quad \text{on } X \times X. \tag{8.9}$$

In particular, the topology induced by ρ^N is coarser than the original topology on X (induced by d).

(ii) *If* $\operatorname{ess-lim\,sup}_{z\to x} N(z) \leq \underline{N}(x)$ *for all $x \in X$ then the topology induced by ρ^N is finer than the original topology on X (induced by d) and the length metric $\overline{\rho^N}$ derived from ρ^N satisfies*

$$\overline{\rho^N} \geq d \quad \text{on } X \times X. \tag{8.10}$$

Proof. (i) Assume $\operatorname{ess-lim\,inf} N \geq \overline{N}$ on X. That is, for all $x \in X$ and for all $u \in C^*$,

$$\begin{aligned}&\operatorname*{ess-lim\,inf}_{z\to x} N(z) \operatorname*{ess-lim\,sup}_{z\to x} \operatorname*{lim-inf}_{r\to 0} \frac{1}{\sqrt{m(B_r(z))}} \\ &\quad\times \int_{B_r(z)} \left| \frac{u(z)-u(y)}{d(z,y)} \right|^2 m_r(dy) \geq [\operatorname{dil}_x u]^2.\end{aligned} \tag{8.11}$$

Now consider such $u \in C^*$ for which $\mu^N_{\langle u \rangle} \le m$. That is, for which

$$\lim_{r \to 0} \int_X \varphi(z)\, N(z) \int_{B_r(z)} \left| \frac{u(z) - u(y)}{d(z, y)} \right|^2 m_r(dy)\, m_r(z) \le \int_X \varphi(z)\, m(dz)$$

for all nonnegative $\varphi \in C_0(X)$. By Fatou's lemma, the latter implies

$$\liminf_{r \to 0} \frac{N(z)}{\sqrt{m(B_r(z))}} \int_{B_r(x)} \left| \frac{u(x) - u(y)}{d(x, y)} \right|^2 m_r(dy) \le 1$$

for m-a.e. $z \in X$ and thus

$$\operatorname*{ess-lim\,inf}_{z \to x} N(z)\ \operatorname*{ess-lim\,sup}_{z \to x} \liminf_{r \to 0} \frac{1}{\sqrt{m(B_r(z))}}$$

$$\times \int_{B_r(x)} \left| \frac{u(x) - u(y)}{d(x, y)} \right|^2 m_r(dy) \le 1$$

for all $x \in X$. Together with (8.11) this yields

$$\operatorname{dil}_x u \le 1 \tag{8.12}$$

for all $x \in X$. This, however, easily implies

$$|u(y) - u(z)| \le d(y, z)$$

for all $y, z \in X$ and thus

$$\rho^N(y, z) \le d(y, z) \tag{8.13}$$

for all $y, z \in X$.

(ii) Assume $\underline{N}(z) \ge \operatorname{ess-lim\,inf}_{x \to z} N(x)$ for all $z \in X$. Fix $z \in X$ and a number $\alpha > 1$. Choose $R > 0$ such that

$$\liminf_{r \to 0} \left(\frac{1}{\sqrt{m(B_r(x))}} \int_{B_r(x)} \left| \frac{d(x, z) - d(y, z)}{d(x, y)} \right|^2 m_r(dy) \right)^{-1} \ge \frac{1}{\alpha} N(x)$$

for m-a.e. $x \in B_R(z)$. That is,

$$\limsup_{r \to 0} \frac{N(x)}{\sqrt{m(B_r(x))}} \int_{B_r(x)} \left| \frac{d(x, z) - d(y, z)}{d(x, y)} \right|^2 m_r(dy) \le \alpha \tag{8.14}$$

for m-a.e. $x \in B_r(z)$. Therefore (by Fatou's lemma),

$$\begin{aligned} &\int_X \varphi(x)\, \mu^N_{\langle d(z, \cdot) \rangle}(dx) \\ &\quad = \lim_{r \to 0} \int_X \varphi(x)\, N(x) \int_{B_r(x)} \left| \frac{d(x, z) - d(y, z)}{d(x, y)} \right|^2 m_r(dy)\, m_r(dx) \\ &\quad \le \alpha \int_X \varphi(x)\, m(dx) \end{aligned}$$

for all nonnegative $\varphi \in C_0(B_R(z))$. That is,

$$\mu^N_{\langle d(z,\cdot)\rangle} \leq \alpha m \quad \text{on } B_R(x). \tag{8.15}$$

Now consider the function $u\colon x \mapsto (1/\sqrt{\alpha})(R - d(z,x))_+$. Then $\mu^N_{\langle u\rangle} \leq m$ on X. That is,

$$\mu^N_{\langle u\rangle} \leq m \quad \text{on } X. \tag{8.16}$$

Therefore, for all $x, y \in B_R(z)$,

$$\rho^N(x,y) \geq \frac{1}{\alpha} d(x,y). \tag{8.17}$$

Since the whole space is covered by such balls $B_R(z)$ we get (8.17) for all $x, y \in X$ with $d(x,y)$ sufficiently small. In particular, the topology induced by ρ^N on X is finer than the original one (induced by d).

Now let $\gamma\colon [0,1] \to X$ be any arc in X and let $L_{\rho^N}(\gamma)$ and $L_d(\gamma)$ be its lengths w.r.t. the metrics ρ^N and d, resp. Then (8.17) implies

$$L_{\rho^N}(\gamma) \geq \frac{1}{\sqrt{\alpha}} L_d(\gamma) \geq \frac{1}{\sqrt{\alpha}} d(\gamma_0, \gamma_1) \tag{8.18}$$

uniformly for all $\alpha > 1$. That is,

$$\overline{\rho^N}(x,y) \geq \frac{1}{\sqrt{\alpha}} \bar{d}(x,y) = d(x,y) \tag{8.19}$$

for all $x, y \in X$.

Remark 8.7. Let again N be a bounded Borel function on X with $N \in C(Y)$ for some open set Y with $m(X \setminus Y) = 0$.

(i) In order to deduce (8.9) it suffices that

$$\operatorname*{ess\text{-}lim\,sup}_{x \to z} \left(\sqrt{N(x)}\, \underline{D}^0_x u\right) \geq \operatorname{dil}_z u \tag{8.20}$$

for all $z \in X$ and all $u \in C^*$.

(ii) In order to deduce (8.10) it suffices that

$$\operatorname*{ess\text{-}lim\,sup}_{x \to z} \left(\sqrt{N(x)}\, \overline{D}^0_x (d_z)\right) \leq 1 \tag{8.21}$$

for all $z \in X$.

Theorem 8.8. *Assume that $\overline{N} = \underline{N} \in C(X)$ and choose $N = \overline{N}$. Then the intrinsic metric ρ^N coincides with the original metric d.*

Proof. Proposition 8.6 implies that $\rho^N \geq d$ on $X \times X$ and $\bar{\rho}^N \leq d$ on $X \times X$. Moreover, ρ^N and d induce the same topology on X. The latter implies that (X, ρ^N) is a length space [Sturm (1995a)], that is, $\bar{\rho}^N = \rho^N$ on $X \times X$. This proves the claim. $\square$

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REFERENCES

BALLMANN, W. (1995). *Lectures on Spaces of Nonpositive Curvature. DMV Seminar* **25**. Birkhäuser, Boston.

BIROLI, M. and MOSCO, U. (1995). A Saint-Vernant principle for Dirichlet forms on discontinuous media. *Ann. Mat. Pura Appl.* **169** 125–181.

BOULEAU, N. and HIRSCH, F. (1991). *Dirichlet Forms and Analysis on Wiener Space.* de Gruyter, Berlin.

CHAVEL, I. (1993). *Riemannian Geometry: A Modern Introduction*. Cambridge Univ. Press.

DAL MASO, G. (1993). *An Introduction to Γ-Convergence.* Birkhäuser, Boston.

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes.* Wiley, New York.

FUKUSHIMA, M., OSHIMA, Y. and TAKEDA, M. (1994). *Dirichlet Forms and Symmetric Markov Processes.* de Gruyter, Berlin.

GHYS, E. and DE LA HARPE, P. (1990). *Sur les Groupes Hyperboliques d'apres Mikhael Gromov.* Birkhäuser, Boston.

GROMOV, M. (1981). *Structures Métriqués pour les Variétés Riemanniennes* (J. Lafontaine and P. Pansu, eds.). Cedic/F. Nathan.

JERISON, D. (1986). The Poincaré inequality for vector fields satisfying an Hörmander's condition. *Duke J. Math.* **53** 503–523.

JOST, J. (1994). Equilibrium maps between metric spaces. *Calc. Var.* **2** 173–204.

JOST, J. (1996). Generalized harmonic maps between metric spaces. In *Geometric Analysis and the Calculus of Variations for Stefan Hildebrandt* (J. Jost, ed.). International Press, Boston.

JOST, J. (1997). Generalized Dirichlet forms and harmonic maps. *Calc. Var.* **5** 1–19.

KOREVAAR, N. J. and SCHOEN, R. M. (1993). Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.* **1** 561–659.

MA, Z. and RÖCKNER, M. (1992). *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms.* Universitext. Springer, Berlin.

MOSCO, U. (1994). Composite media and asymptotic Dirichlet forms. *J. Funct. Anal.* **123** 368–421.

MOSER, J. (1964). A Harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math.* **17** 101–134.

RINOW, M. (1961). *Die innere Geometrie der metrischen Räume. Grundlehren* **105**. Springer, Berlin.

STURM, K. T. (1994). Analysis on local Dirichlet spaces I. Recurrence, conservativeness and L^p-Liouville properties. *J. Reine Angew. Math.* **456** 173–196.

STURM, K. T. (1995a). Analysis on local Dirichlet spaces II. Upper Gaussian estimates for the fundamental solution of parabolic equations. *Osaka J. Math.* **32** 275–312.

STURM, K. T. (1995b). On the geometry defined by Dirichlet forms. In *Seminar on Stochastic Analysis, Random Fields and Applications* (E. Bolthausen et al., eds.) 231–242. Birkhäuser, Boston.

STURM, K. T. (1996). Analysis on local Dirichlet spaces III. The parabolic Harnack inequality. *J. Math. Pures Appl.* **75** 273–297.

STURM, K. T. (1997). How to construct diffusion processes on metric spaces. *Potential Analysis.* To appear.

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