shall discuss these two forms in the following chapters XI and XII, first the adjustment by correlates whose rules it is easiest to deduce. In practice we prefer adjustment by correlates when m is nearly as large as n, adjustment by elements when m is small.

XI. ADJUSTMENT BY CORRELATES.

§ 47. We suppose we have ascertained that the whole theory is expressed in the equations $[au] = A, \dots [cu] = C$, where the adjusted values u of the n observations are the only unknown quantities; we prefer in doubtful cases to have too many equations rather than too few, and occasionally a supernumerary equation to check the computation. The first thing the adjustment by correlates then requires is that the functions $[ao] \dots [co]$, corresponding to these equations, are made free of one another by the schedule in § 42.

Let $[ao], \dots [c''o]$ indicate the $n-m$ mutually free functions which we have got by this operation, and let us, beside these, imagine the system of free functions completed by m other arbitrarily selected functions, $[d'''o], \dots [g^{\nu}o]$, representatives of the empiric functions; the adjustment is then principally made by introducing the theoretical values into this system of free functions. It is finally accomplished by transforming back from the free modified functions to the adjusted observations. For this inverse transformation, according to (62), the n equations are:

$$o_i = \left\{\frac{a_i}{[aa\lambda_2]}[ao] + \dots + \frac{c''_i}{[c''c''\lambda_2]}[c''o] + \frac{d'''_i}{[d'''d'''\lambda_2]}[d'''o] + \dots + \frac{g^{\nu}_i}{[g^{\nu}g^{\nu}\lambda_2]}[g^{\nu}o]\right\}\lambda_2(o_i) \quad (73)$$

and according to (35) (compare also (63))

$$\left.\begin{aligned} \lambda_2(o_i) &= \left\{\frac{a_i^2\lambda_2^2(o_i)}{[aa\lambda_2]^2}\cdot\lambda_2[ao] + \dots\dots\dots + \frac{g_i^{\nu 2}\lambda_2^2(o_i)}{[g^{\nu}g^{\nu}\lambda_2]^2}\lambda_2[g^{\nu}o]\right\} \\ &= \left\{\frac{a_i^2}{[aa\lambda_2]} + \dots + \frac{c_i''^2}{[c''c''\lambda_2]} + \frac{d_i'''^2}{[d'''d'''\lambda_2]} + \dots + \frac{g_i^{\nu 2}}{[g^{\nu}g^{\nu}\lambda_2]}\right\}\lambda_2^2(o_i) \end{aligned}\right\} \quad (74)$$

As the adjustment influences only the $n-m$ first terms of each of these equations, we have, because $[au] = A, \dots [c''u] = C''$, and $\lambda_2[au] = \dots = \lambda_2[c''u] = 0$,

$$u_i = \left\{\frac{a_i}{[aa\lambda_2]}A + \dots + \frac{c''_i}{[c''c''\lambda_2]}C'' + \frac{d'''_i}{[d'''d'''\lambda_2]}[d'''o] + \dots + \frac{g^{\nu}_i}{[g^{\nu}g^{\nu}\lambda_2]}[g^{\nu}o]\right\}\lambda_2(o_i) \quad (75)$$

and

$$\lambda_2(u_i) = \left\{\frac{d_i'''^2}{[d'''d'''\lambda_2]} + \dots + \frac{g_i^{\nu 2}}{[g^{\nu}g^{\nu}\lambda_2]}\right\}\lambda_2^2(o_i). \quad (76)$$

Consequently

$$o_i - u_i = \lambda_2(o_i)\left\{a_i \frac{[ao]-A}{[aa\lambda_2]} + \dots + c_i'' \frac{[c''o]-C''}{[c''c''\lambda_2]}\right\} \tag{77}$$

and

$$\lambda_2(o_i) - \lambda_2(u_i) = \lambda_2^2(o_i)\left\{\frac{a_i^2}{[aa\lambda_2]} + \dots + \frac{c_i''^2}{[c''c''\lambda_2]}\right\} = \lambda_2(o_i - u_i). \tag{78}$$

Thus for the computation of all the differences between the observed and adjusted values of the several observations and the squares of their mean errors, and thereby indirectly for the whole adjustment, we need but use the values and the mean errors of the several observations, the coefficients in the theoretically given functions, and the two values of each of these, namely, the theoretical value, and the value which the observations would give them.

The factors in the expression for $o_i - u_i$,

$$K_a = \frac{[ao]-A}{[aa\lambda_2]}, \quad \dots\dots, \quad K_{c''} = \frac{[c''o]-C''}{[c''c''\lambda_2]},$$

which are common to all the observations, are called *correlates*, and have given the method its name. The adjusted, improved values of the observations are computed in the easiest way by the formula

$$u_i = o_i - \lambda_2(o_i)\left\{a_i K_a + \dots + c_i'' K_{c''}\right\}. \tag{79}$$

By writing the equation (78)

$$\frac{\lambda_2(o_i - u_i)}{\lambda_2(o_i)} = \left\{\frac{a_i^2}{[aa\lambda_2]} + \dots + \frac{c_i''^2}{[c''c''\lambda_2]}\right\} \lambda_2(o_i) \tag{80}$$

and summing up for all values of i from 1 to n, we demonstrate the proposition concerning the sum of the scales discussed in the preceding chapter, viz.

$$\left[1 - \frac{\lambda_2(u)}{\lambda_2(o)}\right] = \frac{[aa\lambda_2]}{[aa\lambda_2]} + \dots + \frac{[c''c''\lambda_2]}{[c''c''\lambda_2]} = n - m. \tag{81}$$

§ 48. It deserves to be noticed that all these equations are homogeneous with respect to the symbol λ_2. Therefore it makes no change at all in the results of the adjustment or the computation of the scales, if our assumed knowledge of the mean errors in the several observations has failed by a wrong estimate of the unity of the mean errors if only the proportionality is preserved; we can adjust correctly if we know only t - relative weights of the observations. The homogeneousness is not broken till we reach the equations of the criticism:

$$\left.\begin{aligned}&\frac{([ao]-A)^2}{[aa\lambda_2]}+\dots+\frac{([c''o]-C'')^2}{[c''c''\lambda_2]}=\\&=K_a^2[aa\lambda_2]+\dots+K_{c''}^2[c''c''\lambda_2]=\\&=\left[\frac{(o-u)^2}{\lambda_2(o)}\right]=[(aK_a+\dots+c''K_{c''})^2\lambda_2(o)]=n-m\pm\sqrt{2(n-m)}\end{aligned}\right\}\quad(82)$$

It follows that criticism in this form, the "summary criticism", can only be used to try the correctness of the hypothetical unity of the mean errors, or to determine this if it has originally been quite unknown. The special criticism, on the other hand, can, where the series of observations is divided into groups, give fuller information through the sums of squares

$$\Sigma\frac{(o-u)^2}{\lambda_2(o)}=\Sigma\left(1-\frac{\lambda_2(u)}{\lambda_2(o)}\right),\qquad(83)$$

taken for each group. We may, for instance, test or determine the unities of the mean errors for one group by means of observations of angles, for another by measurements of distances, etc.

The criticism has also other means at its disposal. Thus the differences $(o-u)$ ought to be small, particularly those whose mean errors have been small, and they ought to change their signs in such a way that approximately

$$\Sigma\frac{o_i-u_i}{\lambda_2(o_i)}=0\qquad(84)$$

for natural or accidentally selected groups, especially for such series of observations as are nearly repetitions, the essential circumstances having varied very little.

If, ultimately, the observations can be arranged systematically, either according to essential circumstances or to such as are considered inessential, we must expect frequent and irregular changes of the signs of $o-u$. If not, we are to suspect the observations of systematical errors, the theory proving to be insufficient.

§ 49. It will not be superfluous to present in the form of a schedule of the adjustment by correlates what has been said here, also as to the working out of the free functions. We suppose then that, among 4 unbound observations o_1, o_2, o_3, and o_4, with the squares on their mean errors $\lambda_2(o_1)$, $\lambda_2(o_2)$, $\lambda_2(o_3)$, and $\lambda_2(o_4)$, there exist relations which can be expressed by the three theoretical equations

$$\begin{aligned}[][au]&=a_1u_1+a_2u_2+a_3u_3+a_4u_4=A\\[][bu]&=b_1u_1+b_2u_2+b_3u_3+b_4u_4=B\\[][cu]&=c_1u_1+c_2u_2+c_3u_3+c_4u_4=C.\end{aligned}$$

The schedule is then as follows:

		The given			Free functions			Adjusted values				Scales
		A	B	C	B'	C'	C''					
o_1	$\lambda_2(o_1)$	a_1	b_1	c_1	b'_1	c'_1	c''_1	o_1-u_1	u_1	$\lambda_2(o_1-u_1)$	$\lambda_2(u_1)$	$1-\lambda_2(u_1):\lambda_2(o_1)$
o_2	$\lambda_2(o_2)$	a_2	b_2	c_2	b'_2	c'_2	c''_2	o_2-u_2	u_2	$\lambda_2(o_2-u_2)$	$\lambda_2(u_2)$	$1-\lambda_2(u_2):\lambda_2(o_2)$
o_3	$\lambda_2(o_3)$	a_3	b_3	c_3	b'_3	c'_3	c''_3	o_3-u_3	u_3	$\lambda_2(o_3-u_3)$	$\lambda_2(u_3)$	$1-\lambda_2(u_3):\lambda_2(o_3)$
o_4	$\lambda_2(o_4)$	a_4	b_4	c_4	b'_4	c'_4	c''_4	o_4-u_4	u_4	$\lambda_2(o_4-u_4)$	$\lambda_2(u_4)$	$1-\lambda_2(u_4):\lambda_2(o_4)$
		$[ao]$	$[bo]$	$[co]$	$[b'o]$	$[c'o]$	$[c''o]$					$=3$ as proof.
		$[aa\lambda]$	$[ab\lambda]$	$[ac\lambda]$						Criticism		
		$[ba\lambda]$	$[bb\lambda]$	$[bc\lambda]$	$[b'b'\lambda]$	$[b'c'\lambda]$				$(o_1-u_1)^2:\lambda_2(o_1)$		
		$[ca\lambda]$	$[cb\lambda]$	$[cc\lambda]$	$[c'b'\lambda]$	$[c'c'\lambda]$	$[c''c''\lambda]$			$(o_2-u_2)^2:\lambda_2(o_2)$		
		$\beta=\frac{[ba\lambda]}{[aa\lambda]}$, $\gamma=\frac{[ca\lambda]}{[aa\lambda]}$			$\gamma'=\frac{[c'b'\lambda]}{[b'b'\lambda]}$					$(o_3-u_3)^2:\lambda_2(o_3)$		
										$(o_4-u_4)^2:\lambda_2(o_4)$		
Correlates		$K_a=\frac{[ao]-A}{[aa\lambda]}$			$K_{b'}=\frac{[b'o]-B'}{[b'b'\lambda]}$		$K_{c''}=\frac{[c''o]-C''}{[c''c''\lambda]}$			Sum for proof and summary criticism		

The free functions are computed by means of:

$B'=B-\beta A$	$C'=C-\gamma A$	$C''=C'-\gamma' B'$
$b'_i=b_i-\beta a_i$	$c'_i=c_i-\gamma a_i$	$c''_i=c'_i-\gamma' b'_i$
$[b'o]=[bo]-\beta[ao]$	$[c'o]=[co]-\gamma[ao]$	$[c''o]=[c'o]-\gamma'[b'o]$
....	....	....
$[c'b'\lambda]=[cb\lambda]-\beta[ca\lambda]$	$[c'c'\lambda]=[cc\lambda]-\gamma[ca\lambda]$	$[c''c''\lambda]=[c'c'\lambda]-\gamma'[c'b'\lambda]$

By the adjustment properly so called we compute

$$o_i-u_i=(a_iK_a+b'_iK_{b'}+c''_iK_{c''})\,\lambda_2(o_i)$$

$$\lambda_2(o_i-u_i)=\left(\frac{a_i^2}{[aa\lambda]}+\frac{b_i'^2}{[b'b'\lambda]}+\frac{c_i''^2}{[c''c''\lambda]}\right)(\lambda_2(o_i))^2$$

$$\lambda_2(u_i)=\lambda_2(o_i)-\lambda_2(o_i-u_i),$$

and for the summary criticism

$$K_a^2[aa\lambda_2]+K_{b'}^2[b'b'\lambda_2]+K_{c''}^2[c''c''\lambda_2]=\left[\frac{(o-u)^2}{\lambda_2(o)}\right]=3\pm\sqrt{6}.$$

In order to get a check we ought further to compute $[au]=A$, $[bu]=B$, and $[cu]=C$, with the values we have found for u_1, u_2, u_3, and u_4. Moreover it is useful to add a superfluous theoretical equation, for instance $[(a+b+c)u]=A+B+C$, through the computation of the free functions, which is correct only if such a superfluity leads to identical results.

§ 50. It is a deficiency in the adjustment by correlates that it cannot well be employed as an intermediate link in a computation that goes beyond it. The method is good as far as the determination of the adjusted values of the several observations and the criticism on the same, but no farther. We are often in want of the adjusted values with determinations of the mean errors of certain functions of the observations; in order to solve such problems the adjustment by correlates must be made in a modified form. The simplest course is, I think, immediately after drawing up the theoretical equations of condition to annex the whole series of the functions that are to be examined, for instance $[do], \dots [eo]$, and include them in the computation of the free functions. In doing so we must take care not to mix up the theoretically and the empirically determined functions, so that the order of the operation must unconditionally give the precedence to the theoretical functions; the others are not made free till the treatment of these is quite finished. The functions $[d'''o], \dots [e^{v}o]$, which are separated from these — it is scarcely necessary to mention it — remain unchanged by the adjustment both in value and in mean error. And at last the adjusted functions $[du], \dots [eu]$, by retrograde transformation, are determined as linear functions of A, B', C'', $[d'''o], \dots [e^{v}o]$.

Example 1. In a plane triangle each angle has been measured several times, all measurements being made according to the same method, bondfree and with the same (unknown) mean error:

for angle A has been found 70° 0′ 5″ as the mean number of 6 measurements
» » B » » » 50° 0′ 3″ » » » » » 10 »
» » C » » » 60° 0′ 2″ » » » » » 15 »

The adjusted values for the angles are then 70°, 50°, and 60°, the mean error for single measurement $= \sqrt{300} = 17''3$, the scales 0·5, 0·3, and 0·2.

Example 2. (Comp. example § 42.) Five equidistant tabular values, 12, 19, 29, 41, 55, have been obtained by taking approximate round values from an exact table, from which reason their mean errors are all $= \sqrt{\frac{1}{12}}$. The adjustment is performed under the successive hypotheses that the table belongs to a function of the 3rd, 2nd, and 1st degree, and the hypothesis of the second degree is varied by the special hypothesis that the 2nd difference is exactly $= 2$, in the following schedule marked (or). The same schedule may be used for all four modifications of the problem, so that in the sums to the right in the schedule, the first term corresponds to the first modification only, and the sum of the two first terms to the second modification:

o	$\lambda_2(o)$	Δ^4	$V\Delta^3$	Δ^2	$(V\Delta^3)'$	$(\Delta^3)'-(\Delta^2)''$	$o-u$	$\lambda_2(o-u)$
		0	0	0 (or 2)	0	0 (or 2)		
12	$\frac{1}{12}$	1	0	0	$-\frac{1}{2}$	$\frac{2}{7}$	$\frac{1}{70}(\ \ 1+\ 7+160$ (or $+20$)),	$\frac{1}{840}(\ 1+\ 7+20)$
19	$\frac{1}{12}$	-4	-1	1	1	$-\frac{1}{7}$	$\frac{1}{70}(-4-14-\ 80$ (or -10)),	$\frac{1}{840}(16+28+\ 5)$
29	$\frac{1}{12}$	6	3	-2	0	$-\frac{2}{7}$	$\frac{1}{70}(\ \ 6+\ 0-160$ (or -20)),	$\frac{1}{840}(36+\ 0+20)$
41	$\frac{1}{12}$	-4	-3	1	-1	$-\frac{1}{7}$	$\frac{1}{70}(-4+14-\ 80$ (or -10)),	$\frac{1}{840}(16+28+\ 5)$
55	$\frac{1}{12}$	1	1	0	$\frac{1}{2}$	$\frac{2}{7}$	$\frac{1}{70}(\ \ 1-\ 7+160$ (or $+20$)),	$\frac{1}{840}(\ 1+\ 7+20)$
		1	0	2	$-\frac{1}{2}$	$\frac{16}{7}$		
		$\frac{70}{12}$	$\frac{35}{12}$	$-\frac{20}{12}$				
		$\frac{35}{12}$	$\frac{20}{12}$	$-\frac{10}{12}$	$\frac{5}{24}$	0		
		$-\frac{20}{12}$	$-\frac{10}{12}$	$\frac{6}{12}$	0	$\frac{1}{42}$		

$\beta = \frac{1}{2}$, $\gamma = -\frac{2}{7}$ $\qquad$ $\gamma' = 0$

$\frac{1}{12}K_a = \frac{1}{70}$, $\frac{1}{12}K_{b'} = -\frac{1}{5}$, $\frac{1}{12}K_{c''} = 8$ (or 1)

For the summary criticism:

$$\left[\frac{(o-u)^2}{\lambda_2(o)}\right] = \frac{6}{35}+\frac{42}{35}+\frac{7680 \text{ (or } 120)}{35}$$

The hypothesis of the third degree, $\Delta^4 = 0$, where the values of $70u_i$ and their differences are:

839 1334 2024 2874 3849
495 690 850 975
195 160 125
-35 -35,

agrees too well with the observations, and must be suspected of being underadjusted, for the sum of the squares of the summary criticism is only

$\frac{6}{35}$, where we might expect $1 \pm \sqrt{2}$.

The hypothesis of the second degree, $\Delta^4 = 0$, $V\Delta^3 = 0$, gives for $70u_i$ and differences:

832 1348 2024 2860 3856
516 676 836 996
160 160 160.

The adjustment is here good, the sum of the squares is

$\frac{48}{35}$, and we might expect $2 \pm \sqrt{4}$.

The hypothesis of the first degree, $\Delta^4 = 0$, $V\Delta^3 = 0$, $\Delta^2 = 0$, gives for the adjusted values and their differences:

9·6 20·4 31·2 42·0 52·8
10·8 10·8 10·8 10·8.

The deviations are evidently too large ($o - u$ is $+2{\cdot}4$, $-1{\cdot}4$, $-2{\cdot}2$, $-1{\cdot}0$, $+2{\cdot}2$) to be due to the use of round numbers; the sum of the squares is also

$$220{\cdot}8 \text{ instead of } 3 \pm \sqrt{6},$$

consequently, no doubt, an over-adjustment.

The special adjustment of the second degree, $\Delta^4 = 0$, $\nabla\Delta^3 = 0$, and $\Delta^2 = 2$, gives for u_i and its differences:

$$\begin{array}{ccccccccc} 11{\cdot}6 & & 19{\cdot}4 & & 29{\cdot}2 & & 41{\cdot}0 & & 54{\cdot}8 \\ & 7{\cdot}8 & & 9{\cdot}8 & & 11{\cdot}8 & & 13{\cdot}8 & \end{array}$$

The deviations $o - u =$ $\cdot\, 0{\cdot}4$, $-0{\cdot}4$, $-0{\cdot}2$, $0{\cdot}0$, $+0{\cdot}2$

nowhere reach $\frac{1}{2}$, and may consequently be due to the use of round numbers; the sum of the squares

$$4{\cdot}8 \text{ instead of } 3 \pm \sqrt{6}$$

also agrees very well. Indeed, a constant subtraction of $0{\cdot}04$ from u_i would lead to $(3{\cdot}4)^2$, $(4{\cdot}4)^2$, $(5{\cdot}4)^2$, $(6{\cdot}4)^2$, and $(7{\cdot}4)^2$, from which the example is taken.

Example 3. Between 4 points on a straight line the 6 distances

$$\begin{array}{ccc} o_{12}, & o_{13}, & o_{14} \\ & o_{23}, & o_{24} \\ & & o_{34}\,. \end{array}$$

are measured with equal exactness without bonds. By adjustment we find for instance

$$u_{12} = \tfrac{1}{2}o_{12} + \tfrac{1}{4}(o_{13} - o_{23}) + \tfrac{1}{4}(o_{14} - o_{24});$$

we notice that every scale $= \frac{1}{2}$. It is recommended actually to work the example by a millimeter scale, which is displaced after the measurement of each distance in order to avoid bonds.

XII. ADJUSTMENT BY ELEMENTS.

§ 51. Though every problem in adjustment may be solved in both ways, by correlates as well as by elements, the difficulty in so doing is often very different. The most frequent cases, where the number of equations of condition is large, are best suited for adjustment by elements, and this is therefore employed far oftener than adjustment by correlates.

The adjustment by elements requires the theory in such a form that *each observation is represented by one equation* which expresses the mean value $\lambda_1(o)$ explicitely as linear functions of unknown values, the "*elements*", x, y, … z: