NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

A NOTE CONCERNING HOTELLING'S METHOD OF INVERTING A PARTITIONED MATRIX

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Professor Hotelling recently presented several methods of computing the inverse of a matrix.[1] Among these was a method of partitioning a square matrix of $2p$ rows into four square matrices, a, b, c and d, of p rows each, resulting in the partitioned matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The inverse of this matrix can also be written as a partitioned matrix,

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix}.$$

Then, multiplying the original matrix by its inverse we get four matrix equations,

$$\begin{aligned} aA + bB &= 1 \qquad & aC + bD &= 0 \\ cA + dB &= 0 \qquad & cC + dD &= 1. \end{aligned}$$

These equations can be solved for A, B, C, and D.

Professor Hotelling's solution requires the inversion of four p-rowed matrices. It is possible, however, to solve these equations by formulas involving only two inversions. The formulas are

$$\begin{aligned} D &= (d - ca^{-1}b)^{-1} \qquad & B &= -Dca^{-1} \\ C &= -a^{-1}bD \qquad & A &= a^{-1} - a^{-1}bB. \end{aligned}$$

As an example of the procedure let the given matrix be

$$\left[\begin{array}{rr:rr} 26 & -10 & 15 & 32 \\ 19 & 45 & -14 & -8 \\ \hdashline -12 & 16 & 27 & 13 \\ 32 & 29 & -35 & 28 \end{array}\right].$$

[1] Harold Hotelling. "Some new methods of matrix calculation," *Annals of Math. Stat.*, Vol. 14 (1943), pp. 1–34.

The necessary steps in computation are

$$a^{-1} = \begin{bmatrix} .03309 & .00735 \\ -.01397 & .01912 \end{bmatrix} \qquad a^{-1}b = \begin{bmatrix} .39345 & 1.00008 \\ -.47723 & -.60000 \end{bmatrix}$$

$$ca^{-1} = \begin{bmatrix} -.62060 & .21772 \\ .65375 & .78968 \end{bmatrix} \qquad ca^{-1}b = \begin{bmatrix} -12.35708 & -21.60096 \\ -1.24927 & 14.60256 \end{bmatrix}.$$

Note that a convenient check at this point is to compute both $(ca^{-1})b$ and $c(a^{-1}b)$

$$d - ca^{-1}b = \begin{bmatrix} 39.35708 & 34.60096 \\ -33.75073 & 13.39744 \end{bmatrix}$$

$$(d - ca^{-1}b)^{-1} = D = \begin{bmatrix} .00790 & -.02041 \\ .01991 & .02322 \end{bmatrix}$$

$$-a^{-1}bD = C = \begin{bmatrix} -.02302 & -.01519 \\ .01572 & .00419 \end{bmatrix}$$

$$-Dca^{-1} = B = \begin{bmatrix} .01825 & .01440 \\ -.00282 & -.02267 \end{bmatrix}$$

$$a^{-1} - a^{-1}bB = A = \begin{bmatrix} .02873 & .02436 \\ -.00696 & .01239 \end{bmatrix}.$$

The last four of these matrices are the four parts of the inverse, which can be written

$$\begin{bmatrix} .02873 & .02436 & -.02302 & -.01519 \\ -.00696 & .01239 & .01572 & .00419 \\ .01825 & .01440 & .00790 & -.02041 \\ -.00282 & -.02267 & .01991 & .02322 \end{bmatrix}.$$

The accuracy of the computations can be checked by multiplying the original matrix by the computed inverse matrix. The product should, of course, be a close approximation of the identity matrix. If further accuracy is called for we can use Hotelling's iterative formula,

$$C_1 = C_0(2 - AC_0)$$

where C_0 is the estimated inverse; A is the original matrix; and C_1 is a second approximation of the inverse.