

DISTRIBUTION TABLE FOR THE DEVIATION BETWEEN TWO SAMPLE CUMULATIVES¹

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Suppose $x_1 < x_2 < \cdots < x_{N_1}$, $y_1 < y_2 < \cdots < y_{N_2}$ are the ordered results of two random samples, of sizes N_1 and N_2 , taken from two populations with cumulative distribution functions $F_1(x)$ and $F_2(y)$.

Let $d = \max |S_{N_1}(x) - S_{N_2}(x)|$, where $S_{N_1}(x)$ is the proportion of the N_1 observed values x_i less than or equal to x , and $S_{N_2}(x)$ is the proportion of the N_2 observed values y_i less than or equal to x . If $F_1(x) \equiv F_2(x) \equiv F(x)$ and if $F(x)$ is continuous then the distribution of d is independent of $F(x)$. Table 1 lists, for $N_1, N_2 \leq 10$, all possible values of d_α and α such that

$$P\{d \leq d_\alpha\} = \alpha.$$

Table 2 lists certain other selected values of N_1, N_2 , and d_α .

The method used in finding the values is that described in [1].

REFERENCE

- [1] FRANK J. MASSEY, JR., "The distribution of the maximum deviation between two sample cumulative step functions," *Annals of Math. Stat.*, Vol. 22 (1951), pp. 125-128.

¹ The method of computing the values tabled was developed under contract N6-onr-218/IV with the Office of Naval Research (see [1]). The main part of the table was made possible by a grant from the Graduate School of the University of Oregon.

TABLE 1

Values of α and d_α such that for $N_1, N_2 \leq 10$, $P\{d \leq d_\alpha\} = \alpha$. Headings are in the form (N_1, N_2, d_α) with the d_α expressed as a fraction with the least common multiple of N_1 and N_2 in the denominator. All possible values of d_α are listed in the table.

(1, 1, $h/1$)	(1, 2, $h/2$)	(1, 3, $h/3$) ^a	(1, 4, $h/4$)	(1, 5, $h/5$)	(1, 6, $h/6$)	(1, 7, $h/7$)
h α	h α	h α	h α	h α	h α	h α
1 1	1 .33 2 1	2 .5 3 1	2 .2 3 .6 4 1	3 .33333 4 .66667 5 1	3 .14286 4 .42857 5 .71429 6 1	4 .25 5 .50 6 .75 7 1
(1, 8, $h/8$)	(1, 9, $h/9$)	(1, 10, $h/10$)		(2, 2, $h/2$)	(2, 3, $h/6$)	(2, 4, $h/4$)
h α	h α	h α		h α	h α	h α
4 .11111 5 .33333 6 .55556 7 .77778 8 1	5 .2 6 .4 7 .6 8 .8 9 1	5 .09091 6 .27273 7 .45455 8 .63636 9 .81818 10 1		1 .66667 2 1	2 .1 3 .4 4 .8 6 1	1 .06667 2 .60000 3 .86667 4 1
(2, 5, $h/10$)	(2, 6, $h/6$)	(2, 7, $h/14$)	(2, 8, $h/8$)	(2, 9, $h/18$)	(2, 10, $h/10$)	
h α	h α	h α	h α	h α	h α	
3 .04762 4 .19048 5 .42857 6 .71429 8 .90476 10 1	2 .14286 3 .57143 4 .78571 5 .92857 6 1	4 .02778 5 .11111 6 .25000 7 .44444 8 .66667 10 .83333 12 .94444 14 1	2 .02222 3 .20000 4 .55556 5 .73333 6 .86667 7 .95556 8 1	5 .01818 6 .07273 7 .16364 8 .29091 9 .45455 10 .63636 12 .78182 14 .89091 16 .96364 18 1	3 .06061 4 .24242 5 .54545 6 .69697 7 .81818 8 .90909 9 .96970 10 1	
(3, 3, $h/3$)	(3, 4, $h/12$)	(3, 5, $h/15$)	(3, 6, $h/6$)	(3, 7, $h/21$)	(3, 8, $h/24$)	(3, 9, $h/9$)
h α	h α	h α	h α	h α	h α	h α
1 .4 2 .9 3 1	3 .02857 4 .11429 5 .34286 6 .60000 8 .77143 9 .94286 12 1	4 .03571 5 .14286 6 .32143 7 .53571 9 .71429 10 .85714 12 .96429 15 1	1 .01191 2 .32143 3 .66667 4 .90476 5 .97619 6 1	5 .00833 6 .06667 7 .15000 8 .30000 9 .46667 11 .60000 12 .75000 14 .83333 15 .93333 18 .98333 21 1	5 .00606 6 .02424 7 .07273 8 .16364 9 .29091 10 .43636 12 .57576 13 .69697 15 .80606 16 .87879 18 .95152 21 .98788 24 1	2 .03636 3 .29091 4 .54545 5 .76364 6 .90909 7 .96364 8 .99091 9 1

TABLE 1—Continued

(3, 10, $h/30$)		(3, 10, $h/30$) (cont.)		(4, 4, $h/4$)		(4, 5, $h/20$)		(4, 6, $h/12$)		(4, 7, $h/28$)		(4, 7, $h/28$) (cont.)	
h	α	h	α	h	α	h	α	h	α	h	α	h	α
6	.00350	18	.81119	1	.22857	5	.03175	2	.00476	5	.00303	17	.87879
7	.01399	20	.83566	2	.77143	6	.12698	3	.07619	6	.01212	20	.93333
8	.04196	21	.93007	3	.97143	7	.25397	4	.30476	7	.04848	21	.96970
9	.09441	24	.97203	4	1	8	.43651	5	.44762	8	.10909	24	.99394
10	.16783	27	.99301			10	.57143	6	.70476	9	.21818	28	1
11	.27972	30	1			11	.71429	7	.81905	10	.33939		
12	.40210					12	.85714	8	.90476	12	.46364		
14	.50699					15	.92063	9	.95238	13	.58485		
15	.62937					16	.98413	10	.99048	14	.69091		
17	.71329					20	1	12	1	16	.78788		

(4, 8, $h/8$)		(4, 9, $h/36$)		(4, 10, $h/20$)				(5, 5, $h/5$)		(5, 6, $h/30$)		(5, 7, $h/35$)	
h	α	h	α	h	α			h	α	h	α	h	α
1	.00202	6	.00140	3	.00100			1	.12698	5	.00216	6	.00253
2	.16364	7	.00559	4	.01518			2	.64286	6	.00866	7	.01010
3	.48687	8	.02238	5	.08092			3	.92064	7	.03463	8	.04040
4	.77778	9	.05035	6	.22478			4	.99206	8	.10390	9	.09091
5	.91515	10	.11329	7	.31269			5	1	9	.18182	10	.16162
6	.97980	11	.18881	8	.50050					10	.31169	11	.26263
7	.99596	12	.29231	9	.59041					12	.40909	13	.36111
8	1	14	.38182	10	.74026					13	.52597	14	.45455
		15	.48252	11	.81219					14	.64286	15	.56566
		16	.59580	12	.87413					15	.76190	16	.67172
		18	.66434	13	.91608					18	.82251	18	.76263
		19	.74825	14	.95405					19	.89177	20	.83333
		20	.83497	15	.97003					20	.95238	21	.88384
		23	.88531	16	.99001					24	.97403	23	.93434
		24	.93846	18	.99800					25	.99567	25	.96970
		27	.95804	20	1					30	1	29	.98485
		28	.98601									30	.99747
		32	.99720									35	1
		36	1										

(5, 8, $h/40$)		(5, 8, $h/40$) (cont.)		(5, 9, $h/45$)		(5, 9, $h/45$) (cont.)		(5, 10, $h/10$)				(6, 6, $h/6$)	
h	α	h	α	h	α	h	α	h	α			h	α
6	.00078	20	.76535	7	.00100	21	.69930	1	.00033			1	.06926
7	.00311	22	.83528	8	.00400	22	.77423	2	.08092			2	.52597
8	.01243	24	.87413	9	.01598	25	.82717	3	.34532			3	.85714
9	.03730	25	.92075	10	.03596	26	.88112	4	.64935			4	.97403
10	.08392	27	.95804	11	.08092	27	.91409	5	.83417			5	.99784
11	.14918	29	.97980	12	.13487	30	.94406	6	.93939			6	1
12	.23077	32	.99068	13	.20879	31	.97203	7	.98069				
14	.31624	35	.99845	15	.28571	35	.98601	8	.99600				
15	.41414	40	1	16	.37363	36	.99401	9	.99933				
16	.50039			17	.45854	40	.99900	10	1				
17	.59984			19	.53946	45	1						
19	.68376			20	.61938								

TABLE 1—*Continued*

(6, 7, $h/42$)		(6, 7, $h/42$) (<i>cont.</i>)		(6, 8, $h/24$)		(6, 8, $h/24$) (<i>cont.</i>)		(6, 9, $h/18$)		(6, 10, $h/30$)		(6, 10, $h/30$) (<i>cont.</i>)	
h	α	h	α	h	α	h	α	h	α	h	α	h	α
6	.00058	28	.93240	3	.00033	20	.99467	2	.00020	4	.00050	20	.96853
7	.00233	29	.96154	4	.00533	21	.99933	3	.01279	5	.00799	21	.98102
8	.00932	30	.98485	5	.04795	24	1	4	.10230	6	.04046	22	.99101
9	.03730	35	.99126	6	.14685			5	.22058	7	.11239	24	.99600
10	.07459	36	.99883	7	.21612			6	.43716	8	.16459	25	.99825
11	.12821	42	1	8	.36264			7	.59161	9	.28746	27	.99975
12	.21970			9	.53979			8	.72647	10	.42258	30	1
14	.28846			10	.62371			9	.82418	11	.49451		
15	.37762			11	.69830			10	.90529	12	.62887		
16	.47203			12	.80753			11	.93886	13	.69505		
17	.56643			13	.86081			12	.97203	14	.75125		
18	.66725			14	.90743			13	.98601	15	.82992		
21	.72261			15	.93939			14	.99401	16	.87488		
22	.78788			16	.95738			15	.99720	17	.90784		
23	.85315			17	.97736			16	.99960	18	.93357		
24	.90909			18	.99068			18	1	19	.95804		

(7, 7, $h/7$)		(7, 8, $h/56$)		(7, 8, $h/56$) (<i>cont.</i>)		(7, 9, $h/63$)		(7, 9, $h/63$) (<i>cont.</i>)		(7, 10, $h/70$)		(7, 10, $h/70$) (<i>cont.</i>)	
h	α	h	α	h	α	h	α	h	α	h	α	h	α
1	.03730	7	.00016	28	.85859	8	.00017	29	.78234	8	.00005	29	.68511
2	.42483	8	.00062	32	.88190	9	.00070	31	.83129	9	.00021	30	.71858
3	.78788	9	.00249	33	.91298	10	.00280	33	.87325	10	.00082	32	.76743
4	.94697	10	.00995	34	.93629	11	.01119	35	.90210	11	.00329	33	.81062
5	.99184	11	.02984	35	.96737	12	.02517	36	.92133	12	.00987	35	.84461
6	.99942	12	.05221	40	.97576	13	.04476	38	.94493	13	.02221	36	.88348
7	1	13	.08951	41	.98695	14	.07710	40	.96591	14	.03949	39	.91310
		14	.15338	42	.99534	15	.12483	42	.97902	15	.06911	40	.92842
		16	.18943	48	.99751	17	.17238	45	.98514	16	.10587	42	.94591
		17	.26434	49	.99969	18	.22500	47	.99248	18	.14937	43	.96390
		18	.33986	56	1	19	.28636	49	.99720	19	.19616	46	.97830
		19	.41538			20	.35245	55	.99860	20	.24666	49	.98416
		20	.49231			21	.42727	56	.99983	21	.30769	50	.98900
		21	.57824			22	.50122	63	1	22	.37392	53	.99393
		24	.62720			23	.56976			23	.43979	56	.99825
		25	.65703			26	.63199			25	.50273	61	.99918
		26	.71795			27	.67622			26	.56659	63	.99990
		27	.77762			28	.72990			28	.62176	70	1

TABLE 1—*Concluded*

(8, 8, $h/8$)		(8, 9, $h/72$)		(8, 9, $h/72$) (<i>cont.</i>)		(8, 9, $h/72$) (<i>cont.</i>)		(8, 10, $h/40$)		(8, 10, $h/40$) (<i>cont.</i>)		(8, 10, $h/40$) (<i>cont.</i>)	
h	α	h	α	h	α	h	α	h	α	h	α	h	α
1	.01989	8	.00004	22	.35985	40	.94406	5	.00037	18	.79633	32	.99918
2	.33986	9	.00016	23	.42390	45	.95311	6	.00585	19	.83404	35	.99959
3	.71733	10	.00066	24	.49720	46	.96643	7	.02340	20	.87975	36	.99995
4	.91298	11	.00263	27	.53970	47	.97976	8	.06913	21	.90123	40 1	
5	.98135	12	.01053	28	.59301	48	.98881	9	.10176	22	.92317		
6	.99751	13	.02106	29	.64829	54	.99169	10	.17729	23	.95014		
7	.99985	14	.03620	30	.70358	55	.99572	11	.27725	24	.96270		
8 1		15	.06203	31	.75755	56	.99860	12	.39760	25	.96974		
		16	.10629	32	.80424	63	.99926	13	.45775	26	.97934		
		18	.13957	36	.82682	64	.99992	14	.51469	27	.98798		
		19	.18326	37	.85767	72 1		15	.60825	28	.99301		
		20	.23990	38	.89058			16	.67045	30	.99502		
		21	.29988	39	.92143			17	.75246	31	.99758		
(9, 9, $h/9$)		(9, 10, $h/90$)		(9, 10, $h/90$) (<i>cont.</i>)		(9, 10, $h/90$) (<i>cont.</i>)		(9, 10, $h/90$) (<i>cont.</i>)		(9, 10, $h/90$) (<i>cont.</i>)		(10, 10, $h/10$)	
h	α	h	α	h	α	h	α	h	α	h	α	h	α
1	.01054	9	.00001	20	.09616	32	.55754	45	.91608	70	.99725	1	.00554
2	.26989	10	.00004	21	.12626	33	.60742	50	.92537	71	.99864	2	.21307
3	.64829	11	.00017	22	.16572	34	.65713	51	.93966	72	.99959	3	.58248
4	.87413	12	.00069	23	.21307	35	.70580	52	.95544	80	.99978	4	.83218
5	.96643	13	.00277	24	.26042	36	.74847	53	.96564	81	.99998	5	.94755
6	.99371	14	.00831	25	.30864	40	.76993	54	.97902	90 1		6	.98766
7	.99926	15	.01455	26	.35336	41	.79970	60	.98236			7	.99794
8	.99996	16	.02494	28	.42493	42	.83218	61	.98766			8	.99978
9 1		17	.04274	30	.46141	43	.86465	62	.99281			9	.99999
		18	.07323	31	.50783	44	.89442	63	.99630			10 1	

TABLE 2

Values of α and d_α such that $P\{d \leq d_\alpha\} = \alpha$ for certain values of N_1, N_2 and d_α . Headings are expressed as in Table 1 in the form (N_1, N_2, d_α) .

$(3, 12, h/12)$	$(4, 12, h/12)$	$(4, 16, h/16)$	$(5, 15, h/15)$	$(5, 20, h/20)$	$(6, 12, h/12)$	$(6, 18, h/18)$
h α	h α	h α	h α	h α	h α	h α
2 .00220	5 .59560	2 .00021	6 .65738	2 .00002	1 .00005	7 .69767
3 .05934	6 .78462	3 .01672	7 .79463	3 .00457	2 .03927	8 .81794
4 .27473	7 .88791	4 .12900	8 .89048	4 .05882	3 .24036	9 .89899
5 .47473	8 .95165	5 .29164	9 .94814	5 .17397	4 .53200	10 .94655
6 .66154	9 .98352	6 .47864	10 .97665	6 .33618	5 .74618	11 .97459
7 .81099	10 .99451	7 .65243	11 .99123	7 .50941	6 .88537	12 .98908
8 .91209	11 .99890	8 .78947	12 .99729	8 .66319	7 .95367	13 .99563
9 .95604		9 .87203	13 .99923	9 .77216	8 .98513	
10 .98242		10 .92983		10 .85590	9 .99591	
11 .99560		11 .96615		11 .91519	10 .99925	
		12 .98555		12 .95306	11 .99989	
		13 .99381		13 .97470		
		14 .99794		14 .98769		
		15 .99959		15 .99466		
				16 .99789		
				17 .99921		
				18 .99977		

$(6, 24, h/24)$	$(7, 14, h/14)$	$(7, 28, h/28)$	$(8, 12, h/24)$	$(8, 16, h/16)$	$(8, 32, h/32)$	$(9, 12, h/36)$
h α	h α	h α	h α	h α	h α	h α
3 .00123	2 .01881	3 .00033	7 .40465	2 .00892	3 .00009	12 .57284
4 .02632	3 .16511	4 .01162	8 .55223	3 .11232	4 .00508	13 .65542
5 .10183	4 .43014	5 .05882	9 .67596	4 .34442	5 .03364	14 .70465
6 .23167	5 .65870	6 .15755	10 .78579	5 .57501	6 .10610	15 .79433
7 .39022	6 .82383	7 .29497	11 .85044	6 .76578	7 .22079	16 .84105
8 .54651	7 .91730	8 .44440	12 .90933	7 .87443	8 .35783	17 .88617
9 .67067	8 .96687	9 .57474	13 .94434	8 .94207	9 .48770	18 .92151
10 .77301	9 .98822	10 .68883	14 .96801	9 .97576	10 .60778	19 .93937
11 .85137	10 .99661	11 .78160	15 .98179	10 .99124	11 .71051	20 .95856
12 .90678	11 .99917	12 .85191	16 .99138	11 .99720	12 .79265	21 .97396
13 .94311	12 .99986	13 .90208	17 .99524	12 .99927	13 .85487	22 .98204
14 .96738	13 .99998	14 .93811	18 .99802	13 .99984	14 .90198	23 .98803
15 .98242		15 .96254	19 .99913	14 .99998	15 .93600	24 .99281
16 .99101		16 .97818	20 .99968	15 1.00000	16 .95948	25 .99538
17 .99562		17 .98775	21 .99986	16 1.00000	17 .97512	26 .99752
18 .99806		18 .99351			18 .98534	27 .99857
19 .99923		19 .99675			19 .99170	28 .99901
20 .99972		20 .99846			20 .99547	29 .99956
						30 .99986

TABLE 2—*Concluded*

(9, 15, $h/45$)	(9, 18, $h/18$)	(9, 36, $h/36$)	(10, 15, $h/30$)	(10, 20, $h/20$)	(10, 40, $h/40$)	(12, 15, $h/60$)
h α	h α	h α	h α	h α	h α	h α
14 .51355	2 .00420	3 .00002	8 .40712	2 .00020	3 .00001	21 .72675
15 .60637	3 .07584	4 .00220	9 .53773	3 .05090	4 .00095	22 .76664
16 .67583	4 .27372	5 .01910	10 .66165	4 .21624	5 .01078	23 .81761
17 .73473	5 .49960	6 .07091	11 .74789	5 .43090	6 .04711	24 .84921
18 .78605	6 .69439	7 .16403	12 .82837	6 .63135	7 .12214	25 .87374
19 .83420	7 .82756	8 .28596	13 .88189	7 .77861	8 .22717	26 .89897
20 .86713	8 .91206	9 .41073	14 .92260	8 .87819	9 .34383	27 .92166
21 .89937	9 .95874	10 .53221	15 .95017	9 .93771	10 .46325	28 .93902
22 .92671	10 .98270	11 .64099	16 .97042	10 .97095	11 .57482	29 .94928
23 .94103	11 .99344	12 .73196	17 .98187	11 .98755	12 .67187	30 .96020
24 .95820	12 .99783	13 .80416	18 .98997	12 .99520	13 .75196	31 .97033
25 .97027	13 .99937	14 .86114	19 .99449	13 .99833	14 .81737	32 .97813
26 .97769	14 .99985	15 .90427	20 .99717	14 .99949	15 .86877	33 .98297
27 .98515	15 .99997	16 .93569	21 .99853	15 .99986	16 .90780	34 .98692
28 .98962		17 .95790		16 .99997	17 .93668	35 .99045
29 .99272		18 .97333			18 .95767	36 .99317
30 .99448		19 .98362			19 .97242	37 .99494
31 .99637		20 .99024			20 .98249	38 .99650
32 .99731						39 .99754
33 .99799						40 .99824
34 .99864						41 .99870

(12, 16, $h/48$)	(12, 18, $h/36$)	(12, 20, $h/60$)	(15, 20, $h/60$)	(16, 20, $h/80$)
h α	h α	h α	h α	h α
14 .53765	9 .41949	16 .49357	16 .56529	25 .74728
15 .63564	10 .54635	17 .56236	17 .63688	26 .78477
16 .70027	11 .64503	18 .62623	18 .70099	27 .81939
17 .76291	12 .73908	19 .68783	19 .74983	28 .84849
18 .81604	13 .80850	20 .73647	20 .79723	29 .87017
19 .85183	14 .86428	21 .78375	21 .83880	30 .89148
20 .88682	15 .90549	22 .82582	22 .87092	31 .91112
21 .91630	16 .93737	23 .85490	23 .89808	32 .92759
22 .93670	17 .95829	24 .88521	24 .92103	33 .94026
23 .95309	18 .97374	25 .90937	25 .93891	34 .95111
24 .96637	19 .98371	26 .92760	26 .95393	35 .96069
25 .97573	20 .99029	27 .94447	27 .96528	36 .96864
26 .98335	21 .99429	28 .95708	28 .97389	37 .97489
27 .98848		29 .96699	29 .98044	38 .98032
28 .99190		30 .97431	30 .98586	39 .98458
29 .99423		31 .98093	31 .99007	40 .98790
30 .99624		32 .98554		41 .99051
31 .99775		33 .98915		
		34 .99216		