$$- 567000(n^4 - 6n^3 + 11n^2 - 6n)s_2^4 s_1^2$$
$$+ 2268000(n^3 - 3n^2 + 2n)s_2^3 s_1^4$$
$$- 3175200(n^2 - n)s_2^2 s_1^6$$
$$+ 1814400n\, s_2 s_1^8$$
$$- 362880\, s_1^{10}$$

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THE PROBABILITY INTEGRAL OF RANGE FOR SAMPLES FROM A SYMMETRICAL UNIMODAL POPULATION

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1. Summary. An asymptotic expression is given for the probability integral of range for samples from a symmetrical unimodal population. Its accuracy is investigated for the case of a normal parent population and for sample sizes from 20 to 100. Over this range errors are small, and by using a correction based on values given below the probability integral can be found with a maximum error of 0.0001. Percentage points of range in the normal case are tabled for $n = 20, 40, 60, 80$ and 100.

2. The asymptotic expansion. The parent probability density function $\phi(x)$ is symmetrical about $x = 0$ and its integral from 0 to x is denoted by $\Phi(x)$. The p.d.f. of w, the range for a sample of size n, is

$$p(w) = n(n-1)\int_{-\infty}^{\infty}\{\Phi(x) - \Phi(x-w)\}^{n-2}\phi(x)\phi(x-w)\,dx. \tag{1}$$

Integrating with respect to w from $-\infty$ to w gives

$$F(w) = n\int_{-\infty}^{\infty}\{\Phi(x) - \Phi(x-w)\}^{n-1}\phi(x)\,dx. \tag{2}$$

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[1] This work arose during analyses made by the Board's Statistics Group.

Hartley [2] proves that this can be transformed to

$$(3)\quad F(2u) = \{2\Phi(u)\}^n + 2n \int_0^\infty \{\Phi(x+u) - \Phi(x-u)\}^{n-1}\phi(x+u)\,dx.$$

Since $\phi(x)$ is unimodal the integrand in (3) is greatest when $x = 0$ and decreases rapidly to zero on either side of this point. This suggests the application of the methods used [1] to furnish asymptotic series for similar integrals. We have

$$\Phi(x+u) - \Phi(x-u) = 2\Phi(u)\{1 + A(u)x^4 + \cdots\} \exp\left\{\frac{x^2\phi'(u)}{2\Phi(u)}\right\},$$

$$\phi(x+u) = \phi(u)\{1 + B(u)x^3 + \cdots\} \exp\left\{\frac{x\phi'(u)}{\phi(u)} - \frac{x^2}{2}\left(\frac{\phi'(u)}{\phi(u)}\right)^2 + \frac{x^2\phi''(u)}{2\phi(u)}\right\}.$$

Ignoring all but the first term in each series, we substitute in (3) to obtain

$$(4)\quad \begin{aligned} F(2u) \simeq \{2\Phi(u)\}^n + 2\sqrt{2\pi}\, nk\phi(u)\{2\Phi(u)\}^{n-1}\{\tfrac{1}{2} - \Phi(-k\phi'(u)/\phi(u))\} \\ \exp\{\tfrac{1}{2}k^2(\phi'(u)/\phi(u))^2\}, \end{aligned}$$

where

$$k^{-2} = (\phi'(u)/\phi(u))^2 - (\phi''(u)/\phi(u)) - (n-1)\phi'(u)/\Phi(u).$$

When $\phi(x) = \exp - \tfrac{1}{2}x^2/\sqrt{2\pi}$, we find that (4) reduces to

$$(5)\quad F(2u) \simeq \{2\Phi(u)\}^n + 2nk\{2\Phi(u)\}^{n-1}\{\exp - \tfrac{1}{2}u^2(1-k^2)\}\{\tfrac{1}{2} - \Phi(uk)\},$$

where

$$k^{-2} = 1 + (n-1)u\phi(u)/\Phi(u).$$

In this case it is easy to include a further term, which results in the last bracket of (5) being replaced by

$$(6)\qquad \{\tfrac{1}{2} - \Phi(uk) - (n-1)k^4P(u)Q(uk)\},$$

where

$$P(x) = \frac{x^2}{8}\left(\frac{\phi(x)}{\Phi(x)}\right)^2 + \frac{x^3 - 3x}{24}\frac{\phi(x)}{\Phi(x)},$$

$$Q(x) = (x^4 + 6x^2 + 3)\{\tfrac{1}{2} - \Phi(x)\} - (x^3 + 5x)\phi(x).$$

3. Accuracy in the normal case. While w is not defined when $n = 1$, expression (2) gives $F(w)$ the formal value unity for all w. This is also the value given by (5) or (6). Thus our expression, besides being asymptotically correct, also gives the exact value when $n = 1$. Hence [1], errors will at first rise with increase of n and then fall asymptotically to zero.

The following values of maximum error for (5) and (6) are the differences between exact values obtained by evaluating the p.d.f. using (1) and values of

$F(w)$ then found by quadrature.

Sample size, n:	20	60	100
Maximum error of (5):	+0.0031	+0.0040	+0.0043
Maximum error of (6):	−0.00052	−0.00070	−0.00075

By using (6), results of reasonable accuracy are obtained. Table I gives corrections in units in the fourth decimal place to be added to the approximate value given by (6), for five sample sizes. The corrections are given as functions of the approximate value itself, rather than of w, to make interpolation for n much simpler. By plotting the correction against the approximate probability on

TABLE I

Corrections ($\times 10^4$) *to be applied to approximate value obtained from equation* (6), *for samples of size n*

n	Value obtained from equation (6)									
	.05	.10	.25	.50	.75	.90	.95	.99	.995	.999
20	0	0.1	0.4	1.7	4.2	5.1	4.1	1.5	0.8	0.2
40	0.1	0.3	0.8	2.6	5.3	6.2	4.9	1.6	0.8	0.2
60	0.2	0.4	1.0	3.1	6.1	6.9	5.5	1.6	0.8	0.2
80	0.2	0.5	1.2	3.5	6.6	7.2	5.9	1.6	0.8	0.2
100	0.2	0.5	1.2	3.6	6.9	7.4	6.1	1.6	0.8	0.2

TABLE II

Percentage points of range (w) *for samples of various sizes* (n) *from normal populations of unit standard deviation*

n	Percentage points													Mean $\bar{w}$
	.001	.005	.010	.050	.100	.250	.500	.750	.900	.950	.990	.995	.999	
20	1.88	2.13	2.25	2.63	2.84	3.22	3.69	4.20	4.69	5.01	5.65	5.89	6.40	3.73
40	2.62	2.85	2.97	3.31	3.50	3.85	4.27	4.74	5.20	5.50	6.09	6.32	6.81	4.32
60	3.03	3.24	3.35	3.68	3.86	4.19	4.59	5.04	5.48	5.76	6.34	6.55	7.04	4.64
80	3.30	3.50	3.61	3.92	4.10	4.42	4.81	5.24	5.67	5.95	6.51	6.73	7.20	4.85
100	3.50	3.71	3.80	4.11	4.28	4.59	4.97	5.39	5.81	6.09	6.64	6.85	7.31	5.02

arithmetical probability paper, we can interpolate graphically for n and the approximate value. This will enable the probability integral to be found with an error that should not exceed 0.0001, and will usually be less than 0.00005.

Table II, giving percentage points of range found by quadrature, will assist in making preliminary estimates. Plotting ($w - \bar{w}$) against $n^{-1/2}$ for a given percentage level, permits interpolation for other values of n to be made with accuracy. Values of $\bar{w}$ are tabled in [3].

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ABSTRACTS OF PAPERS

(*Abstracts of papers presented at the Montreal meeting of the Institute, September 10–13, 1954*)

1. On Quadratic Estimates of Variance Components in Balanced Models, A. W. Wortham, Chance Vought Aircraft and Oklahoma A and M College.

A balanced model is defined as a model whose analysis of variance mean squares are symmetric in the squares of the observations. Included in this class of models are: (1) Completely Randomized, (2) Randomized Blocks, (3) Latin Squares, (4) Graeco-Latin Squares, (5) Split Plots, (6) Factorial Arrangements, etc.

The "analysis of variance estimates" of the variance components are the estimates obtained by solving the system of equations which result when the observed and expected mean squares in the analysis of variance table are equated. For any infinite population let the general balanced model be $y_{i_1 i_2 \cdots i_n} = \mu + \sum_{k=1}^{n} A_{ki_k} + e_{i_1 i_2 \cdots i_n}$, where μ is a constant, A_{ki_k} and $e_{i_1 i_2 \cdots i_n}$ are independent random variables with zero means, finite fourth moments, and variances σ_k^2 and σ_0^2 respectively. Let $\hat{\sigma}_k^2$ and $\hat{\sigma}_0^2$ be "the analysis of variance estimates" of the variance components σ_k^2 and σ_0^2. It is shown that the quadratic estimate of $\sum_{k=0}^{p} g_k \sigma_k^2$ (g_k known) which is unbiased, independent of μ, and has minimum variance is given by $\sum_{k=0}^{p} g_k \hat{\sigma}_k^2$. That is, the best quadratic unbiased estimate of the linear combination of the variance components is given by the same linear combination of "the analysis of variance estimates" of the variance components.

2. The Coefficients in the Best Linear Estimate of the Mean in Symmetric Populations, A. E. Sarhan, University of North Carolina.

In a previous paper ("Estimation of the Mean and Standard Deviation by Order Statistics" by A. E. Sarhan, *Ann. Math. Stat.* Vol. 25 (1954), pp. 317–328) the best linear estimate of the mean of a rectangular, triangular and double exponential population were worked out. By considering some other symmetric distributions with different shapes, it is found that the coefficients in the estimates form a sequence. From the sequence, it is observed that the coefficients in the estimates are influenced by the shape of the distribution. The variances of the estimates are also so affected.

3. Distribution of Linear Contrasts of Order Statistics, Jacques St. Pierre, University of North Carolina.

Consider $n + 1$ independent normal populations with unknown means, $m_0, m_1, \cdots, m_n$, respectively, and with a common known variance $\sigma^2 = 1$ (say). Suppose a sample of size N is available from each population; and let $x_{(0)} > x_{(1)} > \cdots > x_{(n)}$ be the ordered sample means. Consider the linear contrasts $z = x_{(0)} - c_1 x_{(1)} - \cdots - c_n x_{(n)}$, where $\sum_{i=1}^{n} c_i = 1$, $c_i \geqq 0$, $(i = 1, 2, \cdots, n)$. The probability density function of the contrasts z is derived under the null hypothesis $H_0: m_0 = m_1 = \cdots = m_n$. The density of the contrasts z is also