ON THE NONCENTRAL BETA-DISTRIBUTION

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1. Summary. A computing formula for the noncentral beta-distribution recently published by Nicholson is simplified. A table of figurate numbers is provided to facilitate the use of the simplified formula. It is compared for ease of use with the method of Tang in several situations.

2. Simplification of Nicholson's formula. Let $Y_1 - \theta_1, \cdots, Y_{2a} - \theta_{2a}$, $Z_1, \cdots, Z_{2b}$ be independent normal random variables of zero expectation and unit variance. Let $S = \sum Y_i^2$, $T = \sum Z_i^2$, $2\lambda = \sum \theta_i^2$. If $\lambda = 0$, the random variable $X = T / (S + T)$ has the beta-distribution $\Pr(X \leqq x) = I_x(b, a)$. We shall say that X has the noncentral beta-distribution if $\lambda > 0$, and in general denote $\Pr(X \leqq x)$ by $B(x; a, b, \lambda)$.

Nicholson [1] has recently derived a closed expression for B in case b is an integer. In our notation, his result is

$$B(x; a, b, \lambda) = 1 - e^{-\lambda x}\left\{I_{1-x}(a, b) + (1 - x)^a \sum_{j=1}^{b-1} [x(1 - x)\lambda]^j (P_j/j!)\right\} \tag{1}$$

where

$$P_j = \sum_{k=0}^{b-j-1}\left[(-1)^k\binom{b - j - 1}{k}\frac{(a+b-1)(a+b-2)\cdots(a+j)}{(b-j-1)!(a+j+k)}\right](1-x)^k. \tag{2}$$

It should be noted that our x, in conformity with the notation of [2], [3], is the complement of the x used by [1], [4].

A main purpose of this paper is to point out the simplification effected in (2) if it is expressed as a polynomial in x instead of $(1 - x)$. Since

$$(1 - x)^k = \sum_{t=0}^{k} (-1)^t \binom{k}{t} x^t,$$

the coefficient of x^t in P_j is

$$\sum_{k=t}^{b-j-1} (-1)^{k+t}\binom{b - j - 1}{k}\binom{k}{t}\frac{(a + b - 1)(a + b - 2)\cdots(a + j)}{(b - j - 1)!(a + j + k)}$$

$$= \frac{(a + b - 1)(a + b - 2)\cdots(a + j)}{(b - j - t - 1)!t!}\sum_{u=0}^{b-j-t-1}(-1)^u \binom{b - j - t - 1}{u}\Big/(a + j + t + u). \tag{3}$$

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If we now use the easily checked identity

$$\sum_{u=0}^{n} (-1)^u \binom{n}{u} \Big/ (c + u) = 1/c \binom{c+n}{n} \tag{4}$$

we have

$$P_j = \sum_{t=0}^{b-j-1} \binom{A+t}{t} x^t, \qquad A = a + j - 1. \tag{5}$$

A comparison of (5) with (2) shows the greater simplicity of the new form. In particular, the coefficients now depend on but two arguments and are readily tabled. If a is an integer, the coefficients are the widely available combinatorials; if a is a half-integer, they are figurate numbers. A table of $\binom{A+t}{t}$ to 7 significant figures is provided for $A = .5(1)19.5$, $t = 1(1)18$, adequate for computing all of the P_j when $2b \leqq 40$, $2a + 2b \leqq 43$, and for the initial P_j for larger values of $2a$.

3. Comparison with Tang. The formula (1) may be regarded as the summed form of a recursion formula due to Tang [4]:

$$B(x; a, b, \lambda) = 1 - e^{-\lambda x}(1 - x)^{a+b-1} \sum_{j=0}^{b-1} T_j \tag{6}$$

where

$$\begin{aligned} &T_0 = 1, \qquad T_1 = \frac{x}{1-x}[(1 - x)\lambda + a + b - 1] \\ &T_j = \frac{x}{j(1-x)}[\{(1 - x)\lambda + a + b - j\}T_{j-1} + x\lambda T_{j-2}]. \end{aligned} \tag{7}$$

The choice between the polynomial method (1), (5) and the recursion method (6), (7) will depend on just what computation is in hand.

(i) Consider first the problem of computing an isolated value of (1), corresponding to given values of x, a, b, λ not covered by existing tables and charts. This is the most familiar use of the noncentral beta-distribution as it provides values of the power of the analysis of variance test. It corresponds to the tables of Tang, and also arises in computing the power of the test of the hypothesis that $\lambda \leqq \lambda_0$. The polynomial method (1), (5), even with the coefficients of (5) available, involves $(b^2 + 3b - 4)/2$ multiplications and divisions. Recursion method (6), (7) requires $4b - 2$. Thus, Tang's method appears to be superior for $b \geqq 6$, with the advantage increasing rapidly with b.

The comparison just made is however not quite fair to the polynomial method for several reasons. The computation of (5) is somewhat simpler than that of (7), so that it proceeds more rapidly with less risk of a mistake. Further, a recursion computation is particularly subject to the accumulation of error, so more figures must be carried with (7). Finally, it may not be necessary to compute all of the P_j, as the tail of the sum in (1) may be negligible. Nicholson has given a λ-uniform bound to the error committed in neglecting this tail, though in fact the smaller λ is, the fewer terms are needed. With (7), however, all terms are needed even when $\lambda = 0$.

This last advantage of the polynomial method is not as great as it may at first seem, since the neglected P_j are just those easiest to compute. To illustrate this point, consider the example given in [1], with $b = 15$, where the first 8 P_j are adequate for two-figure accuracy for all λ. Here, 103 multiplications are required to calculate all of the P_j, of which only 15 are saved by neglecting those beyond P_8. The entire truncated computation requires 113 multiplications and divisions, compared with 60 needed with Tang's formula. However, for small λ, fewer than 8 P_j's would suffice.

As a rough summary statement, the polynomial method should be chosen for the direct computation of an isolated B if either b or λ is small; otherwise, the older method will be quicker.

(ii) Consider next the problem of finding x corresponding to given a, b, λ, and B. This problem arises for example, in determining the critical value of the test for the hypothesis $\lambda \leqq \lambda_0$, a problem met by the Incomplete Beta-Function tables if $\lambda_0 = 0$. It is well known that the noncentral beta-distribution provides the significance levels of this test. As (1) or (6) cannot be explicitly inverted, we must essentially calculate B for several values of x, with a, b, $\lambda = \lambda_0$ fixed, and then interpolate. The comparison between the two methods is as before, with Tang's preferred unless b or λ is small.

(iii) Finally, consider the problem of finding λ for given a, b, x, and B. This arises if we seek to find the smallest λ_0 for which on the basis of given x we should accept the hypothesis $\lambda \leqq \lambda_0$, or if we wish to provide confidence intervals for λ. It is the problem solved in computing the Lehmer tables. As in (ii), we must calculate B for several values of λ with a, b, x fixed. For this purpose method (1, 5) will usually be much superior to (6, 7), since we can compute the quantities $Q_j = (1 - x)^a[x(1 - x)]^j P_j/j!$ once and for all, as they do not involve λ. Then it is relatively easy to find

$$B = 1 - e^{-\lambda x}\left\{I_{1-x}(a, b) + \sum_{j=1}^{b-1} Q_j\lambda^j\right\}$$

for several values of λ.

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Table of $\binom{A+t}{t}$

t	A = 0.5	1.5	2.5	3.5
1	0-1.5	0-2.5	0-3.5	0-4.5
2	0-1.875	0-4.375	0-7.875	1-1.237 5
3	0-2.187 5	0-6.562 5	1-1.443 75	1-2.681 25
4	0-2.460 938	0-9.023 438	1-2.346 094	1-5.027 344
5	0-2.707 032	1-1.173 047	1-3.519 141	1-8.546 484
6	0-2.932 617	1-1.466 309	1-4.985 449	2-1.353 193
7	0-3.142 090	1-1.780 518	1-6.765 967	2-2.029 790
8	0-3.338 470	1-2.114 365	1-8.880 331	2-2.917 823
9	0-3.523 941	1-2.466 759	2-1.134 709	2-4.052 532
10	0-3.700 138	1-2.836 773	2-1.418 386	2-5.470 918
11	0-3.868 326	1-3.223 605	2-1.740 747	2-7.211 665
12	0-4.029 506	1-3.626 556	2-2.103 402	2-9.315 068
13	0-4.184 487	1-4.045 005	2-2.507 903	3-1.182 297
14	0-4.333 933	1-4.478 398	2-2.955 743	3-1.477 871
15	0-4.478 398	1-4.926 238	2-3.448 366	3-1.822 708
16	0-4.618 348	1-5.388 072	2-3.987 174	3-2.221 425
17	0-4.754 182	1-5.863 491	2-4.573 523	3-2.678 778
18	0-4.886 242	1-6.352 114	2-5.208 734	3-3.199 651

t	A = 4.5	5.5	6.5	7.5
1	0-5.5	0-6.5	0-7.5	0-8.5
2	1-1.787 5	1-2.437 5	1-3.187 5	1-4.037 5
3	1-4.468 75	1-6.906 25	2-1.009 375	2-1.413 125
4	1-9.496 094	2-1.640 234	2-2.649 609	2-4.062 734
5	2-1.804 258	2-3.444 492	2-6.094 102	3-1.015 684
6	2-3.157 451	2-6.601 943	3-1.269 604	3-2.285 288
7	2-5.187 241	3-1.178 918	3-2.448 523	3-4.733 811
8	2-8.105 064	3-1.989 425	3-4.437 948	3-9.171 759
9	3-1.215 760	3-3.205 185	3-7.643 132	4-1.681 489
10	3-1.762 852	3-4.968 036	4-1.261 117	4-2.942 606
11	3-2.484 018	3-7.452 054	4-2.006 322	4-4.948 928
12	3-3.415 525	4-1.086 758	4-3.093 080	4-8.042 008
13	3-4.597 822	4-1.546 540	4-4.639 620	5-1.268 629
14	3-6.075 693	4-2.154 109	4-6.793 730	5-1.947 536
15	3-7.898 401	4-2.943 949	4-9.737 679	5-2.921 304
16	4-1.011 983	4-3.955 932	5-1.369 361	5-4.290 665
17	4-1.279 860	4-5.235 793	5-1.892 940	5-6.183 605
18	4-1.599 825	4-6.835 618	5-2.576 502	5-8.760 107

Table of $\binom{A+t}{t}$—(*Continued*)

t	A: 8.5	9.5	10.5	11.5
1	0–9.5	1–1.05	1–1.15	1–1.25
2	1–4.987 5	1–6.037 5	1–7.187 5	1–8.437 5
3	2–1.911 875	2–2.515 625	2–3.234 375	2–4.078 125
4	2–5.974 609	2–8.490 234	3–1.172 461	3–1.580 273
5	3–1.613 145	3–2.462 168	3–3.634 629	3–5.214 902
6	3–3.898 433	3–6.360 601	3–9.995 229	4–1.521 013
7	3–8.632 244	4–1.499 284	4–2.498 807	4–4.019 821
8	4–1.780 400	4–3.279 685	4–5.778 492	4–9.798 313
9	4–3.461 889	4–6.741 574	5–1.252 007	5–2.231 838
10	4–6.404 495	5–1.314 607	5–2.566 614	5–4.798 451
11	5–1.135 342	5–2.449 949	5–5.016 563	5–9.815 014
12	5–1.939 543	5–4.389 492	5–9.406 055	6–1.922 107
13	5–3.207 706	5–7.597 199	6–1.700 325	6–3.622 432
14	5–5.155 242	6–1.275 244	6–2.975 569	6–6.598 002
15	5–8.076 546	6–2.082 899	6–5.058 468	7–1.165 647
16	6–1.236 721	6–3.319 620	6–8.378 088	7–2.003 456
17	6–1.855 082	6–5.174 701	7–1.355 279	7–3.358 735
18	6–2.731 092	6–7.905 794	7–2.145 858	7–5.504 593

t	A: 12.5	13.5	14.5	15.5
1	1–1.35	1–1.45	1–1.55	1–1.65
2	1–9.787 5	2–1.123 75	2–1.278 75	2–1.443 75
3	2–5.056 875	2–6.180 625	2–7.459 375	2–8.903 125
4	3–2.085 961	3–2.704 023	3–3.449 961	3–4.340 273
5	3–7.300 863	4–1.000 489	4–1.345 485	4–1.779 512
6	4–2.251 100	4–3.251 588	4–4.597 073	4–6.376 585
7	4–6.270 920	4–9.522 508	5–1.411 958	5–2.049 617
8	5–1.606 923	5–2.559 174	5–3.971 132	5–6.020 749
9	5–3.838 761	5–6.397 935	6–1.036 907	6–1.638 982
10	5–8.637 213	6–1.503 515	6–2.540 422	6–4.179 403
11	6–1.845 223	6–3.348 737	6–5.889 159	7–1.006 856
12	6–3.767 330	6–7.116 067	7–1.300 523	7–2.307 379
13	6–7.389 762	7–1.450 583	7–2.751 106	7–5.058 484
14	7–1.398 776	7–2.849 359	7–5.600 465	8–1.065 895
15	7–2.564 423	7–5.413 783	8–1.101 425	8–2.167 320
16	7–4.567 879	7–9.981 662	8–2.099 591	8–4.266 911
17	7–7.926 614	8–1.790 828	8–3.890 418	8–8.157 329
18	8–1.343 121	8–3.133 948	8–7.024 367	9–1.518 170

Table of $\binom{A+t}{t}$—(*Concluded*)

t	A			
	16.5	17.5	18.5	19.5
1	1-1.75	1-1.85	1-1.95	1-2.05
2	2-1.618 75	2-1.803 75	2-1.998 75	2-2.203 75
3	3-1.052 188	3-1.232 562	3-1.432 438	3-1.652 812
4	3-5.392 461	3-6.625 023	3-8.057 461	3-9.710 273
5	4-2.318 758	4-2.981 261	4-3.787 007	4-4.758 034
6	4-8.695 343	5-1.167 660	5-1.546 361	5-2.022 164
7	5-2.919 151	5-4.086 811	5-5.633 172	5-7.655 337
8	5-8.939 900	6-1.302 671	6-1.865 988	6-2.631 522
9	6-2.532 972	6-3.835 643	6-5.701 631	6-8.333 153
10	6-6.712 375	7-1.054 802	7-1.624 965	7-2.458 280
11	7-1.678 094	7-2.732 895	7-4.357 860	7-6.816 140
12	7-3.985 473	7-6.718 368	8-1.107 623	8-1.789 237
13	7-9.043 957	8-1.576 232	8-2.683 855	8-4.473 092
14	8-1.970 291	8-3.546 523	8-6.230 378	9-1.070 347
15	8-4.137 610	8-7.684 133	9-1.391 451	9-2.461 798
16	8-8.404 521	9-1.608 865	9-3.000 317	9-5.462 115
17	9-1.656 185	9-3.265 050	9-6.265 367	10-1.172 748
18	9-3.174 355	9-6.439 405	10-1.270 477	10-2.443 225