N-DIMENSIONAL DISTRIBUTIONS CONTAINING A NORMAL COMPONENT[1]

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In this paper we obtain necessary and sufficient conditions for an n-dimensional distribution function $F(x_1, \cdots, x_n)$ to contain as a factor the distribution function of n independent normal random variables having common mean zero and variance 1. That is we obtain conditions for $F(x_1, \cdots, x_n)$ to be of the form

$$(1)\quad F(x_1, \cdots, x_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G(x_1 - u_1, \cdots, x_n - u_n)\, dP(u_1, \cdots, u_n),$$

where $P(u_1, \cdots, u_n)$ is a distribution function and

$$G(x_1, \cdots, x_n) = \left(\frac{1}{\sqrt{\pi}}\right)^n \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \exp\,[-(u_1^2 + \cdots + u_n^2)]\, du_1 \cdots du_n.$$

If we denote $\partial^n/\partial x_1 \cdots \partial x_n F(x_1, \cdots, x_n)$ by $f(x_1, \cdots, x_n)$, the problem becomes that of representing $f(x_1, \cdots, x_n)$ in the form

$$(2)\quad \begin{aligned} f(x_1, \cdots, x_n) = \left(\frac{1}{\sqrt{\pi}}\right)^n \int_{-\infty}^{\infty} \\ \cdots \int_{-\infty}^{\infty} \exp\,\{-[(x_1 - u_1)^2 + \cdots (x_n - u_n)^2]\}\, dP(u_1 \cdots, u_n). \end{aligned}$$

The one-dimensional case has been treated by Pollard [1] employing properties of the heat equation. We use a different approach to prove the following

Theorem. *$f(x_1, \cdots, x_n)$ is representable in the form* (2) *with $P(u_1, \cdots, u_n)$ a distribution function if and only if*

$$\text{(i)}\qquad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n = 1$$

(ii) *$f(x_1, \cdots, x_n)$ is bounded and has mixed partial derivatives of all orders satisfying*

$$\left| \frac{\partial^{k_1} \cdots \partial^{k_n}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} f(x_1, \cdots, x_n) \right| \leqq A^n 2^{\frac{k_1 + \cdots + k_n}{n}} \sqrt{k_1! \cdots k_n!}, \qquad k_1, \cdots, k_n = 1, 2, \cdots.$$

$$\text{(iii)}\quad \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \frac{(-1)^{k_1+\cdots+k_n} t_1^{k_1} \cdots t_n^{k_n}}{4^{k_1+\cdots+p_n} k_1! \cdots k_n!} \frac{\partial^{k_1} \cdots \partial^{k_n}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} f(x_1, \cdots, x_n) \geqq 0, \qquad |t_1| < 1, \cdots, |t_n| < 1.$$

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PROOF. We carry out the proof for $n = 2$, the proof for $n > 2$ proceeding in exactly the same fashion. The necessity of (i) is obvious. As for (ii) we have

$$\left| \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} f(x_1, x_2) \right| \leqq \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | H_{k_1}(x_1 - u_1) H_{k_2}(x_2 - u_2) \cdot \exp \{ -[(x_1 - u_1)^2 + (x_2 - u_2)^2] \} \, dP(u_1, u_2) \, |,$$

where $H_k(x)$ is the kth Hermite polynomial which satisfies

$$| H_k(x) | \leqq A 2^{k/2} \sqrt{k!} \exp \frac{x^2}{2} \tag{3}$$

([2], p. 236). Hence the integral above is majorized by

$$A^2 2 \frac{k_1 + k_2}{2} \sqrt{k_1! \, k_2!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \{ -\tfrac{1}{2}[(x_1 - u_1)^2 + (x_2 - u_2)^2] \} \, dP(u_1, u_2),$$

which is $\leqq A^2 \, 2 \frac{k_1 + k_2}{2} \sqrt{k_1! k_2!}$. To establish the necessity of (iii) we observe that we have formally

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(-1)^{k_1+k_2} t_1^{k_1} t_2^{k_2}}{4^{k_1+k_2} k_1! \, k_2!} \frac{\partial^{2k_1}}{\partial x_1^{2k_1}} \frac{\partial^{2k_2}}{\partial x_2^{2k_2}} f(x_1, x_2)$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{(-1)^{k_1+k_2} t_1^{k_1} t_2^{k_2}}{4^{k_1+k_2} k_1! \, k_2!} H_{2k_1}(x_1 - u_1) H_{2k_2}(x_2 - u_2) \times \exp \{ -[(x_1 - u_1)^2 + (x_2 - u_2)^2] \} \, dP(u_1, u_2).$$

From (3) it is seen that the double series in the integrand converges if all terms are replaced by their absolute values provided $|t_1| < 1$, $|t_2| < 1$, and the integral may be written as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\sum_{k_1=0}^{\infty} \frac{(-1)^{k_1} t_1^{k_1}}{4^{k_1} k_1!} H_{2k_1}(x_1 - u_1) \right] \left[\sum_{k_2=0}^{\infty} \frac{(-1)^{k_2} t_2^{k_2}}{4^{k_2} k_2!} H_{2k_2}(x_2 - u_2) \right] \times \exp \{ -[(x_1 - u_1)^2 + (x_2 - u_2)^2] \} \, dP(u_1, u_2), \tag{4}$$

but

$$\sum_{k_1=0}^{\infty} \frac{(-1)^{k_1} t_1^{k_1}}{4^{k_1} k_1!} H_{2k_1}(x) = \frac{1}{\sqrt{1 - t_1}} \exp \left(-\frac{x^2 t_1}{1 - t_1} \right)$$

([1], p. 580), and (4) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{(1 - t_1)(1 - t_2)}} \exp - \left\{ \left[\frac{(x_1 - u_1)^2}{1 - t_1} + \frac{(x_2 - u_2)^2}{1 - t_2} \right] \right\} dP(u_1, u_2), \tag{5}$$

which for fixed t_1 and t_2 is $\leqq$ constant $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |dP(u_1, u_2)|$. This justifies the formal manipulations above and (5) is clearly non-negative establishing the necessity of (iii). For the sufficiency we need a couple of lemmas.

LEMMA 1. Denoting the left-hand side of (iii) by $T_{t_1,t_2}f(x_1, x_2)$ we have for functions $f(x_1, x_2)$ satisfying (ii)

$$\lim_{\substack{t_1\to 1\\ t_2\to 1}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-[(x_1 - u_1)^2 + (x_2 - u_2)^2]\} T_{t_1,t_2}f(u_1, u_2)\, du_1\, du_2 = f(x_1, x_2).$$

PROOF. The estimates furnished by (ii) enable us to write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-[(x_1 - u_1)^2 + (x_2 - u_2)^2]\} \frac{\partial^{2k_1}\partial^{2k_2}}{\partial u_1^{2k_1}\partial u_2^{2k_2}} f(u_1, u_2)\, du_1\, du_2$$

$$= \int_{-\infty}^{\infty} \exp[-(x_2 - u_2)^2]\, du_2 \int_{-\infty}^{\infty} \exp[-(x_1 - u_1)^2] \frac{\partial^{2k_1}\partial^{2k_2}}{\partial u_1^{2k_1}\partial u_2^{2k_2}} f(u_1, u_2)\, du_1,$$

and upon integrating the inner integral $2k_1$ times by parts we have

$$\int_{-\infty}^{\infty} \exp[-(x_2 - u_2)^2]\, du_2 \int_{-\infty}^{\infty} \frac{\partial^{2k_1}}{\partial u_1^{2k_1}} \exp[-(x_1 - u_1)^2] \frac{\partial^{2k_2}}{\partial u_2^{2k_2}} f(u_1, u_2)\, du_1$$

$$= \int_{-\infty}^{\infty} \frac{\partial^{2k_1}}{\partial u_1^{2k_1}} \exp[-(x_1 - u_1)^2]\, du_1 \int_{-\infty}^{\infty} \exp[-(x_2 - u_2)^2] \frac{\partial^{2k_2}}{\partial u_2^{2k_2}} f(u_1, u_2)\, du_2.$$

We integrate $2k_2$ more times by parts and obtain finally

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^{2k_1}}{\partial u_1^{2k_1}} \exp[-(x_1 - u_1)^2] \frac{\partial^{2k_2}}{\partial u_2^{2k_2}} \exp[-(x_2 - u_2)^2] f(u_1, u_2)\, du_1\, du_2.$$

Thus

$$\lim_{\substack{t_1\to 1\\ t_2\to 1}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-[(x_1 - u_1)^2 + (x_2 - u_2)^2]\} T_{t_1,t_2}f(u_1, u_2)\, du_1\, du_2$$

$$= \lim_{\substack{t_1\to 1\\ t_2\to 1}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\sum_{k_1=0}^{\infty} \frac{(-1)^{k_1} t_1^{k_1}}{4^{k_1} k_1!} H_{2k_1}(x_1 - u_1)\right]\left[\sum_{k_2=0}^{\infty} \frac{(-1)^{k_2} t_2^{k_2}}{4^{k_2} k_2!} H_{2k_2}(x_2 - u_2)\right]$$

$$\times \exp\{-[(x_1 - u_1)^2 + (x_2 - u_2)^2]\} f(u_1, u_2)\, du_1\, du_2.$$

By (4) and (5) this becomes

$$\left(\lim_{t_1\to 1} \frac{1}{\sqrt{1 - t_1}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x_1 - u_1)^2}{1 - t_1}\right] du_1\right)$$

$$\times \left(\lim_{t_2\to 1} \frac{1}{\sqrt{1 - t_2}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x_2 - u_2)^2}{1 - t_2}\right] f(u_1, u_2)\, du_2\right) = f(x_1, x_2).$$

LEMMA 2.

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_{t_1,t_2}f(u_1, u_2)\, du_1\, du_2 = 1, \quad |t_1| < 1, |t_2| < 1.$$

PROOF.

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_{t_1,t_2}f(u_1, u_2)\,du_1\,du_2$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1\,dx_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp - [(x_1 - u_1)^2 + (x_2 - u_2)^2]T_{t_1,t_2}f(u_1, u_2)\,du_1\,du_2$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1\,dx_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{(1-t_1)(1-t_2)}} \exp\left\{-\left[\frac{(x_1-u_1)^2}{1-t_1} + \frac{(x_2-u_2)^2}{1-t_2}\right]\right\} \times f(u_1, u_2)\,du_1\,du_2$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u_1 u_2)\,du_1\,du_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{(1-t_1)(1-t_2)}} \times \exp\left\{-\left[\frac{(x_1-u_1)^2}{1-t_1} + \frac{(x_2-u_2)^2}{1-t_2}\right]\right\} dx_1\,dx_2 = 1.$$

By the above lemma and (iii) the family of functions

$$P_{t_1,t_2}(x_1, x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} T_{t_1,t_2}f(u_1, u_2)\,du_1\,du_2$$

is monotone in the sense of Bochner ([3], p. 383) and uniformly bounded; hence there exist sequences $\{t_{1n}\}\{t_{2n}\}$ such that $t_{1n} \to 1$, $t_{2n} \to 1$ and a function $P(x_1, x_2)$ monotone and bounded such that

$$\lim_{n\to\infty} P_{t_{1n},t_{2n}}(x_1, x_2) = P(x_1, x_2)$$

([3], p. 389–390). By Lemma 1,

$$f(x_1, x_2) = \lim_{n\to\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-[(x_1-u_1)^2 + (x_2-u_2)^2]\}\,dP_{t_{1n},t_{2n}}(u_1, u_2).$$

By the formula for integration by parts in two dimensions [4] the above integral becomes

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_{t_{1n},t_{2n}}(u_1, u_2)\frac{\partial^2}{\partial u_1\,\partial u_2}\exp\{-[(x_1-u_1)^2 + (x_2-u_2)^2]\}\,du_1\,du_2,$$

and integrating by parts again

$$f(x_1, x_2) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-[(x_1-u_1)^2 + (x_2-u_2)^2]\}\,dP(u_1, u_2).$$

To complete the proof that $P(u_1, u_2)$ is a distribution function we observe that by condition (1)

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2)\,dx_1 dx_2$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dP(u_1 u_2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-[(x_1-u_1)^2 + (x_2-u_2)^2]\}\,dx_1\,dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dP(u_1, u_2).$$

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A CERTAIN CLASS OF TESTS OF FIT[1]

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1. Summary and introduction. Suppose $X_1, X_2, \cdots, X_n$ are known to be independently and identically distributed, each with the density function $f(x)$, with $\int_0^1 f(x)\,dx = 1$. Let $Y_1 \leqq Y_2 \leqq \cdots \leqq Y_n$ be the ordered values of $X_1, X_2, \cdots, X_n$, and define $W_1 = Y_1$, $W_2 = Y_2 - Y_1, \cdots, W_n = Y_n - Y_{n-1}$, and $W_{n+1} = 1 - Y_n$, so that $W_1 + \cdots + W_{n+1} = 1$. Finally, define $Z_1, \cdots, Z_{n+1}$ as the ordered values of $W_1, \cdots, W_{n+1}$, so that $0 \leqq Z_1 \leqq Z_2 \leqq \cdots \leqq Z_{n+1}$, with $Z_1 + \cdots + Z_{n+1} = 1$. We are going to test the hypothesis that $f(x) = 1$ for $0 < x < 1$, and we are going to consider only tests based on $Z_1, Z_2, \cdots, Z_n$. The intutitive justification for this is that, roughly speaking, deviations from the hypothesis on any part of the unit interval are treated alike. Several authors have discussed tests based on $Z_1, \cdots, Z_n$. (See references [1], [2], [3].)

If u is a number greater than unity, it is shown that the test of the form "reject the hypothesis if $Z_1^u + \cdots + Z_{n+1}^u > K$" is consistent against a very wide class of alternatives. When $u = 2$, the resulting test has some desirable properties with respect to alternatives with linear density functions.

2. The distribution of $Z_1, \cdots, Z_n$. It is easily seen that $P[Z_i = Z_j$ for any $i \neq j]$ is equal to zero. We want to find the joint density function $h(z_1, \cdots, z_n)$ of $Z_1, \cdots, Z_n$. The joint density function of $W_1, \cdots, W_n$ is equal to $n!\, f(w_1)f(w_1 + w_2) \cdots f(w_1 + w_2 + \cdots + w_n)$ in the region $w_i \geqq 0$, $w_1 + \cdots + w_n \leqq 1$, and is equal to zero elsewhere. Let $\{j(1), j(2), \cdots, j(n+1)\}$ be any permutation of the first $n + 1$ integers, and let $\sum_p$ denote summation over all the $(n + 1)!$ permutations. Given any set of numbers $0 < z_1 < z_2 < \cdots < z_n < 1 - (z_1 + \cdots + z_n)$, we denote by $Q[j(1), j(2), \cdots, j(n + 1)]$ the conditional probability that $W_i = z_{j(i)}$ for $i = 1, \cdots, n + 1$, given that $Z_i = z_i$ for $i = 1, \cdots, n + 1$. It is understood that if $j(i) = n + 1$, then $z_{j(i)} = 1 -$

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