Let $f(n)$ denote the total number of decision patterns for n means. Clearly,

$$f(n) = f_0(n) + f_1(n), \tag{2.4}$$

since s_0 and s_1 are the only possible first steps.

Since $f(n)$ depends only on $f_0(n)$ and $f_1(n)$, equations (2.1), (2.2), and (2.3), together with the boundary conditions

$$f_0(1) = f_0(2) = f_1(2) = 1, \tag{2.5}$$

will lead to (1.1).

Using standard techniques for solving difference equations, it can be shown that[3]

$$f_e(k) = \frac{2e+1}{e+k}\binom{2k-2}{k+e-1}. \tag{2.6}$$

This result can be verified by substituting (2.6) into equations (2.1), (2.2), (2.3), and (2.5). It follows immediately that

$$f(n) = f_0(n) + f_1(n) = \frac{1}{n+1}\binom{2n}{n}.$$

PERCENTILES OF THE ω_n STATISTIC[1]

By B. Sherman[2]

University of California, Los Angeles

If n points are selected independently from a uniform distribution on a unit interval there arise $n + 1$ subintervals, each of expected length $1/(n + 1)$. If L_k is the length of the kth interval from the left, then

$$\omega_n = \tfrac{1}{2}\sum_{k=1}^{n+1}\left|L_k - \frac{1}{n+1}\right|.$$

The distribution function of ω_n is 0 for $x < 0$, 1 for $x > n/(n + 1)$, and for $0 \leqq x \leqq n/(n + 1)$

$$F_n(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 + 1,$$

where

$$b_k = \sum_{q=0}^{r}(-1)^{q+k+1}\binom{n+1}{q+1}\binom{q+k}{q}\binom{n}{k}\left(\frac{n-q}{n+1}\right)^{n-k},$$

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[2] Present address, Westinghouse Research Laboratories, Pittsburgh 35, Pennsylvania.

the upper limit of summation being determined by

$$\frac{n-r-1}{n+1} \leqq x < \frac{n-r}{n+1}.$$

This distribution function has been derived in [1]. Further discussion may be found in [2]. The statistical uses of ω_n are discussed in [1] and [3].

Using the SWAC, the high-speed automatic computer at Numerical Analysis Research, the 99, 95, and 90 per cent points of ω_n have been obtained for n ranging from 3 to 20. These results appear in Table 1 below, with values for $n = 1$ and $n = 2$ also included. Technical details of the calculation may be found in [4], but we give a brief summary here. In the range $(n-1)/(n+1) \leqq x \leqq n/(n+1)$ the distribution function $F_n(x)$ is given by a single polynomial, and we know that $F_n(n/(n+1)) = 1$. We then calculate $F_n((n-1)/(n+1))$, and if this value is less than .99, we know the 99 per cent point of ω_n lies in the range $(n-1)/(n+1) \leqq x \leqq n/(n+1)$. We solve the polynomial equation $F_n(x) = .99$ to obtain it. If $F_n((n-1)/(n+1))$ is less than .95 or .90, we obtain the corresponding per cent point by solving the equation $F_n(x) = .95$ or .90. If $F_n((n-1)/(n+1))$ is greater than .99, we determine $F_n(x)$ in the range $(n-2)/(n+1) \leqq x < (n-1)/(n+1)$ and calculate $F_n((n-2)/(n+1))$, thus determining which of the per cent points lies in this range. Proceeding in this manner we obtain, for each n, the three per cent points.

TABLE 1

99, 95, 90 *Percentiles of* ω_n

n	99	95	90
1	.49500	.47500	.45000
2	.60893	.53757	.48410
3	.61428	.51792	.46673
4	.58870	.50955	.46850
5	.57442	.50181	.46195
6	.56263	.49398	.45847
7	.55128	.48801	.45434
8	.54241	.48243	.45100
9	.53435	.47772	.44786
10	.52743	.47346	.44510
11	.52126	.46970	.44257
12	.51577	.46630	.44029
13	.51082	.46323	.43820
14	.50634	.46043	.43628
15	.50225	.45786	.43452
16	.49851	.45550	.43288
17	.49506	.45332	.43137
18	.49188	.45130	.42995
19	.48892	.44942	.42863
20	.48617	.44766	.42739

TABLE 2

99, 95, 90 *Percentiles of* $\omega_n^{(1)}$ *and* $\omega_n^{(2)}$

n	$\omega_n^{(1)}(99)$	$\omega_n^{(2)}(99)$	$\omega_n^{(1)}(95)$	$\omega_n^{(2)}(95)$	$\omega_n^{(1)}(90)$	$\omega_n^{(2)}(90)$
5	2.4723	1.9000	1.7228	1.2321	1.3114	0.8654
10	2.4417	2.0758	1.6969	1.3736	1.3055	1.0046
15	2.4238	2.1415	1.6874	1.4338	1.3004	1.0618
16	2.4206	2.1497	1.6861	1.4420	1.2997	1.0697
17	2.4181	2.1574	1.6848	1.4494	1.2991	1.0769
18	2.4158	2.1644	1.6837	1.4561	1.2985	1.0835
19	2.4137	2.1707	1.6827	1.4622	1.2980	1.0895
20	2.4117	2.1764	1.6817	1.4678	1.2976	1.0950
∞	2.3264	2.3264	1.6449	1.6449	1.2816	1.2816

The random variable ω_n has mean

$$E_n = \left(\frac{n}{n+1}\right)^{n+1} \to \frac{1}{e}$$

and variance

$$D_n^2 = \frac{2n^{n+2} + n(n-1)^{n+2}}{(n+2)(n+1)^{n+2}} - \left(\frac{n}{n+1}\right)^{2n+2} \sim \frac{2e-5}{e^2} \cdot \frac{1}{n},$$

and it is proved in [1] that $\omega_n^{(1)} = 1/D_n(\omega_n - E_n)$ tends to be normally distributed as $n \to \infty$. It is therefore also true that

$$\omega_n^{(2)} = \sqrt{\frac{e^2}{2e-5}} \cdot n\left(\omega_n - \frac{1}{e}\right)$$

tends to be normally distributed as $n \to \infty$. The rapidity with which this convergence occurs is indicated in Table 2 below, which gives the 99, 95, and 90 percentiles of $\omega_n^{(1)}$ and $\omega_n^{(2)}$ for $n = 5$, $n = 10$, n ranging from 15 to 20, and $n = \infty$, i.e., the corresponding percentiles of a normal variate with mean 0 and variance 1. The percentiles of $\omega_{20}^{(1)}$ are reasonably close to the limiting values but those of $\omega_{20}^{(2)}$ are not. In either case the convergence is slow. It appears that n would have to be greater than 100 before the 99, 95, 90 percentiles of $\omega_n^{(1)}$ and $\omega_n^{(2)}$ are within one per cent of the limiting normal values. We note that the percentiles of $\omega_n^{(1)}$ and $\omega_n^{(2)}$ are, respectively, decreasing and increasing to the limiting values.

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