ABSTRACTS OF PAPERS

(Abstracts of papers presented for the Cambridge, Massachusetts Meeting of the Institute, August 25–28, 1958.)

48. On the Relationship Algebra and the Association Algebra of the Partially Balanced Incomplete Block Design. JUNJIRO OGAWA, University of North Carolina.

A. T. James (1957) defined the so-called "relationship algebra of a design" and showed that the partition of the total sum of squares into partial sums of squares can be characterized by the structure of the relationship algebra. He constructed the relationship algebras and analyzed their algebraic structures for the randomized block, the latin-square and the balanced incomplete block design. The purpose of this note is to construct the relationship algebra for the partially balanced incomplete block design and analize its algebraic structure. The main result of this paper is that the second degree irreducible representations of the relationship algebra are completely determined by the irreducible (linear) representations of the association algebra defined by R. C. Bose. (Received June 2, 1958.)

49. Estimation of the Medians for Dependent Variables. OLIVE JEAN DUNN, University of California.

The problem considered in this paper is that of using non-parametric methods to estimate the unknown medians of two dependent variables. In various types of research, it is convenient to consider a sample of n individuals and to take measurements at two different times or at two different levels of treatment. These $2n$ measurements are then a sample of size n from a bivariate distribution. For independent variables, two confidence intervals of the classic type using order statistics may be used as simultaneous confidence intervals for the two medians by simply multiplying the two probabilities to obtain the new confidence level. In this paper it is shown that for two dependent variables these same confidence intervals may be used as a set with bounded confidence level. Comparisons are made on the basis of average length between these intervals and other joint intervals for the means of a bivariate normal distribution. It is also shown that the result of this paper does not generalize, at any rate in the most obvious way, to three or more dependent variables. (Received June 16, 1958.)

50. On the Problem of Incomplete Data. (Preliminary report) JUNJIRO OGAWA and BERNARD S. PASTERNACK, University of North Carolina.

Consider a sample of size n, $x_1, x_2, \cdots, x_n$, drawn from $N(\mu, \sigma^2)$ (σ^2 known). Suppose only $n - k$ observations are available, $x_1, x_2, \cdots, x_{n-k}$ (say). Let

$$\bar{x} = 1/n \textstyle\sum_{i=1}^{n} x_i, \ \bar{x}^* = 1/(n-k) \sum_{i=1}^{n-k} x_i, \ u = (\sqrt{n}\, \bar{x})/\sigma, \ u^* = (\sqrt{n-k}\, \bar{x}^*)/\sigma$$

and define $u(\alpha)$ by $P_{H_0}(|u| \leqq u(\alpha)) = 1 - \alpha$ and $u^*(\alpha)$ by $P_{H_0}(|u^*| \leqq u^*(\alpha)) = 1 - \alpha$, where $H_0{:}\mu = \mu_0$. We define $P_H\{|u| \leqq u(\alpha) \mid |u^*| > u^*(\alpha)\}$, α being prefixed, as the *reversal function* of this test procedure. The reversal function has been tabulated for various values of k/n. When σ^2 is unknown, the test procedure depends upon t, or, for incomplete data, t^*. A least upper bound for t given t^* has been obtained, i.e., the minimum value of τ such that $P_{H_0}(|t| > \tau \mid t^*) = 0$. Similar bounds (both l.u.b. and g.l.b.), in probability, have also been obtained for situations involving one-way classifications, the general linear model and Hotelling's T^2. Another approach to the problem of missing data involves the introduction of a chance mechanism according to which observations are missed. Research along these lines is now in progress and the authors hope to present some of these results in the near future. (Received June 20, 1958.)

51. Aids for Fitting the Pearson Type III Curve by Maximum Likelihood. (Preliminary report) J. ARTHUR GREENWOOD, Iowa State College and DAVID DURAND, M.I.T.

New tables and formulas of approximation are given for the function $\rho = y\phi(y)$, where $\phi(y)$ is the inverse function to $y = \ln\rho - d/d\rho \ln\Gamma(\rho)$. With the aid of these tables, one may obtain by direct interpolation the maximum likelihood estimate (joint) of the exponent in a Type III distribution with known lower limit. Application of the tables to the Type III with unknown lower limit and to the Type V are briefly discussed. (Received June 20, 1958.)

52. Admissible Estimates and Maximum Likelihood Estimates (Preliminary report) ALLAN BIRNBAUM, Columbia University.

A definition of admissibility of a point-estimate of a real-valued parameter θ is formulated on the basis of a slightly generalized form of the Neyman-Pearson theory of confidence regions, using *Ann. Math. Stat.*, Vol. 27 (1956), pp. 544–545, without introduction of loss functions. Necessary and sufficient conditions for existence of such estimates are given under mild regularity conditions. By extending methods developed in *Ann. Math. Stat.*, Vol. 26 (1955), pp. 21–36, it is shown that each admissible estimate is obtainable as the (unique) solution of an equation $\partial/\partial\theta \log L(x, \theta) = G(\theta)$, where $G(\theta)$ is a known function and $L(x, \theta)$ is the likelihood function. Setting $G(\theta) = 0$ gives the maximum likelihood estimate $\hat{\theta}$, which is thus shown to be admissible. In the case of non-existence of admissible estimates, asymptotically admissible estimates are defined and shown under certain conditions to exist and to include $\hat{\theta}$. An estimate $\tilde{\theta}$ is called median-unbiased if Prob $\{\tilde{\theta} \leqq \theta \mid \theta\} = \frac{1}{2}$ for all θ. $\hat{\theta}$ is shown under general conditions to be asymptotically median-unbiased, and to be a convenient approximation (often close for moderate sample sizes) to the median-unbiased admissible estimate (which is often difficult to compute). Relations to sufficiency and to multi-parameter estimation problems are discussed. (Received June 24, 1958.)

53. Stochastic Models for the Electron Multiplier Tube (Preliminary report) EDWARD K. DALTON, WILLARD D. JAMES AND HOWARD G. TUCKER, University of California.

Four stochastic models are proposed for the electron multiplier tube, two being branching processes involving Poisson distributions and two being branching processes involving binomial distributions. In each model there are two unknown parameters. It is desired to determine the best model among these and to estimate the parameters for it. Although the probability generating functions in each case are easy to derive, explicit formulas for the probability distributions in each case could not be found. A method for testing these models is presented which is based on the following theorem. THEOREM. *Let X be a random variable which takes on non-negative integer values, and let $X_1, \cdots, X_n, \cdots$ denote an infinite sequence of independent observations on X. Let $g(u \mid \alpha) = E(u^X)$ be the probability generating function of X which depends on a (vector) parameter α and is continuous in α. Let α_0 be the true value of α, and assume that there exists a sequence $\{\hat{\alpha}_n\}$ of random variables which converges to α_0 with probability one. Then for any value of u for which $u^2 - u \neq 0$ and $g(u^2 \mid \alpha) < \infty$ there exists a subsequence $\{\hat{\alpha}_{N_n}\}$ of $\{\hat{\alpha}_n\}$ such that the* *limiting distribution of the ratio of* $\sum\{u^{X_k} \mid 1 \leqq k \leqq n\} - ng(u \mid \hat{\alpha}_{N_n})$ *to either the square root of* $n\{g(u^2 \mid \hat{\alpha}_{N_n}) - g^2(u \mid \hat{\alpha}_{N_n})\}$ *or to the square root of*

$$n(n^{-1}\sum\{u^{2X_k} \mid 1 \leqq k \leqq n\} - (n^{-1}\sum\{u^{X_k} \mid 1 \leqq k \leqq n\})^2)$$

is normal with mean zero and variance one. A resumé of numerical results is included for three different sets of data corresponding to three different energy inputs. (Received June 26, 1958.)

54. On the Choice of Sample Size in the Kolmogorov-Smirnov Tests. Judah Rosenblatt, Purdue University.

If F_n is the empirical distribution based on independent random variables $X_1, \cdots, X_n$, with common c.d.f. F, it is well known that a test of the hypothesis H_0: $F = F_0$ having asymptotic probability of type one error not exceeding α is to reject H_0 if and only if $n^{1/2} d_1(F_n, F_0) \equiv n^{1/2} \sup_x |F_n(x) - F_0(x)| > h_{1\alpha}$, where

$$\lim_{n\to\infty} P_F\{n^{1/2} d_1(F_n, F) > h_{1\alpha}\} = \alpha$$

if F is continuous. Massey has suggested that the sample size n needed to achieve

$$P_F\{\text{Reject } H_0\} \geqq 1 - \beta \text{ when } d_1(F_0, F) \geqq l$$

be chosen as follows: n is the smallest integer such that $2[n^{1/2} l - h_{1\alpha}] \geqq \varphi_\beta$, where

$$\int_{-\infty}^{\varphi_\beta} (1/2\pi)^{1/2} e^{-t^2/2}\, dt = 1 - \beta.$$

This suggestion is motivated by the normal approximation to the binomial distribution. A thorough investigation is made of this suggested procedure, and a completely justified, still rather simple technique is devised for choosing n such that

$$P_F\{\text{Reject } H_0\} \geqq 1 - \beta \text{ when } d_1(F_0, F) \geqq l.$$

The investigation is in two parts. First a region (near $p = \frac{1}{2}$) is determined where

$$\sum_{\nu=0}^{[n(p+l)-n^{1/2}h_{1\alpha}]} \binom{n}{\nu} p^{\nu}(1-p)^{n-\nu}$$

takes on its minimum value. This, together with the Uspensky version of the normal approximation to the binomial (with correction and error term) leads to the justified procedure for choosing n with the desired properties. This n is not much larger than that suggested by Massey and is far smaller than the one derivable from Chebychev's inequality. (Received July 2, 1958.)

55. The Use of Sample Quasi-Ranges in Estimating Population Standard Deviation. H. Leon Harter, Wright Air Development Center.

The use of sample quasi-ranges in estimating the standard deviation of normal, rectangular, and exponential populations is discussed. For the normal population, the expected value, variance, and standard deviation of the rth quasi-range for samples of size n are tabulated for $r = 0(1)8$ and $n = (2r + 2)(1)100$. The efficiency of the unbiased estimate of population standard deviation based on one sample quasi-range is tabulated for the same values of r and n. Estimates based on a linear combination of two quasi-ranges are considered, and a method is given for determining the weighting factor which maximizes the efficiency. The most efficient unbiased estimates based on one quasi-range for $n = 2(1)100$ and on linear combinations of two adjacent quasi-ranges and of any two quasi-ranges ($r < r' \leqq 8$) for $n = 4(1)100$ are tabulated, along with their efficiencies. An example illustrates the use of these estimates. For rectangular and exponential populations, the most efficient unbiased estimates based on one quasi-range are tabulated, together with their efficiencies, also the bias when estimates which assume normality are used. (Received July 2, 1958.)

56. On a Limiting Distribution Due to Renyi. D. G. Chapman, University of Washington.

Let X be a real valued random variable with distribution function (d.f.) $F(x)$. Let $F_n(x)$ denote the emperical d.f. based on n independent observations $x_1, x_2, \cdots, x_n$ of X. Renyi ("On the theory of order statistics," *Acta Math.*, Acad. Sci. Hungary, Vol. 4 (1953), pp. 191–231) has given the limiting distribution of $n^{1/2}R_n(a) = n^{1/2} \sup_{F(x) \geqq a} [F_n(x) - F(x)]/F(x)$ as n tends to infinity, a being an arbitrary positive constant. It is therefore of interest to determine the limiting distribution of $R_n(0)$, i.e., without the arbitrary restriction $F(x) \geqq a$. The result is obtained that $P_r[R_n(0) \leqq \epsilon] = \epsilon/1 + \epsilon$ for all n, so that the limiting distribution of $R_n(0)$ has the same form. Also studied in this paper are the limiting distributions of some slight generalizations of $R_n(a)$. The method used is that due to Doob which is simpler than Renyi's and may also be used to determine the asymptotic power of the Smirnov test of goodness-of-fit for certain alternatives. (Received July 2, 1958.)

57. Power and Control of Size of Some Optimal Welch-type Statistics. Roger S. McCullough and John Gurland.

A Welch-type statistic (Welch, *Biometrika*, 1938) is considered for testing equality of means in two normal populations with unknown variances which may be unequal. For various combinations of small sample sizes a nearly perfect one-sided control of size is possible, that is, optimal statistics are available which keep the size extremely close to a preassigned level if one population has a larger variance than the other. For two-sided control of size, that is with no restriction on the direction of inequality of variances, optimal statistics are available which keep the size below a pre-assigned level but arbitrarily close to the level over an infinite range of variance values. A table giving the optimal statistics for various combinations of small sample sizes has been prepared with the aid of an electronic computer. Tables of the power are also included. (Received July 2, 1958.)

58. A Note on Estimating Translation and Scalar Parameters. Joseph A. Dubay, University of Oregon.

Let $X = (X_1, \cdots, X_n)$ be a random variable whose distribution depends on an unknown real valued parameter θ. Let $\delta(X)$ be an estimator of θ, Γ be the class of all maximal translation invariant functions of X and assume the loss in estimating θ by $\delta(X)$ is $k(\delta(X)-\theta)^2$. A necessary and sufficient condition that among all estimators of the form $\delta(X) + u\gamma(X)$, $\gamma \varepsilon \Gamma$, u constant, $\delta(X)$ uniquely minimize the risk is given and an explicit construction of the minimum risk estimator is derived therefrom. In the particular case where $\delta(X)$ has the translation property, the class of estimators of the form $\delta(X) + u\gamma(X)$ is the class of all estimators having the translation property. Thus, a construction of the minimum risk estimator having the translation property is exhibited of which the constructions given by Pitman (1939) and Blackwell and Girshick (1954) in the case where θ is a translation parameter are special cases. An example is given in which θ is not a translation parameter in the usual sense but estimators having the translation property are naturally admitted. Under an appropriate transformation the results are applicable to the estimation of scalar parameters. (Received July 2, 1958.)

59. The Moments of the Maximum of Partial Sums of Independent Random Variables. JOHN S. WHITE, Minneapolis-Honeywell Regulator Co.

Let $X_1, \cdots, X_n$ be independent identically distributed random variables. Let $S_k = \sum_{i=1}^{k} X_i$, $S_k^+ = \max(0, S_k)$, $\bar{S}_n - \max_{k \leq n} (S_k^+)$, $m_i(k) = E((S_k^+)^i)$ and $M_j(n) = E(\bar{S}_n^j)$. By successive differentiation of Spitzer's Theorem (*Trans. Amer. Math. Soc.*, Vol. 82, 1956) the following recursion relation for the moments of $\bar{S}_n$ is obtained:

$$E(\bar{S}_n^{j+1}) = M_{j+1}(n) = \sum_{k=1}^{n} \sum_{i=0}^{j} \binom{j}{i} [m_{i+1}(k)/k] M_{j-i}(n - k).$$

(Received July 2, 1958)

60. A Characterization of Triangular Association Scheme. S. S. SHRIKHANDE, University of North Carolina. (By title)

If a partially balanced design with two associate classes for $v = n(n-1)/2$ is triangular, (Bose and Shimamoto, *J. Amer. Stat. Assn.*, Vol. 47 (1952) pp. 151–190) then its parameters are given by $v = n(n-1)/2$, $n_1 = 2n - 4$, $p_{11}^1 = n - 2$, $p_{11}^2 = 4$. Connor (*Ann. Math. Stat.*, Vol. 29 (1958), pp. 262–266) has proved that if $n \geqq 9$, a design with above parameters is necessarily triangular. The following Lemma is established and it is utilized to prove that Connor's result is true for $n = 5, 6$ as well.

LEMMA: If for a design with above parameters, the 1-associates of any treatment x can be divided into two sets $(y_1, y_2, \cdots, y_{n-2})$, $(z_1, z_2, \cdots, z_{n-2})$ such that $(y_i, y_j) = (z_i, z_j) = (y_i, z_i) = 1$ and $(y_i, z_j) = 2$, $i \neq j = 1, 2, \cdots, n-2$, then the design is triangular. (Received July 2, 1958.)

61. A Problem in Two-Stage Experimentation. (Preliminary Report) DONALD L. RICHTER, University of North Carolina.

Let N_1 and N_2 be two normal populations with unknown variances and an unknown but common mean μ; it is desired to estimate μ using a fixed number n of observations. For this problem, a two-stage sampling procedure is proposed in which m observations are taken from each of N_1 and N_2 in the first stage and, depending on the observed values, $n - 2m$ observations are taken from one or the other population in the second stage. Associated with an estimator of μ, a risk function is defined which is equal to the variance of the estimator multiplied by a suitable stabilizing factor. For a particular unbiased estimator, it is shown that the minimax value of m is asymptotically equal to $cn^{2/3}$. Extensions in several directions are being studied. (Received July 2, 1958.)

62. Tests for the Validity of an Exponential Distribution of Life. BENJAMIN Epstein, Stanford University. (By title)

In this paper a number of procedures are given for testing, on the basis of life test data, whether there are substantial departures from an exponential distribution of life. The particular procedures that one should adopt depends on the class of alternatives one is testing against. A number of the tests are based in an essential way on fundamental properties of Poisson processes. (Received July 2, 1958.)

63. Stochastic Models for Length of Life. BENJAMIN EPSTEIN, Stanford University. (By title)

Various models for length of life are considered in this paper. Among these are (1) models which we call exponential (these involve Poisson processes and appropriate generalizations of such processes); (2) models based on the conditional probability of failure function; (3) extreme value models. Implications of and interrelations among the various models are discussed. Many examples are given. As examples of models (1) and (2) one may cite the recent paper by Z. W. Birnbaum and S. C. Saunders (*J. Amer. Stat. Assn.*, Vol. 53 (1958), pp. 151–160) in which they give a statistical model for the life length of structures under dynamic loading (i.e., fatigue) and a recent report by George H. Weiss in which it is shown that some kinds of mechanical failure, such as creep failure of oriented polymenic filaments under tensile stresses, can be viewed as "pure death" processes. An example of where model (3) may be relevant is in phenomena involving corrosion. (Received July 2, 1958.)

64. **Truncation and Tests of Hypotheses.** Om P. Aggarwal and Irwin Guttman, Purdue University and Princeton University.

Consider a normal distribution with variance σ^2 and a sample from the distribution obtained from this normal distribution by truncating it at the same distance a on both sides of the mean. The distribution of the sample mean for sample sizes up to 4 is obtained explicitly and the results of applying the usual tests of hypotheses for one-sided testing of the mean of a normal distribution are examined when a and σ^2 are known. Some tables are given and it is found that the loss in power decreases very rapidly with the distance of the alternative value of the mean from the one tested and also with the distance of the truncation from the mean. (Received July 2, 1958.)

65. **Mathematical Outline of Polyvariable Analysis (Including Random Balance).** F. E. Satterthwaite, Statistical Engineering Institute.

A polyvariable technique for statistical analysis is defined as any estimation procedure applied to the linear model, $Y = BZ + E = BZ + EI = AX$, $A = (B, E)$, $X = (Z, I)$, which gives estimates for all (or of some) of the A unknowns with associated confidence limits that are *valid* and *finite* without restrictions on the number of A unknowns in the model. Specifically the number of unknowns may exceed, and often will greatly exceed, the number of data sets. The theoretical minimum number of data sets is 2. The necessary minimum for a specific application to give useful precision depends primarily on the signal-noise ratio for the available data. In many types of applications satisfactory precisions are obtained with 5 to 30 data sets for models containing a large number of unknowns. This paper is a mathematical outline of method and justification (including, in most cases, formal proofs) for the more important classes of polyvariable methods: (1) Polygression, (2) Bigression, (3) Quadratic, (4) Homovariance, (5) Hetervariance, (6) Random Balance, (7) Split Data. (Received July 3, 1958.)

66. **Statistical Theory of Some Quantal Response Models.** Allan Birnbaum, Columbia University. (By title)

Let $V = (S_1, \cdots S_k)$, where S_g's are independent Bernoulli observations: Prob $\{S_g = 1\} = P_g(y)$, a known strictly-increasing function of the unknown real-valued parameter y, Prob $\{S_g = 0\} = Q_g(y) = 1 - P_g(y)$, for $g = 1, \cdots k$. If $P_g(y)$ depends on known parameters a_g, b_g, $\cdots$, whose values the experimenter may determine, these are called "design parameters." Fisher's (*Phil. Trans. Roy. Soc. London*, A, Vol. 222(1922), pp. 363–366) method in treating estimation and design problems in the dilution series model ($P_g(y) = 1 - \exp(-a_g y)$, $g = 1, \cdots k$) is formulated more explicitly, particularly the use of the practical equivalence of designs having similar information curves $I(y) = \sum I_g(y)$, where $I_g(y) = (\partial/\partial y P_g(y))^2/P_g(y)Q_g(y)$. The "information area" $\int I(y)\, dy$ is introduced and used in various design problems. Point- and interval-estimation, hypothesis-testing, and other inference problems, and related problems of design and comparison of experiments, are treated, using efficient or simpler less efficient statistics, with examples from mental tests, industrial gauging, genetics and special analytical bioassays. It is shown that a necessary and sufficient condition for existence of a sufficient statistics is that, in terms of $z = z(y) = \log P_1(y)/Q_1(y)$, the model have the logistic form:

$$P_g(y) = (1 + \exp(-a_g z - b_g)^{-1} \text{ for } g = 1, \cdots k;$$

then $\sum a_g S_g$ is sufficient. (Invited address given at Los Angeles meeting, December, 1957. Received July 7, 1958.

67. Statistical Theory of Tests of a Mental Ability. ALLAN BIRNBAUM, Columbia University. (Invited paper)

Several writers (F. Lord, *Psychometrika*, Vol. 18(1953), pp. 57–76, and references therein) have studied the following model of a mental-ability test consisting of k items: Let $S_g = 1$ or 0 as a subject's response to item g is correct or not, $g = 1, \cdots k$. Then if a subject has ability y, the probability that his response pattern will be $V = (S_1, \cdots S_k)$ is

$$\Pi_{g=1}^{k} P_g(y)^{S_g} Q_g(y)^{1-S_g},$$

where $P_g(y) = \Phi(a_g y - b_g)$, $g = 1, \cdots k$, and $\Phi(u)$ is the standard normal c.d.f. Assuming item-parameters a_g, b_g known, problems of inference and design (choice of k, a_g's, b_g's) have been treated, as have Bayesean problems with y distributed according to $\Phi(y)$. Replacing $\Phi(u)$ by the logistic c.d.f. $\Psi(u) = (1 + \exp(-u))^{-1}$ gives a more tractable, perhaps equally valid, "logistic test model": $t = \sum a_g S_g$ is a sufficient statistic, typically nearly normal for each y; hence a design $(a_1, b_1; \cdots; a_k, b_k)$ is practically characterized by its "information curve" $I(y) \equiv \partial/\partial y E(t \mid y) \equiv \text{var}(t \mid y)$. If $I(y) \doteqdot c\Psi(ay - b)$ for some a, b, c (as in cases of principal interest), properties of Bayes estimates $E(y \mid t)$ are given as functionals of the c.d.f. of a weighted sum of two independent (Fisher's) z variables; numerical illustrations are given. A simple efficient method of estimating a_g's, b_g's is given. (Received July 7, 1958.)

68. On Logistic Order Statistics. ALLAN BIRNBAUM, Columbia University. (By title)

Plackett (*Ann. Math. Stat.*, Vol. 29(1958), pp. 131–142) has demonstrated the usefulness and tractability of logistic order statistics in treating problems involving order statistics from various distributions. The present more descriptive investigation of logistic order statistics, a by-product of development of statistical theory of a "logistic model" of ability tests, is a contribution to the comparative study of order statistics initiated by Hastings et al. (*Ann. Math. Stat.*, Vol. 18(1947), pp. 413–426). Because with suitable choice of scale parameter the logistic c.d.f. approximates the standard normal c.d.f. with error $< .01$, the logistic model is of interest, and may be sometimes preferred, when equally plausible, to the more usual (but less tractable, as regards order statistics) normal model of the population sampled. The presentation illustrates the effect on order statistics of such a change of parametric assumptions. Tables and graphs compare means and variances of logistic and normal order statistics for various sample sizes. The tractability of asymptotic variance and covariance formulae, and of some distributions related to extreme values, is illustrated. The distribution of each logistic order statistic coincides (to within a scale-factor) with a certain Fisher's—z distribution, for which extensive tables and approximation methods are available. (Received July 7, 1958.)

69. Industrial Experience with Fractional Replicates. CUTHBERT DANIEL, (Invited paper)

Typical conditions of industrial experimentation (including numbers of factors simultaneously studied, number and magnitude of effects sought, and restrictions on time and costs) are reviewed. For meeting these, the sequential use of nested sub-fractions of fractional replicate designs in the 2^{p-q} series is described. Since generally choice of a most informative initial sub-fraction is incompatible with choice of a most informative complete fractional replicate, the relative merits of each type, and of intermediate types, are discussed. A number of sequential designs are given. Methods are recommended and illustrated for inspection and criticism of data from 2^{p-q} experiments by using the graph (on appropriate probability paper) of the empirical c.d.f. of absolute values of contrasts, to detect one or two mavericks, inadvertent plot-splitting, antilognormal data, and the presence of several real effects. The distribution of this c.d.f. is studied under several hypotheses, and the use is described of the operating characteristics of a related statistic given by A. Birnbaum. Partial duplication is recommended when an unbiased estimate of error variance is required at an early stage. (Received July 7, 1958.)

70. On the Analysis of Factorial Experiments without Replication. ALLAN BIRNBAUM, Columbia University. (By title)

Inferences from factorial experiments without replication are usually based on a formal assumption that certain interactions are zero. In an altogether exploratory research situation, any statistical model giving a formal basis for informative inferences will typically be too schematic and restrictive of unknown conditions to be claimed "valid," or a basis for inferences which are "valid" except in the hypothetical formal sense; such a model is, perhaps along with other models, a basis for "plausible inferences," i.e., inferences drawn in a formally-valid manner, based on a model which is more or less plausible. Under some conditions (which are reviewed), the following schematic model is usefully plausible: The m contrasts a_i are independent, normal, homoscedastic; at most (any) r of their means are non-zero. For $r = 1$, to decide which, if any, mean is non-zero, the statistic $\max_i a_i^2/\sum_i a_i^2$ is optimal. An alternative graphical procedure developed by C. Daniel, which has important advantages, is related to the ratio of $\max_i |a_i|$ to another ordered $|a_i|$. Critical values, power and related properties, comparisons with more conventional statistics, and discussion of cases $r > 1$, are given. (Received July 7, 1958.)

71. Linear Regression in the Multivariate Normal Case. CHARLES STEIN, University of California, Berkeley.

The problem of estimating the regression vector of one random variable on p others when all have a joint normal distribution is considered. There are $n \geqslant p + 2$ observations on the whole vector, the mean is assumed 0 for simplicity, and the loss is taken to be the mean squared error of prediction when the estimated regression vector is used to make a prediction on the basis of a new random observation on the predictors, divided by the residual variance. The usual (maximum likelihood) estimate of the regression vector is minimax. It is admissible for $p = 1$, $n \geqslant 4$ and for $p = 2$ and n sufficiently large. For $p \geqslant 3$ it is intuitively clear (by analogy with the problem of estimating the mean of a multivariate normal distribution) that the usual estimate is not admissible. One possible method of improvement is to multiply the usual estimate by a constant depending on the population multiple correlation coefficient, which can be estimated from the sample coefficient. This will be more helpful if a guessed regression or a regression on a small selected set of predictors is first subtracted out. Other possible improvements are suggested when the covariance matrix of the predictors is known. It should also be possible to make further improvements when the covariance matrix of the predictors is not known but guessed or estimated on the basis of an additional sample. (Received September 4, 1958).

72. Some Population Estimation Models and Related Limit Distributions. RONALD PYKE AND N. DONALD YLVISAKER, Stanford University.

The following two stage tag-and-sample model is studied. During stage I, $J + 1$ samples of sizes $m_0, m_1, \cdots, m_J$ are taken from a population of size S_1. In each sample, all untagged members are tagged and the sample replaced. During a later stage II, $K + 1$ samples of sizes $(n_0, n_1, \cdots, n_K)$ are taken in each of which all tagged members are tagged with a different tag than that used in stage I. The time interval between stages is assumed to be large relative to the time required to perform the tagging and sampling. Constant deterministic birth and death rates μ, and ρ are assumed during the intermediary time period. Maximum Likelihood estimates of S_1, μ and ρ are obtained under both Poisson and Binomial assumptions on the distribution of the recovery random variables (r.v.). Some general limit theorems are derived and applied to show that under a suitable reparametrization (corresponding to large sample and population sizes) the recovery r.v.'s and the Maximum Likelihood estimates are asymptotically normally distributed. A further generalization in which the sample sizes are assumed to be r.v.'s is considered. These results are then applied to data obtained from actual field experiments. (Received July 7, 1958.)

73. Applications of a certain Representation of the Wishart Matrix. ROBERT A. WIJSMAN, University of Illinois.

Let X be a $p \times n$ matrix ($p \leqq n$) whose columns are independent and distributed like $N(0, \Sigma)$. It is known (e.g., J. G. Mauldon, *J. Roy. Stat. Soc.*, Ser. B, Vol. 17 (1955) pp. 79–85) that the Wishart matrix XX' can be written as $CTT'C'$, where $CC' = \Sigma$, T is lower triangular with independent elements T_{ij}, T_{ii} is χ_{n-i+1} ($i = 1, \cdots, p$) and all T_{ij} with $i > j$ are $N(0, 1)$. This allows representation of any function of the Wishart matrix in terms of independent normal and χ variables. If the population correlation between two variates is ρ, the sample correlation r can be represented by $r/(1 - r^2)^{1/2} = (T_{21} + T_{11}\,\rho/(1 - \rho^2)^{1/2})/T_{22}$ (this representation was also obtained by G. Elfving, *Skand. Aktuarietids.*, Vol. 30 (1947), pp. 56–74). This can be described as a non-central t_{n-1} variable, with random non-centrality parameter $T_{11}\,\rho/(1 - \rho^2)^{1/2}$. If the population multiple correlation between one variate and the remaining $p - 1$ is $\bar{R}$, the sample multiple correlation R can be represented by $R^2/(1 - R^2) = ([T_{p1} + T_{11}\,\bar{R}/(1 - \bar{R}^2)^{1/2}]^2 + \sum_{i=2}^{p-1} T_{pi}^2)/T_{pp}^2$. This is a non-central $F_{p-1, n-p+1}$ variable, with random non-centrality parameter $T_{11}^2\,\bar{R}^2/(1 - \bar{R}^2)$. The sphericity criterion Z (T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*, Wiley, New York, 1958, section 10.7) in a bivariate population, when the hypothesis is true, can be represented by $Z/(1 - Z) = 2T_{11}T_{22}/((T_{11} - T_{22})^2 + T_{21}^2)$, which is an $F_{2n-2,2}$ variable. (Received July 7, 1958.)

74. Order Statistics and Estimation. M. M. RAO, University of Minnesota. (Introduced by Milton Sobel) (by title)

Let (1) $f(x) = e^{-x}$, if $x > 0$, and zero otherwise, and X_i be the ith order statistic from a sample of N independent observations from the population defined by (1). The following results are proved. (I): Let $1 \leqq r_1 < r_2 < \cdots < r_p \leqq N$ be a set of fixed integers and $X_{r_1}, X_{r_2}, \cdots, X_{r_p}$ be a p-set (subset) of the order statistics $X_1 < X_2 < \cdots < X_N$. Then the order statistics define a Stochastic Process with $r_1, r_2, \cdots$, as the parameter set, which has independent (but not stationary) increments. The finite-dimensional distributions of the process in terms of its log characteristic function are given by $\psi_{r_1, r_2, \ldots, r_p}(\xi_1, \xi_2, \cdots \xi_p)$

$= \log \varphi_{r_1, r_2, \dots, r_p}(\xi_1, \xi_2, \cdots \xi_p) = \log N!/(N - r_p)! - \sum_{j=0}^{p-1} \sum_{m=r_j}^{r_{j+1}-1} \log (N - i\eta_{j+1} - m)$ where $\eta_j = \sum_{t=j}^{p} \xi_t$, $r_0 = 0$, and φ is the ch.f. (II): Let X_{r_i} and r_i be defined as in (I). Then $\{X_{r_i}, 1 \leqq r_i \leqq N\}$ forms a Markov Process in the strict sense as well as in the wide sense (in either case, the Process is non-stationary). (III): Some problems of interest (in physiological data) are the following: (i) The r_i are random but N is fixed. Suppose Prob $\{r_k = i_k, k = 1, 2, \cdots, p \mid r_i < r_{i+1}, i = 1, 2, \cdots p - 1\} = p_{i_1, i_2, \dots, i_p}$, and $p_{i_1, \dots, i_p} \geqq 0$, $\sum_{i_1, \dots, i_p} p_{i_1, \dots, i_p} = 1$, where $(i, < i_2 < \cdots < i_p) = 1, 2, \cdots N$, $i_0 = 0$, and the p's depend on a set of constants, $(\lambda_1, \lambda_2, \cdots, \lambda_k)$. Then the X_{r_i} defined similar to those in (I) still form a Stochastic Process whose finite-dimensional d.f.'s are determined by the ch.f. $\varphi(\xi_1, \cdots, \xi_p; \lambda_1, \cdots \lambda_k) = \sum_{i_1, \dots, i_p} p_{i_1, i_2, \dots, i_p} [\prod_{m=1}^{i_p} (N - m + 1)/\prod_{j=1}^{p} \prod_{m=i_{j-1}}^{i_{j+1}} (N - i\eta_j + m)]$. (ii) The case when N is a random variable. Specifying the d.f.'s in some cases of interest for the r_i, the limit d.f.'s of some linear combinations of X_i are considered. The estimation of the constants (A, θ) and the distribution of the $(\hat{A}, \hat{\theta})$ are treated using the above results when in (1) x is replaced by $(x - A)/\theta$, $\theta > 0$. (Received July 7, 1958.)

75. A Note on Order Statistics and Stochastic Independence.

GERALD S. ROGERS, University of Arizona.

The following theorem is proved. Let x be a continuous or discrete type real random variable. Let $x_1 \leqq \cdots \leqq x_n$ be the order statistics based on a random sample of size n from this x distribution. Let $z = z(x_1, \cdots, x_j)$ be a statistic based on the first $j < n$ items only. If z is stochastically independent of x_j, then z is stochastically independent of all x_k, $j < k \leqq n$; if z is stochastically independent of some x_k, $j < k \leqq n$, then z is stochastically independent of x_j and hence of all x_k, $j \leqq k \leqq n$. The first result is direct, since in terms of the conditional probability density functions, $g(z \mid x_j) = g(z \mid x_j, \cdots, x_n)$. For the second part, in $g(x_1, \cdots, x_{k-1} \mid x_k)$, let x_k be considered as a "parameter." Then, $(x_{k-1} \mid x_k)$ is a "complete single sufficient statistic" for x_k; also, the distribution of $(z \mid x_k)$ is free of the "parameter x_k." By a well known theorem, (Basu, *Sankhya*, Vol. 15 (1955), pp. 377–380), $(z \mid x_k)$ and $(x_{k-1} \mid x_k)$ are stochastically independent. It follows that z and x_{k-1} are stochastically independent; similarly, with an induction, z and x_k, $j \leqq k \leqq n$, are stochastically independent. (Received July 7, 1958.)

76. A model for Failure Data and its Applications. (Preliminary report)

ANDRE G. LAURENT, Wayne State University.

When a "ageing process" takes place, the response pattern of a "system" to a stimulus X does not follow an exponential distribution. The model $S(t) = \exp [1 + t - \exp (t)]$, where $S(t)$ is the "survival function," i.e., the integral of the "X-to-failure" distribution and $t = (X - X_0)/\tau$, has been proposed to meet this situation and tables provided for its use (*Oper. Res.*, February, 1957, p. 150; *Oper. Res. 13th National Meeting*, p. 35.) The present paper describes the more important features of the model above and gives the formulas for the expected values and the covariance matrix of the order statistics of a sample of size n. Tables of the expected values and the variances for $n = 1$ to 15, of the covariances for $n = 2$ to 5 are provided. The minimum variance linear unbiased estimates of the parameters of the distribution based on order statistics are studied for small samples and compared to other estimates from the viewpoint of efficiency. Related models are considered. (Received July 7, 1958.)

77. A Convolutive Class of Monotone Likelihood Ratio Families.

S. G. GHURYE AND DAVID L. WALLACE, University of Chicago.

A one-dimensional family $f(x, \theta)$ of densities on the real line or of probabilities on the integers, with the real parameter θ, is called a monotone likelihood ratio family if the ratio

$f(x, \theta')/f(x, \theta)$ is nondecreasing in x for $\theta \leqq \theta'$. If several monotone likelihood ratio families each have all probability on two points which are the same for all families and all parameter values, then their convolution is a monotone likelihood ratio family. The extent to which similar results hold for distributions on three and more points and, with appropriate extensions of definitions, for multidimensional distributions on the vertices of the simplex and the cube is determined. A sufficient condition that the convolution of monotone likelihood ratio families be a monotone likelihood ratio family is that for each family, the ratio $f(x + h, \theta)/f(x, \theta)$ be non-increasing in x for all $h > 0$. (Received July 7, 1958.)

78. On the Exact Joint Distribution of the First Two Serial Correlation Coefficients. V. K. MURTHY, University of North Carolina.

Any test of the hypothesis that up to a particular lag the true serial correlation coefficients are zero against some suitable alternative seems to necessitate knowledge of the joint distribution of serial correlation coefficients. As far as the author is aware even in the case of the first two serial correlation coefficients, the joint distribution has not so far been obtained in a simple closed form. In this note the joint distribution of r_1 and r_2 has been obtained for samples of independent normal variates assuming the sample size to be of the form $4n + 1$ where n is a positive integer and adopting the circular definition suggested by Hotelling. This result has been obtained using a result of R. L. Anderson on the characteristic roots of the serial covariance, and inversion formulae for the distribution of ratios of quadratic forms given by Gurland. Some properties of the joint distribution are obtained. The case of more than two serial correlation coefficients will be dealt with in a subsequent paper. (Research under ONR contract Nonr 855(06)). (Received July 7, 1958; revised July 28, 1958.)

79. Confidence Bounds Associated with a Test for Symmetry. R. GNANADESIKAN, The Procter and Gamble Company.

In a p-variate nonsingular normal distribution $N[\mu, \Sigma]$, one may be interested in testing a hypothesis of symmetry in the means, viz., that the p variates have the same mean. The tests obtained by using either the extended Type I union-intersection principle or the likelihood ratio are identical and it is well known that they are equivalent to an F-test with appropriate degrees of freedom. However, from the standpoint of confidence procedures, it is shown that the usual elliptical region can be replaced by simultaneous interval statements on parametric functions which are measures of departure from the null hypothesis. Also using a "truncation" procedure it is shown that one can study contrasts which are of particular interest and are components of the null hypothesis. The interval statements, which have a joint confidence coefficient $\geqq (1 - \alpha)$, are easier to use than the elliptical regions which have an exact confidence coefficient $(1 - \alpha)$. Received July 7, 1958.)

80. On Stochastic Approximation. C. DERMAN AND J. SACKS, Columbia University.

A very general theorem was proved by Dvoretzky ("On stochastic approximation", *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*) on the convergence, with probability one and in mean square, of stochastic approximation procedures. Wolfowitz ("On stochastic approximation methods," *Ann. Math. Stat.*, Vol. 27, 1956) presented a different proof. In this paper a third and simpler proof of the probability one convergence is given. Also, the probability one version is extended directly to the multi-dimensional case with absolute values of real numbers replaced by lengths of vectors. The one-dimensional theorem is a consequence of the following easily proved lemma. *If* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\delta_n\}$ *and* $\{\xi_n\}$ *are sequences of real numbers satisfying the following conditions*: (i) $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *are positive*, (ii) $\{\xi_n\}$ *are non-negative*, (iii) $\lim a_n = 0$, $\sum b_n < \infty$, $\sum c_n = \infty$, $\sum \delta_n < \infty$, (iv) $\xi_{n+1} \leqq \max (a_n, (1 + b_n)\xi_n + \delta_n - c_n)$ *for all n greater than some N*, *then* $\lim \xi_n = 0$. The multi-dimensional theorem follows from a slightly modified version of the above lemma. (Received July 7, 1958.)

81. A Classification Problem. Oscar Wesler, University of Michigan.

The following version of "the problem of the k-faced die" is considered: Nature's pure strategies make up two sets of states, Ω_1 consisting of the $k!$ states got by permuting a known probability distribution $p = (p_1, p_2, \cdots, p_k)$ over the faces $1, 2, \cdots, k$ of a k-faced die, Ω_2 consisting similarly of the $k!$ states arising from a known distribution $q = (q_1, q_2, \cdots, q_k)$. Classification is made on the basis of N observations given by the sufficient statistic $r = (r_1, r_2, \cdots, r_k)$ representing the number of times each face appears. Let φ be a randomized statistical decision procedure, and let $\alpha(\varphi)$ and $\beta(\varphi)$ be the maxima of the probabilities of errors of the first and second kind, respectively. Then we wish to minimize $\beta(\varphi)$ subject to $\alpha(\varphi) = \alpha_0$. The class of unique symmetric procedures φ^* optimal in this extended Neyman-Pearson sense is found by a game-theoretic, minimax method, and from the invariance of the problem under the symmetric group of permutations on k letters. A simplification is given for large N, in which the φ^* are replaced by *kaleidoscopic* tests, determined by a one-parameter family of hyperplanes and their symmetric images. Finally, it is shown that, for $k = 2$, the φ^* and the kaleidoscopic approximations are in exact agreement for every N. (Received July 7, 1958.)

82. Generalization of Palm's Loss Formula for Telephone Traffic. V. E. Beneš, Bell Telephone Laboratories, Inc.

Let F be a real non-negative function on a space X, let $\mathfrak{F}$ be a Borel field of X-subsets, and let ξ_k, $k = 0, 1, 2, \cdots$ be a stationary Markov process taking values in X, with transition function $p(\xi, A)$ for ξ in X and A in $\mathfrak{F}$. We interpret the numbers $F(\xi_k)$ as the inter-arrival times of telephone calls at a trunk group. There are N trunks, lost calls are cleared, and holdingtimes of trunks are independent, with a negative exponential distribution of mean, γ^{-1}. We prove the following result: If P is the stationary probability measure of ξ_k, then the chance of loss (of finding all N trunks busy) is $\left[\sum_0^N \binom{N}{n} A_n\right]^{-1} A_0 P(X)$, with $A_N = I$ and $A_n = K_N[I - K_N]^{-1} \cdots K_{n+1}[I - K_{n+1}]^{-1}$, where K_n is the operator whose action on a measure μ is defined by $K_n \mu(A) = \int X \int A \exp\{-n\gamma F(\xi)\} p(\eta, d\xi)\mu(d\eta)$. Palm's formula applies to the case $X = (0, \infty)$, $F(\xi) \equiv \xi$, ξ_k independent. Our formula has the same algebraic form as Palm's, but the multiplicative constants have been replaced by operators. The inverses indicated in our formula exist under weak hypotheses. (Received July 14, 1958.)

83. Factorial Analysis of Life-Tests. Marvin Zelen, National Bureau of Standards.

Consider a factorial experiment involving the factors A and B having levels a and b respectively. Let a life-test experiment be planned such that n items are tested for each of the ab factorial combinations and the test is terminated when exactly r ($r \leqq n$) of the test items have failed. Assume that the underlying distribution of failures for the (i, j) factorial combination ($i = 1, 2, \cdots, a$; $j = 1, 2, \cdots, b$) is $p(x_{ij}) = \theta_{ij}^{-1} \exp[-(x_{ij} - A_{ij})\theta_{ij}^{-1}]$ for $x_{ij} \geqq A_{ij}$, where $\theta_{ij} = ma_i b_j c_{ij}$. Maximum likelihood estimates are found for the parameters m, a_i, b_j, and c_{ij}. Likelihood ratio tests are given for testing various hypotheses for these parameters as well as approximations for the small sample distribution of these tests. (Received July 14, 1958.)

84. **Unbiased Estimation for Functions of Location and Scale Parameters.** R. F. TATE.

Integral transform theory is employed to obtain unbiased estimators (which in many cases have the minimum variance property) for functions of a location parameter θ and/or a scale parameter σ. Applications are made to the gamma distribution with parameters considered together and separately, and to truncated distributions in general. A simple formula is presented for estimating any differentiable function of a single location parameter of truncation; no calculation of distributions or conditional expectations is required in order to find a minimum variance unbiased estimator. Special attention is paid throughout the paper to the estimation of the functions $P(X \varepsilon A \mid \theta)$, $P(X \varepsilon A \mid \sigma)$, and $P(X \varepsilon A \mid \theta, \sigma)$, where A is an arbitrary Borel set. (Received July 21, 1958; revised July 25, 1958.)

85. **Theory of Successive Two-Stage Sampling.** (Preliminary report) B. D. TIKKIWAL (By title)

The general theory of Univariate Sampling on Successive Occasions have been studied by the author [J. Ind. Soc. Agric. Stat., Vol. 8 (1956), pp 84–90] under a specified sampling scheme and correlation pattern. Here the sampling units selected for study on various occasions are completely enumerated. The present paper gives the best estimator and its variance under the same sampling scheme when each of the primary units (assumed of the same size M) are not completely enumerated but observed only on a sub-sample of size m. It is shown that the form of the best estimator is the same as in the univariate case, when the pattern of correlation is the same at both the stages. It is further noted, that, for an infinite population and $M = \infty$, the variance of the best estimator on the hth occasion is given by ϕ_h/n_h''. V in the notations of the above paper and where V is the variance of the simple two stage sampling mean when only one primary unit is selected on the hth occasion. (Received August 1, 1958.)

86. **Functions of Markov Chains** (Preliminary Report), MURRAY ROSENBLATT, Indiana University.

Let $\bar{X}_n$, $n = 0, 1, \cdots$ be a Markov Chain with initial distribution $w_i = P[\bar{X}_o = i]$ and stationary transition probability matrix $P = (p_{ij})\ i, j = 1, 2\ \cdots$. Let $\bar{Y}_n = f(\bar{X}_n)$ and let S_α, $\alpha = 1, 2, \cdots$, be the sets of states of $\bar{X}_n$ that f collapses into states of $\bar{Y}_n$. Let class one consist of those sets of states into which one has access with positive probability from at most one set of states. Class two is the complementary class of states. A necessary and sufficient condition that $\bar{Y}_n$ be Markovian (for a fixed f), whatever the initial distribution w_i of the $\bar{X}_n$ process, is given as follows: (i) If S_α belongs to class two, $\sum_{j\varepsilon S_\alpha} p_{ij}\, p_{j,S_\beta} = p_{i,S_\alpha} C_{S_\alpha, S_\beta}$ for all i, β. Here $p_{i,S_\alpha} = \sum_{j\varepsilon S_\alpha} p_{ij}$. (ii) Given any sequence of sets of states S_1, S_2, $\cdots$, S_n where S_1 is of class two and $S_2, \cdots, S_{n-1}$ of class one, $\sum_{j\varepsilon S_{n-1}} p_{ij}^{(n-1)}\, p_{j,S_n} = p_{i,S_1} c^{(n-1)}_{S_{n-1}, S_n}$ for all i if there is positive probability of the path $S_1 \rightarrow S_2 \rightarrow \cdots \rightarrow S_n$. (Received September 8, 1958.)