THE NUMERICAL EVALUATION OF CERTAIN MULTIVARIATE NORMAL INTEGRALS

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0. Summary. As has been noted by several authors, when a multivariate normal distribution with correlation matrix $\{\rho_{ij}\}$ has a correlation structure of the form $\rho_{ij} = \alpha_i \alpha_j$ $(i \neq j)$, where $-1 \leqq \alpha_i \leqq +1$, its c.d.f. can be expressed as a single integral having a product of univariate normal c.d.f.'s in the integrand. The advantage of such a single integral representation is that it is easy to evaluate numerically. In this paper it is noted that the n-variate normal c.d.f. with correlation matrix $\{\rho_{ij}\}$ can always be written as a single integral in two ways, with an n-variate normal c.d.f. in the integrand and the integration extending over a doubly-infinite range, and with an $(n - 1)$-variate normal c.d.f. in the integrand and the integration extending over a singly-infinite range. We shall show that, for certain correlation structures, the multivariate normal c.d.f. in the integrand factorizes into a product of lower-order normal c.d.f.'s. The results may be useful in instances where these lower-order integrals are tabulated or can be evaluated. One important special case is $\rho_{ij} = \alpha_i \alpha_j$, previously mentioned. Another is $\rho_{ij} = \gamma_i/\gamma_j$ $(i < j)$, where $|\gamma_i| < |\gamma_j|$ for $i < j$. Some applications of these two special cases are given.

1. Introduction. Let $Z_1, Z_2, \cdots, Z_n$ be n standardized normal variates with correlation coefficients ρ_{ij}, and denote their joint frequency function by $f(x_1, x_2, \cdots, x_n)$. The c.d.f. (cumulative distribution function) is defined as

$$\begin{aligned} F_n(h_i; \{\rho_{ij}\}) &= \text{Prob}\{Z_i < h_i; \text{all } i\} \\ &= \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} \cdots \int_{-\infty}^{h_n} f(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n . \end{aligned} \tag{1.1}$$

Here and in the sequel, unless otherwise stated, it is to be understood that the suffixes i and j range from 1 to n. We assume that the matrix $\{\rho_{ij}\}$ is non-singular and, without any loss in generality, that no one group of variates is independent of all the others. (If such a group existed, then the n-fold integral in (1.1) would split into a product of lower order integrals and the dimension of the problem could be reduced.)

The values of integrals of the form (1.1) are required in many applications. Unfortunately, it has been tabulated only for $n = 1$ and 2, with the exception of some very special cases, such as the tables computed by the National Bureau of Standards (see Teichroew [19]) and by Gupta [7] for the case $h_i = h$ and $\rho_{ij} = \frac{1}{2}$. In fact, with $\frac{1}{2}n(n + 1)$ parameters involved in (1.1), a comprehensive tabulation

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hardly seems feasible for $n > 2$, although Steck [17] has provided some tables which simplify its computation for $n = 3$. Hence the development of methods for evaluating the multivariate normal integral in (1.1) would be highly desirable. (For one such method, see Plackett [14] and for a recent review of methods for evaluating multivariate normal integrals, see Gupta [8].)

In certain special cases, the integral (1.1) can be reduced to a single integration. For example, suppose the correlation matrix $\{\rho_{ij}\}$ has the structure $\rho_{ij} = \alpha_i \alpha_j$ $(i \neq j)$, where $-1 \leqq \alpha_i \leqq +1$. Then the variates Z_i can be generated from $n + 1$ independent standard normal variates $X_1, X_2, \cdots, X_n$; Y as follows. Let

$$Z_i = (1 - \alpha_i^2)^{\frac{1}{2}} X_i + \alpha_i Y. \tag{1.2}$$

Then the Z_i are normally distributed with zero means, unit variances and correlation coefficients $\rho_{ij} = \alpha_i \alpha_j$. Since the X's are mutually independent,

$$\begin{aligned} F_n(h_i ; \{\rho_{ij}\}) &= \int_{-\infty}^{+\infty} \text{Prob}\,\{X_i < (h_i - \alpha_i y)/(1 - \alpha_i^2)^{\frac{1}{2}}; \text{all } i\}\, f(y)\, dy \\ &= \int_{-\infty}^{+\infty} \left[\prod_{i=1}^{n} F((h_i - \alpha_i y)/(1 - \alpha_i^2)^{\frac{1}{2}})\right] f(y)\, dy, \end{aligned} \tag{1.3}$$

where $f(t) = \exp(-\frac{1}{2}t^2)/(2\pi)^{\frac{1}{2}}$ and $F(t) = \int_{-\infty}^{t} f(t)\,dt$. This expression was derived, in either this form or some special case of it, by Dunnett and Sobel [6], Kendall [10], Ruben [15], Moran [11] and Stuart [18]. It has also been considered by Ihm [9]. Das [2] obtained a more general result by generating the n variates Z_i by $Z_i = (1 - \sum_{s=1}^{k} \beta_{is}^2)^{\frac{1}{2}} X_i + \sum_{s=1}^{k} \beta_{is} Y_s$, where $X_1, X_2, \cdots, X_n$; $Y_1, Y_2, \cdots, Y_k$ are $n + k$ independent standard normal variates. In this case, $\rho_{ij} = \sum_{s=1}^{k} \beta_{is} \beta_{js}$ $(i \neq j)$ and therefore, for a correlation matrix having this structure,

$$\begin{aligned} F_n(h_i ; \{\rho_{ij}\}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} &\left[\prod_{i=1}^{n} F\left(\frac{h_i - \sum_{s=1}^{k} \beta_{is} y_s}{\left(1 - \sum_{s=1}^{k} \beta_{is}^2\right)^{\frac{1}{2}}}\right)\right] \\ &\cdot f(y_1) f(y_2) \cdots f(y_k)\, dy_1\, dy_2 \cdots dy_k . \end{aligned} \tag{1.4}$$

Stuart [18] derived this result when β_{is} is non-zero for only one s for each i and therefore $F_n(h_i ; \{\rho_{ij}\})$ is equal to the product of integrals of the form given on the right-hand side of (1.3). Of course (1.3) is a special case of (1.4) in which $k = 1$.

The advantage of having a single integral representation like (1.3) is that it is very easy to evaluate numerically by replacing the integral by a sum, such as that given by Simpson's Rule, or, when applicable, by a series employing the roots of Hermite polynomials as described by Salzer *et al* [16]. In fact, (1.3) was used in computing the tables for $\rho_{ij} = \frac{1}{2}$ previously referred to [19] and it was also used by Dunnett and Lamm [5] in computing some tables for $\rho_{ij} = \frac{1}{3}$. The accuracy of such approximations has been studied by Moran [11], [12] and by Das [2]. The usefulness of the more general result in (1.4) is likely to be limited to small values of k.

In this paper, it will be shown that (1.3) can be generalized in another direction. The n-variate normal integral can always be written as a single integral with the integrand containing another n-variate normal integral (see (2.3) below), or an $(n - 1)$-variate normal integral (see (2.4) below). We shall show that, for certain correlation structures $\{\rho_{ij}\}$, the multivariate normal integral factorizes into a product of lower-order normal integrals. The results may be useful in instances where these lower-order integrals are tabulated or can be evaluated. An important application, which falls out as a special case of our results, is the structure $\rho_{ij} = \gamma_i/\gamma_j \,(i < j)$, where $|\gamma_i| < |\gamma_j|$ for $i < j$. In this case, the integrand of the single-integral expression for (1.1) contains the product of a univariate and an $(n - 2)$-variate normal integral. Since in this case it turns out that the correlation structure of the $(n - 2)$-variate normal integral in the integrand is of the same form as that of the original integral, this result can be applied repeatedly until, finally, the n-dimensional integral is reduced to either a $\frac{1}{2}n$-dimensional integral (for n even) or a $\frac{1}{2}(n - 1)$-dimensional integral (for n odd) involving only univariate normal integrals in the integrand. This result should be useful in practice for small values of n. Some applications of the two cases $\rho_{ij} = \alpha_i\alpha_j$ and $\rho_{ij} = \gamma_i/\gamma_j$ are given in the final section of the paper.

2. Generating the variates $Z_1, Z_2, \cdots, Z_n$. To obtain our more general results, we shall generate the variates Z_i as in (1.2) but without restricting the variates $X_1, \cdots, X_n$ to be uncorrelated. We consider

$$X_i = (Z_i - \delta_i Y)/(1 - \delta_i^2)^{\frac{1}{2}}, \tag{2.1}$$

where the Z_i are as defined in (1.1) and where Y is a standard normal variate with cov $(Z_i, Y) = \delta_i \,(-1 \leqq \delta_i \leqq 1)$. Then $X_1, \cdots, X_n$ are normally distributed, independently of Y, with zero means, unit variances and

$$\text{cov}\,(X_i, X_j) = \frac{\rho_{ij} - \delta_i\,\delta_j}{(1 - \delta_i^2)^{\frac{1}{2}}(1 - \delta_j^2)^{\frac{1}{2}}} = \rho'_{ij}, \quad \text{say}, \quad (i \neq j). \tag{2.2}$$

We shall distinguish between two sets of results. In the first, $\delta_i \neq 1$ for any i and, in the second, $\delta_i = 1$ for some $i = k$ and hence $Y = Z_k$. These lead, respectively, to a doubly-infinite and to a singly-infinite integral representation of (1.1).

Consider first $\delta_i \neq 1$ for any i. From (2.1) we can write

$$F_n(h_i\,; \{\rho_{ij}\}) = \text{Prob}\,\{X_i < (h_i - \delta_i Y)/(1 - \delta_i^2)^{\frac{1}{2}};\ \text{all } i\}$$

and, since the X_i are distributed independently of Y, we have

$$F_n(h_i\,; \{\rho_{ij}\}) = \int_{-\infty}^{+\infty} F_n((h_i - \delta_i\, y)/(1 - \delta_i^2)^{\frac{1}{2}}; \{\rho'_{ij}\})f(y)\,dy \tag{2.3}$$

where ρ'_{ij} is as defined in (2.2).

Now suppose $\delta_k = 1$ for some k. Without any loss in generality, we shall take $\delta_1 = 1$, so that $Y = Z_1$ and $\delta_i = \rho_{i1}$. Then we obtain from (2.3) the well-known result

$$(2.4) \qquad F_n(h_i\,;\{\rho_{ij}\}) = \int_{-\infty}^{h_1} F_{n-1}((h_i - \rho_{i1}\,y)/(1-\rho_{i1}^2)^{\frac{1}{2}};\{\rho_{ij\cdot 1}\})\,f(y)\,dy,$$

where $\rho_{ij\cdot 1} = (\rho_{ij} - \rho_{i1}\rho_{j1})/(1-\rho_{i1}^2)^{\frac{1}{2}}(1-\rho_{j1}^2)^{\frac{1}{2}}\ (i, j \neq 1)$. This is equivalent to removing a single variate out of the frequency function $f(x_1, x_2, \cdots, x_n)$.

Any multivariate normal c.d.f. can be expressed in either of the forms (2.3) and (2.4). As a special case of (2.4), we have the bivariate normal integral

$$(2.5) \qquad F_2(h_1, h_2\,;\rho) = \int_{-\infty}^{h_1} F((h_2 - \rho y)/(1-\rho^2)^{\frac{1}{2}})f(y)\,dy.$$

This expression was used by Dr. Evelyn Fix to extend K. Pearson's tables of the bivariate normal distribution (see [13]); it was also used by Dunnett [3] to compute tables for the case $\rho = 1/\sqrt{2}$. An alternative expression for this bivariate normal integral is given in (3.4) below.

In the following sections, we consider conditions for the multivariate normal c.d.f. in the integrand of (2.3) or (2.4) to factorize into a product of lower-order integrals.

3. The doubly-infinite integral representation. In this section, we consider the doubly-infinite integral representation (2.3). If $\rho_{ij} = \alpha_i\,\alpha_j\ (i \neq j)$, then by substituting $\delta_i = \alpha_i$ we obtain $\rho'_{ij} = 0\ (i \neq j)$ and the result (1.3) follows. More generally, suppose we have the following correlation structure,

$$(3.1) \qquad \{\rho_{ij}\} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{M}_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \mathbf{O} \\ \mathbf{O} & \cdot & \cdot\,\cdot\,\mathbf{O} & \mathbf{M}_k \end{pmatrix} + \boldsymbol{\alpha}'\boldsymbol{\alpha},$$

where $\mathbf{M}_i$, $i = 1, 2, \cdots, k$, is a covariance matrix of rank r_i ($\sum_{i=1}^k r_i = n$), $\mathbf{O}$ denotes a null matrix and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. Then by taking $\delta_i = \alpha_i$ in (2.3), the integrand will factor into a product of k multivariate normal c.d.f.'s of order $r_1, r_2, \cdots, r_k$ respectively. Ihm [9] has also considered the correlation matrix (3.1). The case $\rho_{ij} = \alpha_i\,\alpha_j\ (i \neq j)$ corresponds to $k = n$ and $r_1 = r_2 = \cdots = r_n = 1$.

With three exceptions to be mentioned later, (3.1) implies relationships among the correlation coefficients of the type $\rho_{13}\rho_{24} = \rho_{14}\rho_{23}$. If relationships of this form exist, it may be possible to reduce the dimension of the distribution function in the integrand of (2.3) by an appropriate choice of the δ's. This may be particularly useful for small values of n. For example, take $n = 4$ and suppose that $\rho_{13}\rho_{24} = \rho_{14}\rho_{23}$. Then values α_1, α_2, α_3 and α_4 can be found such that $\rho_{13} =$

$\alpha_1\alpha_3$, $\rho_{14} = \alpha_1\alpha_4$, $\rho_{23} = \alpha_2\alpha_3$ and $\rho_{24} = \alpha_2\alpha_4$, and the structure of the correlation matrix is

$$\{\rho_{ij}\} = \begin{Bmatrix} 1 & \rho_{12} & \alpha_1\alpha_3 & \alpha_1\alpha_4 \\ \rho_{12} & 1 & \alpha_2\alpha_3 & \alpha_2\alpha_4 \\ \alpha_1\alpha_3 & \alpha_2\alpha_3 & 1 & \rho_{34} \\ \alpha_1\alpha_4 & \alpha_2\alpha_4 & \rho_{34} & 1 \end{Bmatrix}.$$

Then by taking $\delta_i = \alpha_i$ ($i = 1, 2, 3, 4$) in (2.3), the integrand will contain a product of two bivariate distribution functions with correlation coefficients ρ_{12}' and ρ_{34}'.

The three exceptions to the need for relationships among the correlation coefficients are as follows,

(a) $k = 1$, $r = n$. In this case, $\rho_{ij} = m_{ij} + \alpha_i\alpha_j$ and by taking $\delta_i = \alpha_i$ in (2.3) the n-variate normal c.d.f. with correlation matrix $\{\rho_{ij}\}$ is expressed as an integral involving one with correlation matrix $\{m_{ij}/(1 - \alpha_i^2)^{\frac{1}{2}}(1 - \alpha_j^2)^{\frac{1}{2}}\}$. This is of little interest in general unless the latter distribution is one that has already been studied. We mention, however, two special cases. When $n = 1$, we have

$$F(h) = \int_{-\infty}^{+\infty} F((h - \delta y)/(1 - \delta^2)^{\frac{1}{2}})\, f(y)\, dy \tag{3.2}$$

for any δ: $-1 \leqq \delta \leqq 1$. When $n = 2$, we have

$$\begin{aligned} &F_2(h_i\,;\rho) \\ &\quad = \int_{-\infty}^{+\infty} F_2((h_i - \delta_i\, y)/(1 - \delta_i^2)^{\frac{1}{2}}; (\rho - \delta_1\,\delta_2)/(1 - \delta_1^2)^{\frac{1}{2}}(1 - \delta_2^2)^{\frac{1}{2}}) f(y)\, dy \end{aligned} \tag{3.3}$$

for any δ_i: $-1 \leqq \delta_i \leqq 1$. The obvious choice for δ_1 and δ_2 in (3.3) would satisfy $\delta_1\delta_2 = \rho$ and we obtain the following representation for the bivariate normal c.d.f.,

$$F_2(h_i\,;\rho) = \int_{-\infty}^{+\infty} \left[\prod_{i=1}^{2} F((h_i - \delta_i\, y)/(1 - \delta_i^2)^{\frac{1}{2}})\right] f(y)\, dy. \tag{3.4}$$

This is (1.3) for the case $n = 2$.

(b) $k = 2, r_1 = n - 1, r_2 = 1$. In this case, the n-variate c.d.f. in the integrand of (2.3) factorizes into a product of an $(n - 1)$-variate and a univariate c.d.f. This is always possible, since we can choose $\delta_1, \delta_2, \cdots, \delta_n$ so that, for some k, $\max_j \rho_{kj}^2 \leqq \delta_k^2 \leqq 1$ and $\delta_j = \rho_{kj}/\delta_k$ ($j \neq k$). The results with $\delta_k = 1$ are the most interesting ones and will be discussed in the next section.

(c) $k = n = 3$, $r_1 = r_2 = r_3 = 1$. This is a particular case of (1.3) and it requires $\rho_{12} = \alpha_1\alpha_2$, $\rho_{13} = \alpha_1\alpha_3$ and $\rho_{23} = \alpha_2\alpha_3$. Suitable values for α_1, α_2 and α_3 can be found if and only if $\rho_{12}\rho_{13}\rho_{23} > 0$ and the product of any two of the correlation coefficients is less than the third. The integral is

$$F_3(h_i\,;\{\rho_{ij}\}) = \int_{-\infty}^{+\infty} \left[\prod_{i=1}^{3} F((h_i - \alpha_i\, y)/(1 - \alpha_i^2)^{\frac{1}{2}})\right] f(y)\, dy.$$

If the correlation coefficients do not satisfy this condition and hence suitable α's cannot be found we can still fall back on (b) above and write in the integrand a product of a univariate and a bivariate normal distribution function.

4. The singly-infinite integral representation. Now consider the integral representation (2.4). The result (2.4) could be applied to the $(n - 1)$-variate normal distribution function in the integrand and hence may provide a useful reduction formula for determining $F_n(h_i\,;\{\rho_{ij}\})$. In general, this would require n successive integrations. However, it is easy to see that for certain correlation structures this number may be reduced. Suppose

$$(4.1)\qquad \{\rho_{ij\cdot 1}\} = \begin{Bmatrix} \mathbf{R}_1 & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \\ \cdot & & \cdot & \mathbf{O} \\ \mathbf{O} & \cdot & \cdot\,\mathbf{O} & \mathbf{R}_k \end{Bmatrix},$$

where $\mathbf{R}_i (i = 1, 2, \cdots, k)$ is a correlation matrix of rank r_i $(\sum_{i=1}^{k} r_i = n - 1)$ and $\mathbf{O}$ denotes a null matrix. Then the $(n - 1)$-variate normal distribution function in the integrand factors into a product of k normal distribution functions of order $r_1, r_2, \cdots, r_k$.

If $\rho_{ij\cdot 1} = 0$ for all $i \neq j \neq 1$, (2.4) simplifies to

$$(4.2)\qquad F_n(h_i\,;\{\rho_{ij}\}) = \int_{-\infty}^{h_1} \left[\prod_{i=2}^{n} F((h_i - \rho_{i1}\, y)/(1 - \rho_{i1}^2)^{\frac{1}{2}})\right] f(y)\, dy.$$

This is (1.3) with $\alpha_1 = 1$ and therefore $\alpha_i = \rho_{i1}$. It corresponds to the case $r_1 = \cdots = r_k = 1$ in (4.1).

If $\rho_{ij\cdot 1} = 0$ for all $i \neq 1, j$ but for only one particular value of j, then the integrand of (2.4) would contain the product of a univariate and an $(n - 2)$-variate normal distribution function. In particular, suppose $\rho_{i2\cdot 1} = 0$ for all $i \neq 1, 2$. Then

$$(4.3)\qquad \begin{aligned} F_n(h_i\,;\{\rho_{ij}\}) = \int_{-\infty}^{h_1} & F((h_2 - \rho_{21}\, y)/(1 - \rho_{21}^2)^{\frac{1}{2}}) \\ & \cdot F_{n-2}((h_i - \rho_{i1}\, y)/(1 - \rho_{i1}^2)^{\frac{1}{2}};\, \{\rho_{ij\cdot 1}\}, i \neq 1, 2) f(y)\, dy, \end{aligned}$$

since $\rho_{ij\cdot 12} = \rho_{ij\cdot 1}$ $(i, j \neq 1, 2)$.

For example, consider the correlation matrix defined by $\rho_{ij} = \gamma_i/\gamma_j$ for $i < j$, where $|\gamma_i| < |\gamma_j|$ for $i < j$. This correlation structure occurs in many applications as will be described in the next section. The matrix satisfies $\rho_{i1\cdot 2} =$

0 $(i \neq 1, 2)$ since $\rho_{i1} - \rho_{i2}\rho_{12} = (\gamma_1/\gamma_i) - (\gamma_2/\gamma_i)\cdot(\gamma_1/\gamma_2) = 0$. Therefore, from (4.3) with the suffixes 1 and 2 interchanged,

$$(4.4) \qquad F_n(h_i\,;\{\rho_{ij}\}) = \int_{-\infty}^{h_2} F(h_1')F_{n-2}\,(h_i'\,;\{\rho_{ij\cdot 2}\}\,; i \neq 1, 2)f(y)\,dy,$$

where $h_i' = (h_i - \rho_{i2}y)/(1 - \rho_{i2}^2)^{\frac{1}{2}}$. Now, with $i \neq j \neq 1, 2$, $\rho_{ij\cdot 2} = (\rho_{ij} - \rho_{i2}\rho_{j2})/(1 - \rho_{i2}^2)^{\frac{1}{2}}(1 - \rho_{j2}^2)^{\frac{1}{2}} = \gamma_i'/\gamma_j'$, where $\gamma_i' = \gamma_i(1 - \gamma_2^2/\gamma_i^2)$. Therefore the correlation matrix of the $(n - 2)$-variate distribution function in the integrand is of the same form as the original n-variate correlation matrix. By applying (4.4) repeatedly, then, the n-dimensional integral can be reduced to either an $\frac{1}{2}n$-dimensional integral or an $\frac{1}{2}(n - 1)$-dimensional integral according as n is even or odd. The results for small n in a form suitable for computing are given below. The expression for $n = 3$ has been used by Dunnett (unpublished) to compute the operating characteristics of 3-stage procedures for drug screening.

$$n = 3\colon \int_{-\infty}^{h_2} F\left(\frac{h_1 - \gamma_1 y/\gamma_2}{(1 - \gamma_1^2/\gamma_2^2)^{\frac{1}{2}}}\right) F\left(\frac{h_3 - \gamma_2 y/\gamma_3}{(1 - \gamma_2^2/\gamma_3^2)^{\frac{1}{2}}}\right) f(y)\,dy,$$

$$n = 4\colon \int_{-\infty}^{h_2} F\left(\frac{h_1 - \gamma_1 y/\gamma_2}{(1 - \gamma_1^2/\gamma_2^2)^{\frac{1}{2}}}\right) \left[\int_{-\infty}^{h_4'} F\left(\frac{h_3' - \gamma_3' z/\gamma_4'}{(1 - \gamma_3'^2/\gamma_4'^2)^{\frac{1}{2}}}\right) f(z)\,dz\right] f(y)\,dy,$$

$$n = 5\colon \int_{-\infty}^{h_2} F\left(\frac{h_1 - \gamma_1 y/\gamma_2}{(1 - \gamma_1^2/\gamma_2^2)^{\frac{1}{2}}}\right)$$
$$\cdot\left[\int_{-\infty}^{h_4'} F\left(\frac{h_3' - \gamma_3' z/\gamma_4'}{(1 - \gamma_3'^2/\gamma_4'^2)^{\frac{1}{2}}}\right) F\left(\frac{h_5' - \gamma_4' z/\gamma_5'}{(1 - \gamma_4'^2/\gamma_5'^2)^{\frac{1}{2}}}\right) f(z)\,dz\right] f(y)\,dy,$$

where $\gamma_i' = (\gamma_i^2 - \gamma_2^2)^{\frac{1}{2}}$ and $h_i' = (h_i - \rho_{2i}\,y)/(1 - \rho_{2i}^2)^{\frac{1}{2}}$, $i = 3, 4, 5$.

5. The practical importance of the correlation matrices $\rho_{ij} = \alpha_i\alpha_j$ $(i \neq j)$ and $\rho_{ij} = \gamma_i/\gamma_j$ $(i < j)$. In this section, we mention briefly a few applications in which correlation matrices of the structures $\rho_{ij} = \alpha_i\alpha_j$ and $\rho_{ij} = \gamma_i/\gamma_j$ arise. Note that the case of all the ρ_{ij}'s equal to $\rho > 0$ may be obtained from $\rho_{ij} = \alpha_i\alpha_j$ by taking $\alpha_i = \rho^{\frac{1}{2}}$, all i.

Let $Z_1, Z_2, \cdots, Z_n$ be independent normally distributed estimates of a quantity X. We further assume that the quantity X is normally distributed over a population of such quantities with mean zero and variance σ_0^2. Then the correlation matrix of the Z's is given by

$$(5.1) \qquad \rho_{ij} = \sigma_0^2/(\sigma_0^2 + \sigma_i^2)^{\frac{1}{2}}(\sigma_0^2 + \sigma_j^2)^{\frac{1}{2}}, \qquad i \neq j,$$

where σ_i^2 is the variance of Z_i about X. Clearly, ρ_{ij} is of the form $\rho_{ij} = \alpha_i\alpha_j$ $(i \neq j)$. If a selection is made of all quantities in the X-population for which $Z_i < h_i(\sigma_0^2 + \sigma_i^2)^{\frac{1}{2}}$, then the proportion of the X-population selected is $F_n(h_i\,;\{\rho_{ij}\})$. Values of $F_n(h_i\,;\{\rho_{ij}\})$ with $\rho_{ij} = \alpha_i\alpha_j$ $(i \neq j)$ are therefore of considerable importance in the evaluation of selection procedures. In plant selection, for example, X might represent the true yielding ability of a plant variety chosen at random from a population of such varieties and $Z_1, Z_2, \cdots, Z_n$ represent independent estimates of X based on experimental evidence.

Let $Y_1, Y_2, \cdots, Y_n$ be uncorrelated normally distributed variates with means zero and variances $a_1^2, a_2^2, \cdots, a_n^2$, respectively. Define $Z_1, Z_2, \cdots, Z_n$ to be the partial sums, $Z_i = \sum_{s=1}^{i} Y_s$, $i = 1, 2, \cdots, n$. Then the correlation matrix of the Z's has

$$\rho_{ij} = \left(\sum_{s=1}^{i} a_s^2 \Big/ \sum_{s=1}^{j} a_s^2\right)^{\frac{1}{2}}, \qquad i < j, \tag{5.2}$$

and ρ_{ij} is of the form $\rho_{ij} = \gamma_i/\gamma_j$ where $\gamma_i = (\sum_{s=1}^{i} a_s^2)^{\frac{1}{2}}$. This situation arises in multi-stage selection procedures, such as acceptance sampling and drug screening, in which at each stage a decision is made on the basis of the cumulative sum of the observations available to date whether to continue the experimentation or to terminate and either accept or reject the object to which the observations pertain. The formulae given in the previous section are likely to be useful when the number of stages is small, as is usually the case in drug screening (see Dunnett [4]).

The matrix $\rho_{ij} = \gamma_i/\gamma_j$ $(i < j)$ is important also in factor analysis. In Anderson [1] it appears explicitly as the correlation function of a Wiener process. Clearly, any matrix of the form

$$\begin{Bmatrix} 1 & a_1 & a_1 a_2 & \cdots & \prod_{i=1}^{n-1} a_i \\ a_1 & 1 & a_2 & \cdots & \prod_{i=2}^{n-1} a_i \\ a_1 a_2 & a_2 & 1 & \cdot & \\ & & \cdot & 1 & a_{n-1} \\ \prod_{i=1}^{n-1} a_i & \prod_{i=2}^{n-1} a_i & \cdots & a_{n-1} & 1 \end{Bmatrix}$$

is also of the type $\rho_{ij} = \gamma_i/\gamma_j$ $(i < j)$, and hence an alternative representation is $\rho_{ij} = a_i a_{i+1} \cdots a_{j-1}$, $(i < j)$, where $|a_i| \leqq 1$.

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