ON THE INFLUENCE OF MOMENTS ON THE ASYMPTOTIC DISTRIBUTION OF SUMS OF RANDOM VARIABLES

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1. Introduction. Let $\{X_i : i = 1, 2, \cdots\}$ be a sequence of independent, identically distributed, nondegenerate random variables with common distribution function $F(x)$. Let $S_n = \sum_{i=1}^n X_i$; denote by $F_n(x)$ the distribution function of S_n; and let $a_n = P(S_n < 0)$. In this paper it is shown that if $EX_i = 0$ and $E|X_i|^{2+\alpha} < \infty$ for $0 \leqq \alpha < 1$, then $\sum_{n=1}^\infty n^{-(1-\alpha/2)}|a_n - \frac{1}{2}| < \infty$. In [3] Spitzer showed that if $EX_i = 0$ and $EX_i^2 < \infty$ then $\sum_{n=1}^\infty n^{-1}(a_n - \frac{1}{2}) < \infty$, while in [2] Rosén showed that this series was absolutely convergent. The methods of this paper follow closely the methods of [2].

2. Preliminaries. We require the following results of [2].

(1) Let X be a nondegenerate random variable with distribution function $F(t)$ and characteristic function $\varphi(t)$. Then there exist two constants $\delta > 0$ and $C > 0$ such that $|\varphi(t)| \leqq 1 - Ct^2$ for $|t| \leqq \delta$.

(2) Let $\{X_i : i = 1, 2, \cdots\}$ be a sequence of independent, identically distributed, nondegenerate random variables. Let I_n denote an interval of the real line of length $l(I_n)$. Then $l(I_n) \leqq n^p$, $0 < p < \frac{1}{2}$, implies $P\{S_n \varepsilon I_n\} \leqq Cn^{p-\frac{1}{2}}$ where C is some constant independent of n and I_n. Further $\sup_a P\{S_n = a\} \leqq Cn^{-\frac{1}{2}}$ where C is again independent of n.

(3) Let X be a random variable with distribution function $F(x)$ and suppose $\int_{-\infty}^\infty \log(1 + |x|)F(dx) < \infty$. Then

$$\tfrac{1}{2}\{F(x+0) + F(x-0)\}$$

$$= \tfrac{1}{2} + \lim_{N\to\infty} \frac{1}{2\pi i}\int_0^N \frac{1}{t}\{e^{ixt}\varphi(-t) - e^{-ixt}\varphi(t)\}\,dt$$

$$= \tfrac{1}{2} + \frac{1}{2\pi i}\int_0^\delta \frac{1}{t}\{e^{ixt}\varphi(-t) - e^{-ixt}\varphi(t)\}\,dt + R(1, x, \delta)$$

where $\delta > 0$ and $R(1, x, \delta) = (1/\pi)\int_{-\infty}^\infty F(dy)\int_\delta^\infty [\sin(x-y)t/t]\,dt$. The corresponding remainder term for the distribution function $F_n(x)$ will be denoted $R(n, x, \delta)$.

The following improvement of a lemma in [2] is also required.

Lemma. *For any ϵ, $0 < \epsilon < \frac{1}{2}$, there exists a constant C, independent of n and x, such that $|R(n, x, \delta)| \leqq Cn^{\epsilon-\frac{1}{2}}$.*

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PROOF.

$$\pi|R(n, x, \delta)| \leqq \int_{-\infty}^{\infty} F_n(dy) \left| \int_\delta^\infty \frac{\sin (x-y)t}{t} \, dt \right|$$

$$= \int_{|x-y| \leqq n^\epsilon} \left| \int_\delta^\infty \frac{\sin (x-y)t}{t} \, dt \right| F_n(dy)$$

$$+ \sum_{j=1}^{n-1} \int_{jn^\epsilon < |x-y| \leqq (j+1)n^\epsilon} \left| \int_\delta^\infty \frac{\sin (x-y)t}{t} \, dt \right| F_n(dy)$$

$$+ \int_{|x-y| > n^{1+\epsilon}} \left| \int_\delta^\infty \frac{\sin (x-y)t}{t} \, dt \right| F_n(dy).$$

From (2) and the fact that $|\int_\delta^\infty (\sin ut/t) \, dt| \leqq C$ it follows that

$$\int_{|x-y| \leqq n^\epsilon} \left| \int_\delta^\infty \frac{\sin (x-y)t}{t} \, dt \right| F_n(dy) \leqq Cn^{\epsilon - \frac{1}{2}}.$$

From (2), again, and the estimate $|\int_\delta^\infty (\sin ut/t) \, dt| \leqq C/(\delta|u|)$ it follows that

$$\sum_{j=1}^{n-1} \int_{jn^\epsilon < |x-y| \leqq (j+1)n^\epsilon} \left| \int_\delta^\infty \frac{\sin (x-y)t}{t} \, dt \right| F_n(dy)$$

$$\leqq Cn^{\epsilon - \frac{1}{2}} \sum_{j=1}^{n-1} 1/jn^\epsilon \leqq Cn^{-\frac{1}{2}} \ln n.$$

Finally it is clear that

$$\int_{|x-y| > n^{1+\epsilon}} \left| \int_\delta^\infty \frac{\sin (x-y)t}{t} \, dt \right| F_n(dy) \leqq C_n^{-(1+\epsilon)}$$

and the lemma is proved.

3. Theorem. *Let $\{X_i : i = 1, 2, \cdots\}$ be a sequence of independent, identically distributed, nondegenerate random variables. If $EX_i = 0$ and $E|X_i|^{2+\alpha} < \infty$ for some $\alpha \in [0, 1)$ then $\sum_{n=1}^\infty n^{-(1-\alpha/2)} |P\{S_n < 0\} - \frac{1}{2}| < \infty$.*

PROOF. Setting $x = 0$ in (3) it follows that

$$a_n - \tfrac{1}{2} = P\{S_n < 0\} - \tfrac{1}{2}$$

$$= \tfrac{1}{2}\{F_n(0+) + F_n(0-)\} - \tfrac{1}{2}P\{S_n = 0\} - \tfrac{1}{2}$$

$$= \frac{1}{2\pi i} \int_0^\delta \frac{1}{t} \{\varphi^n(-t) - \varphi^n(t)\} \, dt + R(n, 0, \delta) - \tfrac{1}{2}P\{S_n = 0\}$$

$$= \frac{1}{\pi} \int_0^\delta \frac{|\varphi(t)|^n}{t} \sin \{n \arg \varphi(t)\} \, dt + R(n, 0, \delta) - \tfrac{1}{2}P\{S_n = 0\},$$

where $\delta > 0$ is to be determined later. Since $|\sin n\theta| \leqq n|\theta|$, it follows that

$$\sum_{n=1}^{\infty} n^{-1+\alpha/2}|a_n - \tfrac{1}{2}| \leqq \frac{1}{\pi}\int_0^{\delta} \sum_{n=1}^{\infty} n^{\alpha/2}|\varphi(t)|^n \frac{|\arg \varphi(t)|}{t}\, dt + \sum_{n=1}^{\infty} n^{-1+\alpha/2}|R(n, 0, \delta)| + \tfrac{1}{2}\sum_{n=1}^{\infty} n^{-1+\alpha/2}P\{S_n = 0\}. \tag{4}$$

It follows from the lemma with ϵ chosen less than $(1 - \alpha)/2$ that

$$\sum_{n=1}^{\infty} n^{-1+\alpha/2}|R(n, 0, \delta)| < \infty.$$

From (2) it follows that $\sum_{n=1}^{\infty} n^{-1+\alpha/2}P\{S_n = 0\} < \infty$.

By hypothesis, $E|X_i|^{2+\alpha} < \infty$, and thus it follows that $\varphi(t) = 1 - (EX^2/2)t^2 + t^{2+\alpha}[R(t) + iI(t)]$, where $R(t)$ and $I(t)$ are bounded real functions on any finite interval (See [1], p. 199). Therefore,

$$\arg \varphi(t) = \text{arc tan}\, \{t^{2+\alpha}I(t)/[1 - (EX^2/2)t^2 + R(t)t^{2+\alpha}]\}$$

and for $\delta_1 > 0$ chosen sufficiently small one obtains

$$|\arg \varphi(t)| \leqq Ct^{2+\alpha}|I(t)|. \tag{5}$$

Next, by a well known Abelian theorem ([4], p. 182 Corollary 1a) one has

$$\lim_{u\to 1^-} (1 - u)^{1+\alpha/2} \sum_{n=1}^{\infty} n^{\alpha/2}u^n = \text{const.} \tag{6}$$

and thus for $0 < u < 1$

$$\sum_{n=1}^{\infty} n^{\alpha/2}u^n < C(1 - u)^{-(1+\alpha/2)}. \tag{7}$$

Letting $u = |\varphi(t)|$ one obtains for $t \neq 0$ that

$$\sum_{n=1}^{\infty} n^{\alpha/2}|\varphi(t)|^n \leqq C(1 - |\varphi(t)|)^{-(1+\alpha/2)}.$$

Thus from (1), (5), and (7) for sufficiently small δ,

$$\frac{1}{\pi}\int_0^{\delta} \sum_{n=1}^{\infty} n^{\alpha/2}|\varphi(t)|^n|\arg \varphi(t)|t^{-1}\, dt \leqq \int_0^{\delta} |I(t)|t^{-1}\, dt. \tag{8}$$

The proof is concluded by showing that $\int_0^{\delta} |I(t)|t^{-1}\, dt < \infty$ and this is accomplished as in the proof of Lemma 3 of [2].

Finally it may be noted that the theorem fails if $\alpha \geqq 1$; this is easily seen by choosing X_i to be the random variable taking the values $+1$ and -1 each with probability $\frac{1}{2}$.

REFERENCES

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