ASYMPTOTIC EFFICIENCY OF A CLASS OF c-SAMPLE TESTS[1]

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1. Summary. For testing the equality of c continuous probability distributions on the basis of c independent random samples, the test statistics of the form

$$\mathcal{L} = \sum_{j=1}^{c} m_j[(T_{N,j} - \mu_{N,j})/A_N]^2$$

are considered. Here m_j is the size of the jth sample, $\mu_{N,j}$ and A_N are normalizing constants, and

$$T_{N,j} = (1/m_j)\sum_{i=1}^{N} E_{N,i}Z_{N,i}^{(j)}$$

where $Z_{N,i}^{(j)} = 1$, if the ith smallest of $N = \sum_{j=1}^{N} m_j$ observations is from the jth sample and $Z_{N,i}^{(j)} = 0$ otherwise. Sufficient conditions are given for the joint asymptotic normality of $T_{N,j}$; $j = 1, \cdots, c$. Under suitable regularity conditions and the assumption that the ith distribution function is $F(x + \theta_i/N^{\frac{1}{2}})$, the limiting distribution of $\mathcal{L}$ is derived. Finally, the asymptotic relative efficiencies in Pitman's sense of the $\mathcal{L}$ test relative to some of its competitors viz. the Kruskal-Wallis H test (which is a particular case of the $\mathcal{L}$ test) and the classical F test are obtained and shown to be independent of the number c of samples.

2. Introduction. One of the frequently encountered problems in statistics is to decide whether differences in various samples should be regarded as due to differences in the parent populations or due to chance variations which are to be expected among random samples from the same population. A few tests of nonparametric nature have been proposed for this c-sample problem. The Kruskal-Wallis H test [14], Terpestra's c-sample test [26], the Mood and Brown c-sample test [22] and Kiefer's K-sample analogues of the Kolmogorov-Smirnov and Cramér-von Mises tests [12] are a few of them. Tests for two-sample problems have been proposed by Wilcoxon [29], Mann and Whitney [19], Mood and Brown [22], Lehmann [15] and others. Consistency and power properties of some of these tests have been discussed by Lehmann [15], [16], [17], Mood [23], Van Dantzig [5] and others. An exhaustive review of this problem is given n Kruskal and Wallis [14] and Scheffé [25].

The H test of Kruskal and Wallis is a direct generalization of the two-sample

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Wilcoxon test discussed in detail by Mann and Whitney [19], and its limiting distribution has been derived by Kruskal [13] under the null hypothesis and by Andrews [1] under an alternative hypothesis. These results are generalized by those of the present paper concerning the limiting distribution of the $\mathcal{L}$ test.

The problem discussed in this paper originated from the paper of Chernoff and Savage [2] and had its basis in the paper of Hodges and Lehmann [10]. In their paper "The efficiency of some nonparametric competitors of the t-test" [10], Hodges and Lehmann discussed the asymptotic efficiency of the Wilcoxon test with respect to all translation alternatives. In the same paper they conjectured that the normal score test which was known to be as efficient as the t-test for normal alternatives [11a] is at least as efficient as the t-test for all other alternatives. The validity of this conjecture was established by Chernoff and Savage [2], who developed a new theorem for asymptotic normality of normal score test statistics for the two-sample problem and by a variational argument proved the Hodges-Lehmann conjecture. The work presented here is an attempt toward generalizing these results to the c-sample problem.

Formally, we may state the c-sample problem as follows. Let $[X_{ij}, j = 1, \cdots, m_i; i = 1, \cdots, c]$ be a set of independent random variables and let $F^{(i)}(x)$ be the probability distribution of X_{ij}. The set of admissible hypotheses designates that each $F^{(i)}(x)$ belongs to some class of distribution functions Ω. The hypothesis to be tested, say H_0, specifies that $F^{(i)}$ is an element of Ω, for each i, and that furthermore

$$F^{(1)}(x) = \cdots = F^{(c)}(x) \text{ for all real } x. \tag{2.1}$$

The class of alternatives to H_0 can be considered to be all sets $(F^{(1)}(x), \cdots, F^{(c)}(x))$ which belong to Ω but which violate (2.1). To avoid the problem of ties, it is assumed throughout that the class Ω is the class of continuous distribution functions.

After finding sufficient conditions for the joint asymptotic normality of $T_{N,j}$; $j = 1, \cdots, c$, we study the limiting distributions of $\mathcal{L}$ under a sequence of admissible alternative hypothesis H_n^P which specifies that for each $i = 1, 2, \cdots, c$; $F^{(i)}(x) = F(x + \theta_i/n^{\frac{1}{2}})$ with $F \varepsilon \Omega$ but not specified further, and for some pair (i, j), $\theta_i \neq \theta_j$ where the θ_i's are real numbers. Limiting probability distributions of $\mathcal{L}$ will then be found as $n \to \infty$. The problem will be so formulated that $m_i(n)/n$ tends to some limit s_i, $0 < s_i < \infty$, as n tends to ∞.

3. The proposed test and its relationship to other tests. The over-all sample consists of $\sum_{i=1}^{c} m_i = N$ independent random variables X_{ij} $(i = 1, \cdots, c; j = 1, \cdots, m_i)$, where the first subscript refers to the subsample and the second subscript indexes observations within a subsample. Under the null hypothesis all the X's have the same continuous but unknown c.d.f. (cumulative distribution function) $F(x)$.

Let $Z_{N,i}^{(j)} = 1$, if the ith smallest observation from the combined sample of size N is from the jth sample and otherwise let $Z_{N,i}^{(j)} = 0$. Denote

$$m_j T_{N,j} = \sum_{i=1}^{N} Z_{N,i}^{(j)} E_{N,i} \tag{3.1}$$

where $E_{N,i}$ are given numbers. Then we propose to consider the test statistic $\mathfrak{L}$ defined as

$$\mathfrak{L} = \sum_{j=1}^{c} m_j[(T_{N,j} - \mu_{N,j})/A_N]^2 \tag{3.2}$$

where $\mu_{N,j}$ and A_N are normalizing constants for the statistics $T_{N,j}$; $j = 1, \cdots, c$.

The $\mathfrak{L}$ test presented in this paper includes as special cases a number of well-known tests. For example, when $E_{N,i} = i/N$, it becomes the Kruskal-Wallis H test which is a direct generalization of the two-sample Wilcoxon test and is related to Terpestra's K-sample test [26]. When $c = 2$ and $E_{N,i}$ is the expected value of the ith order statistic from the standard normal distribution, then the $\mathfrak{L}$ test coincides with the symmetrical two-tail version of the normal score test, also known as the Fisher-Yates-Terry-Hoeffding c_1 test and which is asymptotically equivalent to Van der Waerden's test [30], [31]. For it is then seen that

$$\mathfrak{L} = [N/(N - m_1)]\left[\sum_{i=1}^{m_2} E(V^{(s_i)} \mid s_i)\right]^2$$

where $V^{(1)} < \cdots < V^{(N)}$ is an ordered sample of size N from a standard normal distribution, and $s_1 < \cdots < s_{m_2}$ are the ranks of $X_{21}, \cdots, X_{2m_2}$ from the combined sample. See Lehmann [17], pp. 236–237. When $c = 2$, and $E_{N,i} = |\frac{1}{2} - i/N|$, the $\mathfrak{L}$-test test reduces to the Freund-Ansari test [8] for testing the equality of dispersion of two populations.

4. Assumptions and notations. Let $X_{i1}, \cdots, X_{im_i}$ be the ordered observations of a random sample from a population with continuous c.d.f. (cumulative distribution function) $F^{(i)}(x)$; $i = 1, \cdots, c$. Let $N = \sum_{i=1}^{c} m_i$ and $\lambda_i = m_i/N$ and assume that for all N, the inequalities $0 < \lambda_0 \leqq \lambda_1, \cdots, \lambda_c \leqq 1 - \lambda_0 < 1$ hold for some fixed $\lambda_0 \leqq 1/c$.

Let

$$S_{m_i}^{(i)}(x) = m_i^{-1} \text{ (number of } X_{ij} \leqq x, j = 1, \cdots, m_i)$$

be the sample c.d.f. of the m_i observations in the ith set. We shall omit the subscript m_i whenever this causes no confusion. Define $H_N(x) = \lambda_1 S_{m_1}^{(1)}(x) + \cdots + \lambda_c S_{m_c}^{(c)}(x)$. Thus $H_N(x)$ is the combined sample c.d.f. The combined population c.d.f. is $H(x) = \lambda_1 F^{(1)}(x) + \cdots + \lambda_c F^{(c)}(x)$. Even though $H(x)$ depends on N through the λ's, our notation suppresses this fact for convenience and also because our limit theorems are uniform with respect to $F^{(1)}, \cdots, F^{(c)}$ and $\lambda_1, \cdots, \lambda_c$.

Let $Z_{N,i}^{(j)} = 1$ if the ith smallest of $N = \sum_{i=1}^{c} m_i$ observations is from the jth set and otherwise let $Z_{N,i}^{(j)} = 0$. Denote

$$\tau_{N,j} = m_j \cdot T_{N,j} = \sum_{i=1}^{N} E_{N,i} Z_{N,i}^{(j)} \tag{4.1}$$

where the $E_{N,i}$ are given numbers. Following Chernoff and Savage [2], we shall use the representation

$$T_{N,j} = \int_{-\infty}^{\infty} J_N[H_N(x)]\, dS_{m_j}^{(j)}(x) \tag{4.2}$$

where $E_{N,i} = J_N(i/N)$. While J_N need be defined only at $1/N, 2/N, \cdots, N/N$, we shall find it convenient to extend its domain of definition to $(0, 1]$ by letting J_N be constant on $(i/N, (i + 1)/N]$.

Let

$$I_N = \{x: 0 < H_N(x) < 1\}.$$

Then I_N is a random interval, given by $I_N = [X^{(1)}, X^{(N)})$, where $X^{(1)} < \cdots < X^{(N)}$ denote the N observations arranged according to size.

Throughout, K will be used as a generic constant which may depend on J_N but will not depend on $F^{(1)}, \cdots, F^{(c)}, m_1, \cdots, m_c$ and N. The methods used in the proof for the asymptotic normality of the $T_{N,j}$'s are mainly adaptations of the methods of Chernoff and Savage [2].

5. Joint asymptotic normality. Before proving the asymptotic normality of the $T_{N,j}$'s we state a few elementary results.

$$H \geqq \lambda_i F^{(i)} \geqq \lambda_0 F^{(i)}; \qquad i = 1, \cdots, c. \tag{5.1}$$

$$1 - F^{(i)} \leqq (1 - H)/\lambda_i \leqq (1 - H)/\lambda_0; \qquad i = 1, \cdots, c. \tag{5.2}$$

$$F^{(i)}(1 - F^{(i)}) \leqq H(1 - H)/\lambda_i^2 \leqq H(1 - H)/\lambda_0^2; \qquad i = 1, \cdots, c. \tag{5.3}$$

$$dH \geqq \lambda_i\, dF^{(i)} \geqq \lambda_0\, dF^{(i)}; \qquad i = 1, \cdots, c. \tag{5.4}$$

LEMMA 5.1. *If*

(1) $J(H) = \lim_{N\to\infty} J_N(H)$ *exists for* $0 < H < 1$ *and is not constant,*

(2) $\int_{I_N}[J_N(H_N) - J(H_N)]\, dS_{m_j}^{(j)}(x) = o_p(N^{-(\frac{1}{2})})$,

(3) $J_N(1) = o(N^{\frac{1}{2}})$

(4) $|J^{(i)}(H(x))| = |d^iJ(H)/dH^i| \leqq K[H(1 - H)]^{-i-(\frac{1}{2})+\delta}$,

for $i = 0, 1, 2,$ *and for some* $\delta > 0$, *and almost all* x (a.a.x),

then, for fixed $F^{(1)}, \cdots, F^{(c)}$ *and* $\lambda_1, \cdots, \lambda_c$,

$$\lim_{N\to\infty} P\left(\frac{T_{N,j} - \mu_{N,j}}{\sigma_{N,j}} \leqq t\right) = \int_{-\infty}^{t} \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-x^2/2}\, dx, \tag{5.5}$$

where

$$\mu_{N,j} = \int_{-\infty}^{+\infty} J[H(x)]\, dF^{(j)}(x) \tag{5.6}$$

and

$$\begin{aligned} N\sigma_{N,j}^2 = {} & 2\sum_{\substack{i=1\\ i\neq j}}^{c} \lambda_i \iint_{-\infty<x<y<\infty} F^{(i)}(x)[1 - F^{(i)}(y)] \\ & \qquad \cdot J'[H(x)]J'[H(y)]\, dF^{(j)}(x)\, dF^{(j)}(y) \\ & + \frac{2}{\lambda_j}\sum_{\substack{i=1\\ i\neq j}}^{c} \lambda_i^2 \iint_{-\infty<x<y<\infty} F^{(j)}(x)[1 - F^{(j)}(y)] \\ & \qquad \cdot J'[H(x)]J'[H(y)]\, dF^{(i)}(x)\, dF^{(i)}(y) \\ & + \frac{1}{\lambda_j} \sum_{\substack{i,k=1\\ i\neq k, i\neq j, k\neq j}}^{c} \lambda_i\lambda_k \Big[\iint_{-\infty<x<y<\infty} F^{(j)}(x)[1 - F^{(j)}(y)] \\ & \qquad \cdot J'[H(x)]J'[H(y)]\, dF^{(i)}(x)\, dF^{(k)}(y) \\ & + \iint_{-\infty<y<x<\infty} F^{(j)}(y)[1 - F^{(j)}(x)]J'[H(x)]J'[H(y)]\, dF^{(i)}(x)\, dF^{(k)}(y). \end{aligned} \tag{5.7}$$

PROOF.

$$T_{N,j} = \int_{x=-\infty}^{x=+\infty} J_N[H_N(x)]\, dS_{m_j}^{(j)}(x)$$

$$= \int_{\{x:0<H_N(x)<1\}} [J_N[H_N(x)] - J[H_N(x)]]\, dS_{m_j}^{(j)}(x)$$

$$+ \int_{\{x:0<H_N(x)<1\}} J[H_N(x)]\, dS_{m_j}^{(j)}(x) + \int_{\{x:H_N(x)=1\}} J_N[H_N(x)]\, dS_{m_j}^{(j)}(x).$$

In the second integral, writing $dS_{m_j}^{(j)}(x) = d(S_{m_j}^{(j)}(x) - F^{(j)}(x) + F^{(j)}(x))$,

$$J[H_N(x)] = J[H(x)] + [H_N(x) - H(x)]J'[H(x)]$$
$$+ \tfrac{1}{2}[H_N(x) - H(x)]^2 J''[\theta H_N(x) + (1-\theta)H(x)], \text{ a.a.}x.,$$

where $0 < \theta < 1$; and $H(x) = \sum_{i=1}^{c} \lambda_i F^{(i)}(x)$, and simplifying, we obtain

$$T_{N,j} = A + B_{1N} + B_{2N} + \sum_{i=1}^{c+4} C_{iN}$$

where

$$A = \int_{\{x:0<H(x)<1\}} J[H(x)]\, dF^{(j)}(x) \tag{5.8}$$

$$B_{1N} = \int_{\{x:0<H(x)<1\}} J[H(x)]\, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)] \tag{5.9}$$

$$B_{2N} = \int_{\{x:0<H(x)<1\}} [H_N(x) - H(x)]J'[H(x)]\, dF^{(j)}(x) \tag{5.10}$$

$$C_{i,N} = \lambda_i \int_{\{x:0<H(x)<1\}} [S_{m_i}^{(i)}(x) - F^{(i)}(x)]J'[H(x)]\, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)] \tag{5.11}$$

$$i = 1, \cdots, c.$$

$$C_{c+1,N} = \int_{I_N} \frac{[H_N(x) - H(x)]^2}{2} J''[\theta H_N + (1-\theta)H]\, dS_{m_j}^{(j)}(x). \tag{5.12}$$

$$C_{c+2,N} = \int_{I_N} [J_N[H_N(x)] - J[H_N(x)]]\, dS_{m_j}^{(j)}(x). \tag{5.13}$$

$$C_{c+3,N} = \int_{H_N=1} J_N[H_N(x)]\, dS_{m_j}^{(j)}(x). \tag{5.14}$$

$$C_{c+4,N} = \int_{H_N=1} [-J[H(x)] - \{H_N(x) - H(x)\}J'[H(x)]]\, dS_{m_j}^{(j)}(x). \tag{5.15}$$

The proof of the lemma is accomplished by showing that (i) the A-term is nonrandom and finite, (ii) $B_{1N} + B_{2N}$ has a Gaussian distribution in the limit and (iii) the C terms are of higher order.

That the term

$$A = \int_{\{x:0<H(x)<1\}} J[H(x)]\, dF^{(j)}(x)$$

is finite and nonrandom follows from Assumption 4 of Lemma 5.1; see also in this connection [2], p. 986, and in the appendix we have shown that the C terms are of higher order. Thus, all that is required is to prove

SUB-LEMMA 5.1. *$B_{1N} + B_{2N}$ has a Gaussian distribution in the limit.*

PROOF. Integrating B_{2N} by parts, replacing $H_N(x) - H(x)$ by $\sum_{i=1}^{c} \lambda_i [S_{m_i}^{(i)}(x) - F^{(i)}(x)]$, and adding B_{1N} to it, we obtain

$$(5.16)\qquad \begin{aligned} B_{1N} + B_{2N} = &-\sum_{\substack{i=1\\ i\neq j}}^{c} \lambda_i \int_{x=-\infty}^{x=+\infty} B(x)\, d[S_{m_i}^{(i)}(x) - F^{(i)}(x)] \\ &+ \int_{x=-\infty}^{x=+\infty} [J[H(x)] - \lambda_j B(x)]\, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)], \end{aligned}$$

$$(5.17)\qquad \begin{aligned} = &-\sum_{\substack{i=1\\ i\neq j}}^{c} \left[\lambda_i \cdot \frac{1}{m_i} \sum_{k=1}^{m_i} \{B(X_{ik}) - EB(X_i)\}\right] \\ &+ \frac{1}{m_j} \sum_{k=1}^{m_j} \{J[H(X_{jk})] - \lambda_j B(X_{jk}) - E[J[H(X_j)] - \lambda_j B(X_j)]\} \end{aligned}$$

where

$$(5.18)\qquad B(x) = \int_{x_0}^{x} J'[H(y)]\, dF^{(j)}(y)$$

with x_0 determined somewhat arbitrarily, say by $H(x_0) = \frac{1}{2}$; E represents the expectation and $X_1, \cdots, X_c$ have the $F^{(1)}, \cdots, F^{(c)}$ distributions respectively.

The c summations given by (5.17) involve independent samples of identically distributed random variables. Therefore, if we show that the first two moments of these random variables exist, then we can apply the central limit theorem, with the result that each sum when properly normalized will have a normal distribution in the limit and hence the sum of c summations will have a normal distribution in the limit.

First, to turn our attention to moments, note that by Assumption 4 of Lemma 5.1 and $dF^{(j)} \leqq (1/\lambda_0)\, dH$,

$$|B(x)| \leqq K \cdot [H(x)[1 - H(x)]]^{-(\frac{1}{2})+\delta}$$

and proceeding as in [2], for any δ' such that $(2 + \delta')(-\frac{1}{2} + \delta) > -1$

$$E_{F^{(i)}}\{|B(X)|\}^{2+\delta'} \leqq K; \qquad i = 1, \cdots, j-1, j+1, \cdots, c.$$

Since

$$|J(H(x)) - \lambda_j B(x)| \leqq K[H(x)(1 - H(x))]^{-(\frac{1}{2})+\delta}$$

the existence of $2 + \delta'$ absolute moments of all the terms in equation (5.17) follows.

To compute the variance of $B_{1N} + B_{2N}$, note that

$$-\lambda_i \int_{-\infty}^{+\infty} B(x)\, d[S_{m_i}^{(i)}(x) - F^{(i)}(x)]$$

$$= \lambda_i \int_{-\infty}^{+\infty} [S_{m_i}^{(i)}(x) - F^{(i)}(x)]J'[H(x)]\, dF^{(j)}(x), \quad i = 1, \cdots, j-1, j+1, \cdots, c,$$

has mean zero and variance

$$\begin{aligned} &E\left\{\lambda_i \int_{-\infty}^{+\infty} [S_{m_i}^{(i)}(x) - F^{(i)}(x)]J'[H(x)]\, dF^{(j)}(x)\right\}^2 \\ &= E\left\{\lambda_i^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [S_{m_i}^{(i)}(x) - F^{(i)}(x)][S_{m_i}^{(i)}(y) - F^{(i)}(y)] \right. \\ &\qquad \left. \cdot J'[H(x)]J'[H(y)]\, dF^{(j)}(x)\, dF^{(j)}(y)\right\} \\ &= \frac{2\lambda_i}{N} \iint_{-\infty<x<y<\infty} F^{(i)}(x)[1 - F^{(i)}(y)]J'[H(x)]J'[H(y)]\, dF^{(j)}(x)\, dF^{(j)}(y), \\ &\qquad i = 1, \cdots, j-1, j+1, \cdots, c. \end{aligned} \tag{5.19}$$

Note that the application of Fubini's theorem permits the interchange of integral and expectation.

By a similar argument, the variance of

$$\int_{-\infty}^{+\infty} [J(H(x)) - \lambda_j B(x)]\, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)]$$

$$= -\sum_{\substack{i=1 \\ i\neq j}}^{c} \lambda_i \int_{-\infty}^{+\infty} [S_{m_j}^{(j)}(x) - F^{(j)}(x)]J'(H(x))\, dF^{(i)}(x)$$

is given by

$$\begin{aligned} &\frac{2}{N\lambda_j} \sum_{\substack{i=1 \\ i\neq j}}^{c} \lambda_i^2 \iint_{-\infty<x<y<\infty} F^{(j)}(x)[1 - F^{(j)}(y)]J'[H(x)]J'[H(y)] \\ &\qquad \cdot dF^{(i)}(x)\, dF^{(i)}(y) \\ &+ \frac{1}{N\lambda_j} \sum_{\substack{i,k=1 \\ i\neq k, i\neq j, k\neq j}}^{c} \lambda_i \lambda_k \iint_{-\infty<x<y<\infty} F^{(j)}(x)[1 - F^{(j)}(y)]J'[H(x)]J'[H(y)] \\ &\qquad \cdot dF^{(i)}(x)\, dF^{(k)}(y) \\ &+ \frac{1}{N\lambda_j} \sum_{\substack{i,k=1 \\ i\neq k, i\neq j, k\neq j}}^{c} \lambda_i \lambda_k \iint_{-\infty<y<x<\infty} F^{(j)}(y)[1 - F^{(j)}(x)]J'[H(x)]J'[H(y)] \\ &\qquad \cdot dF^{(i)}(x)\, dF^{(k)}(y). \end{aligned} \tag{5.20}$$

Adding the c terms given by (5.19) and (5.20) we obtain the variance result stated in (5.7).

Thus we have shown that $B_{1N} + B_{2N}$ is the sum of c independent terms, each of which has mean zero and finite absolute $2 + \delta'$ moments. Hence Sub-Lemma 5.1 follows.

We shall now extend the proof of the above lemma to the case where $F^{(1)}, \cdots, F^{(c)}$ and $\lambda_1, \cdots, \lambda_c$ are not fixed. We want to find a set of sufficient conditions under which the asymptotic normality holds uniformly with respect to $F^{(1)}, \cdots, F^{(c)}$ and $\lambda_1, \cdots, \lambda_c$. For this we need the following theorem of Esseen [6], p. 43.

THEOREM (*Esseen*) 5.1. *Let $X_1, \cdots, X_n$ be independent observations from a population with mean zero, variance σ^2 and finite absolute $2 + \delta'$ moments $\beta_{2+\delta'}$, $0 < \delta' \leqq 1$, then*

$$|F^* - \Phi^*| < c(\delta')[\rho_{2+\delta'}/n^{\delta'/2} + \rho_{2+\delta'}^{1/\delta'}/n^{\frac{1}{2}}]$$

where F^* is the c.d.f. of $\bar{X}$, Φ^* is the approximating normal c.d.f., $c(\delta')$ is a finite positive constant only depending on δ' and $\rho_{2+\delta'} = \beta_{2+\delta'}/\sigma^{2+\delta'}$. (If $\delta' = 1$, then $|F^* - \Phi^*| < c(\delta')\rho_3/n^{\frac{1}{2}}$).

To apply this theorem in our situation, it suffices, since we have shown that the A term is finite and the C terms are uniformly $o_p(N^{-(\frac{1}{2})})$, to prove the uniform convergence of $B_{1N} + B_{2N}$. For this it suffices to bound $\rho_{2+\delta'} = \beta_{2+\delta'}/\sigma^{2+\delta'}$ for $B(X_1), \cdots, B(X_c)$. Since in the above lemma we already bounded the absolute $2 + \delta'$ moments, all that is required is to bound the variances of $B(X_1), \cdots, B(X_c)$ away from zero. Thus we have

COROLLARY 5.1. *If Conditions 1 to 4 of Lemma 5.1 are satisfied, and $F^{(i)}$ and λ_i, $i = 1, \cdots, c$ (where $0 < \lambda_0 \leqq \lambda_1, \cdots, \lambda_c \leqq 1 - \lambda_0 < 1$ holds for some fixed $\lambda_0 \leqq 1/c$) are restricted to a set for which the variances of $B(X_1), \cdots, B(X_c)$ are bounded away from zero, then the asymptotic normality holds uniformly with respect to $F^{(1)}, \cdots, F^{(c)}$ and $\lambda_1, \cdots, \lambda_c$.*

Next we prove

LEMMA 5.2. *Under the assumptions of Lemma 5.1, the random vector $N^{\frac{1}{2}}(T_{N,1} - \mu_{N,1}; \cdots; T_{N,c} - \mu_{N,c})$ has a limiting normal distribution.*

PROOF. The difference $N^{\frac{1}{2}}(T_{N,j} - \mu_{N,j}) - N^{\frac{1}{2}}(B_{1N}^{(j)} + B_{2N}^{(j)})$, where $B_{1N}^{(j)} + B_{2N}^{(j)}$ is the "$B_{1N} + B_{2N}$" term for the jth component $T_{N,j} - \mu_{N,j}$, tends to zero in probability and so, by a well known theorem ([3], p. 299), the vectors $N^{\frac{1}{2}}(T_{N,1} - \mu_{N,1}; \cdots; T_{N,c} - \mu_{N,c})$ and $N^{\frac{1}{2}}(B_{1N}^{(1)} + B_{2N}^{(1)}; \cdots; B_{1N}^{(c)} + B_{2N}^{(c)})$ possess the same limiting distributions. Now since the jth component $B_{1N}^{(j)} + B_{2N}^{(j)}$ can be expressed as $\sum_{i=1}^{c} \{(1/m_i) \sum_{\alpha=1}^{m_i} B_{ij}^*(X_{i\alpha})\}$, the proof of the lemma follows by applying the Central Limit Theorem to each of the c independent vectors

$$(1/m_i) \sum_{\alpha=1}^{m_i} [B_{i1}^*(X_{i\alpha}),\ B_{i2}^*(X_{i\alpha}),\ \cdots,\ B_{ic}^*(X_{i\alpha})];\ i = 1, \cdots, c.$$

6. The Covariance of two B-Statistics.

By definition

$$\begin{aligned} \operatorname{Cov}(B_{1N}^{(j)} + B_{2N}^{(j)}, B_{1N}^{(j')} + B_{2N}^{(j')}) &= E(B_{1N}^{(j)} + B_{2N}^{(j)})(B_{1N}^{(j')} + B_{2N}^{(j')}) \\ &= E(B_{1N}^{(j)} B_{2N}^{(j')}) + E(B_{2N}^{(j)} B_{1N}^{(j')}) + E(B_{2N}^{(j)} B_{2N}^{(j')}) \end{aligned} \tag{6.1}$$

where

$$B_{1N}^{(j')} = \iint_{\{x:0<H(x)<1\}} J[H(x)]\,d[S_{m_{j'}}^{(j')}(x) - F^{(j')}(x)] \tag{6.2}$$

$$B_{2N}^{(j')} = \int_{\{y:0<H(y)<1\}} [H_N(y) - H(y)]J'[H(y)]\,dF^{(j')}(y) \tag{6.3}$$

and $B_{1N}^{(j)}$ and $B_{2N}^{(j)}$ are given by (5.9) and (5.10) respectively.

Now integrating $B_{1N}^{(j)}$ by parts and using the facts that

$$\int_{-\infty}^{+\infty} d[S_{m_j}^{(j)}(x) - F^{(j)}(x)] = 0$$

$$dH(x) = \sum_{i=1}^{c} \lambda_i\,dF^{(i)}(x)$$

and

$$H_N(y) - H(y) = \sum_{i=1}^{c} \lambda_r[S_{m_r}^{(r)}(y) - F^{(r)}(y)]$$

routine computations yield, for $j \neq j'$,

$$B_{1N}^{(j)}B_{2N}^{(j')} = -\sum_{i=1}^{c}\sum_{r=1}^{c} \lambda_i\,\lambda_r \int_{x=-\infty}^{x=+\infty}\int_{y=-\infty}^{y=+\infty} [S_{m_j}^{(j)}(x) - F^{(j)}(x)][S_{m_r}^{(r)}(y) - F^{(r)}(y)] \cdot J'[H(x)]J'[H(y)]\,dF^{(i)}(x)\,dF^{(j')}(y).$$

Therefore,

$$\begin{aligned} E(B_{1N}^{(j)}B_{2N}^{(j')}) = &-\frac{1}{N}\sum_{i=1}^{c}\lambda_i \iint_{-\infty<x<y<\infty} F^{(j)}(x)[1 - F^{(j)}(y)]J'[H(x)] \cdot J'[H(y)]\,dF^{(i)}(x)\,dF^{(j')}(y) \\ &-\frac{1}{N}\sum_{i=1}^{c}\lambda_i \iint_{-\infty<y<x<\infty} F^{(j)}(y)[1 - F^{(j)}(x)]J'[H(x)]J'[H(y)]\,dF^{(i)}(x)\,dF^{(j')}(y). \end{aligned} \tag{6.4}$$

Proceeding analogously

$$\begin{aligned} E(B_{2N}^{(j)}B_{1N}^{(j')}) = &-\frac{1}{N}\sum_{i=1}^{c}\lambda_i \iint_{-\infty<x<y<\infty} F^{(j')}(x)[1 - F^{(j')}(y)]\cdot J'[H(x)] \cdot J'[H(y)]\,dF^{(i)}(x)\,dF^{(j)}(y) \\ &-\frac{1}{N}\sum_{i=1}^{c}\lambda_i \iint_{-\infty<y<x<\infty} F^{(j')}(y)[1 - F^{(j')}(x)]\cdot J'[H(x)] \cdot J'[H(y)]\,dF^{(i)}(x)\,dF^{(j)}(y) \end{aligned} \tag{6.5}$$

and

$$E(B_{2N}^{(j)}B_{2N}^{(j')}) = \frac{1}{N}\sum_{i=1}^{c}\lambda_i \iint_{-\infty<x<y<\infty} F^{(i)}(x)[1 - F^{(i)}(y)]\cdot J'[H(x)] \cdot J'[H(y)]\, dF^{(j)}(x)\, dF^{(j')}(y) \tag{6.6}$$
$$+ \frac{1}{N}\sum_{i=1}^{c}\lambda_i \iint_{-\infty<y<x<\infty} F^{(i)}(y)[1 - F^{(i)}(x)]\cdot J'[H(x)]\cdot J'[H(y)] \cdot dF^{(j)}(x)\, dF^{(j')}(y).$$

Thus

$$N\cdot \mathrm{Cov}\,(B_{1N}^{(j)} + B_{2N}^{(j)}, B_{1N}^{(j')} + B_{2N}^{(j')})$$
$$= -\sum_{i=1}^{c}\lambda_i\Big[\iint_{-\infty<x<y<\infty} F^{(j)}(x)[1 - F^{(j)}(y)]\cdot J'[H(x)]\cdot J'[H(y)] \cdot dF^{(i)}(x)\, dF^{(j')}(y)$$
$$+ \iint_{-\infty<y<x<\infty} F^{(j)}(y)[1 - F^{(j)}(x)]\cdot J'[H(x)]\cdot J'[H(y) \cdot dF^{(i)}(x)\, dF^{(j')}(y)\Big]$$
$$- \sum_{i=1}^{c}\lambda_i\Big[\iint_{-\infty<x<y<\infty} F^{(j')}(x)[1 - F^{(j')}(y)]\cdot J'[H(x)]\cdot J'[H(y)] \cdot dF^{(i)}(x)\, dF^{(j)}(y) \tag{6.7}$$
$$+ \iint_{-\infty<y<x<\infty} F^{(j')}(y)[1 - F^{(j')}(x)]\cdot J'[H(x)]\cdot J'[H(y)] \cdot dF^{(i)}(x)\, dF^{(j)}(y)\Big]$$
$$+ \sum_{i=1}^{c}\lambda_i\Big[\iint_{-\infty<x<y<\infty} F^{(i)}(x)[1 - F^{(i)}(y)]\cdot J'[H(x)]\cdot J'[H(y)] \cdot dF^{(j)}(x)\, dF^{(j')}(y)$$
$$+ \iint_{-\infty<y<x<\infty} F^{(i)}(y)[1 - F^{(i)}(x)]\cdot J'[H(x)]\cdot J'[H(y)] \cdot dF^{(j)}(x)\, dF^{(j')}(y)\Big], \qquad j \neq j'.$$

Combining the material of the previous two sections produces

THEOREM 6.1. *Under the assumptions of Lemma* 5.1, *the random vector* $T = (N^{\frac{1}{2}}(T_{N,1} - \mu_{N,1}), \cdots, N^{\frac{1}{2}}(T_{N,c} - \mu_{N,c}))$ *has a limiting normal distribution with zero mean vector and variance-covariances given by limiting forms of* (5.7) *and* (6.7) *respectively as* $N \to \infty$.

REMARK. The following theorem gives a simple sufficient condition under which Conditions 1, 2, and 3 of Lemma 5.1 hold.

THEOREM 6.2. *If $J_N(i/N)$ is the expectation of the ith order statistic of a sample of size N from a population whose cumulative distribution function is the inverse function of J and $|J^{(i)}[H(x)]| \leqq K[H(1-H)]^{-i-(\frac{1}{2})+\delta}$ for $i = 0, 1, 2$; for some $\delta > 0$ and a.a. x, then*

(i) $\lim_{N\to\infty} J_N(H) = J(H)$.

(ii) $J_N(1) = o(N^{\frac{1}{2}})$.

(iii) $\int_{I_N}[J_N(H_N) - J(H_N)]\, dS_{m_j}^{(j)}(x) = o(N^{-(\frac{1}{2})})$; $j = 1, \cdots, c$.

REMARK 1. The condition $|J^{(i)}[H(x)]| \leqq K[H(1-H)]^{-i-(\frac{1}{2})+\delta}$ a.a. x is weaker than the condition $|J^{(i)}(H)| \leqq K[H(1-H)]^{-i-(\frac{1}{2})+\delta}$ used by Chernoff and Savage [2], otherwise Theorem 6.2 is the generalization of the latter's Theorem 2.

REMARK 2. With the use of this theorem, it is easy to verify that if $J_N(i/N)$ is the expected value of the ith order statistic of a sample of size N from (i) the standard normal distribution, (ii) the logistic distribution, (iii) the double exponential distribution, (iv) the exponential distribution, then the vector $(T_{N,1}; \cdots; T_{N,c})$ has a limiting normal distribution.

7. The limiting distribution of $\mathcal{L}$ under Pitman's shift alternatives. From this section onward, we assume that $m_1, \cdots, m_c$ are nondecreasing functions of a natural number n that tends to infinity. The dependence on n is indicated when necessary, by writing $m_i(n)$, $\mu_{N,i}(n)$, etc. For convenience, it is assumed that, for all i,

$$\lim_{n\to\infty} m_i(n)/n = s_i$$

exists, and there exist two constants a and b such that $0 < a < s_i < b < \infty$.

In subsequent analysis, we shall concern ourselves with a sequence of admissible alternative hypothesis H_n^P which specifies that for each $i = 1, \cdots, c$; $F^{(i)}(x) = F(x + \theta_i/n^{\frac{1}{2}})$ with $F \varepsilon \Omega$ but not specified further, and for some pair (i, j), $\theta_i \neq \theta_j$. The letter n is used to index a sequence of situations in which H_n^P is the true hypothesis. Limiting probability distribution of $\mathcal{L}$ will then be found as $n \to \infty$.

We first prove the following

THEOREM 7.1. *If*

(1) *for all i,*

$$\lim_{n\to\infty} m_i(n)/n = s_i$$

exists,

(2) *Conditions* (1) *to* (4) *of Lemma* 5.1 *are satisfied,*

(3) *$F^{(j)}(x) = F(x + \theta_j/n^{\frac{1}{2}})$ so that for each index n, the hypothesis H_n^P is valid, then the random vector $[m_1^{\frac{1}{2}}(T_{N,1} - \mu_{N,1}), \cdots, m_c^{\frac{1}{2}}(T_{N,c} - \mu_{N,c})]$ has a limiting normal distribution with zero means and covariance matrix whose (j, j')th term is*

$$\left[\delta_{jj'} - (s_j s_{j'})^{\frac{1}{2}} \Big/ \sum_{i=1}^{c} s_i\right] A^2 \tag{7.1}$$

where

$$A^2 = \int_0^1 J^2(x)\,dx - \left(\int_0^1 J(x)\,dx\right)^2 \tag{7.2}$$

and the limit holds uniformly in s_i *provided* $0 < a < s_i < b < \infty;\ i = 1, \cdots, c.$

PROOF. From Equation (5.7)

$$\begin{aligned}\lim_{n\to\infty} N\cdot\sigma_{N,j}^2 &= \left[\sum_{\substack{i=1\\ i\neq j}}^{c} s_i + \frac{1}{s_j}\sum_{\substack{i=1\\ i\neq j}}^{c} s_i^2 + \frac{1}{2s_j}\sum_{\substack{i,k=1\\ i\neq k, i\neq j, k\neq j}}^{c} s_i s_k\right] I_1 \Big/ \sum_{i=1}^{c} s_r \\ &\quad + \frac{1}{2s_j}\left(\sum_{\substack{i,k=1\\ i\neq k, i\neq j, k\neq j}}^{c} s_i s_k\right) I_2 \Big/ \sum_{i=1}^{c} s_r\end{aligned} \tag{7.3}$$

where

$$I_1 = 2\iint_{0<x<y<1} x(1-y)J'(x)J'(y)\,dx\,dy, \tag{7.4}$$

$$= \int_0^1 J^2(x)\,dx - \left(\int_0^1 J(x)\,dx\right)^2 \tag{7.5}$$

and

$$I_2 = 2\iint_{0<y<x<1} y(1-x)J'(x)J'(y)\,dx\,dy, \tag{7.6}$$

$$= \int_0^1 J^2(x)\,dx - \left(\int_0^1 J(x)\,dx\right)^2. \tag{7.7}$$

Thus, omitting the routine algebra,

$$\lim_{n\to\infty} N\cdot\sigma_{N,j}^2 = \left(-1 + \sum_{i=1}^{c} s_i \Big/ s_j\right) A^2.$$

Similarly, from equation (6.7),

$$\lim_{n\to\infty} N \operatorname{Cov}(T_{N,j} - \mu_{N,j},\ T_{N,j'} - \mu_{N,j'}) = -A^2.$$

Hence using Theorem 6.1, we obtain the desired result.

Denoting $m_j^{\frac{1}{2}}(T_{N,j} - \mu_{N,j})/A$ by W_j, it now follows that the random vector $W = (W_1, \cdots, W_c)$ has a limiting normal distribution with zero mean vector and with covariance matrix whose (j, j')th term is

$$\left[\delta_{jj'} - (s_j s_{j'})^{\frac{1}{2}} \Big/ \sum_{i=1}^{c} s_i\right].$$

We now make the analysis of variance transformation

$$S_0 = \sum_{i'=1}^{c} e_{i'}^{\frac{1}{2}} W_{i'} \text{ where } e_{i'} = s_{i'}/\sum_{i=1}^{c} s_i$$

$$S_i = \sum_{i'=1}^{c} a_{ii'} W_{i'}\ ;\ i = 1, 2, \cdots, c-1$$

where the a's are chosen to make the transformation orthogonal. It follows that $\sum_{i=1}^{c} W_i^2$ is asymptotically chi-square with $c - 1$ degrees of freedom.

Now recalling that

$$W_i = m_i^{\frac{1}{2}}[T_{N,i} - \mu_{N,i}(\theta)]/A$$

and letting

$$r_i = m_i^{\frac{1}{2}}[\mu_{N,i}(\theta) - \mu_{N,i}(0)]/A$$

we write $\mathcal{L}$ as $\mathcal{L} = \sum_{i=1}^{c} (W_i + r_i)^2$ and this has the same limiting distribution as $\mathcal{L}^* = \sum_{i=1}^{c} (W_i + r_i^*)^2$ where $r_i^* = \lim_{n\to\infty} r_i$ reduces to

$$r_i^* = \lim_{n\to\infty} m_i^{\frac{1}{2}}\left[\int_{-\infty}^{+\infty}\left[J\left\{\sum_{\alpha=1}^{c} \lambda_\alpha F\left(x + \frac{\theta_\alpha - \theta_i}{n^{\frac{1}{2}}}\right)\right\} - J\{F(x)\}\right]dF(x)\right]\bigg/ A.$$

We assume that the above limit exists and is finite. Noting that $\sum_{i=1}^{c} s_i^{\frac{1}{2}} W_i = 0$ and $\sum_{i=1}^{c} s_i^{\frac{1}{2}} r_i^* = 0$, it follows from a theorem of Mann and Wald [20] that

THEOREM 7.3. *Suppose that for all i, $\lim_{n\to\infty} m_i/n = s_i$ exists and is positive. Then under the hypothesis H_n^P, if for any real numbers $t_1, \cdots, t_c$,*

$$\lim_{n\to\infty} m_i^{\frac{1}{2}}\left[\int_{-\infty}^{+\infty}\left[J\left\{\sum_{i=1}^{c} \lambda_i F\left(x + \frac{t_i}{n^{\frac{1}{2}}}\right)\right\} - J\{F(x)\}\right]dF(x)\right]\bigg/ A$$

exists and is finite, then for $n \to \infty$, the limiting distribution of the statistic $\mathcal{L}$ is $X_{c-1}^2(\lambda^L(H_n^P))$ where $\lambda^L(H_n^P)$ is the noncentrality parameter given by

$$\lambda^{\mathcal{L}}(H_n^P) = \sum_{j=1}^{c}\left[\lim_{n\to\infty} m_j^{\frac{1}{2}} \int_{-\infty}^{+\infty}\left[J\left\{\sum_{\alpha=1}^{c} \lambda_\alpha F\left(x + \frac{\theta_\alpha - \theta_j}{n^{\frac{1}{2}}}\right)\right\} - J\{F(x)\}\right] dF(x)\right]^2 \bigg/ A^2. \tag{7.8}$$

REMARK. If the function J is such that $J(u) = u$, then from (7.8), letting $m_j = n \cdot s_j$, we obtain for $\lambda^{\mathcal{L}}(H_n^P)$ the expression

$$\left[12 \bigg/ \left(\sum_{i=1}^{c} s_i\right)^2\right] \sum_{j=1}^{c} s_j \left(\sum_{\alpha=1}^{c} s_\alpha \lim_{n\to\infty} \int_{x=-\infty}^{x=+\infty} n^{\frac{1}{2}} \cdot \left\{F\left(x + \frac{\theta_\alpha - \theta_j}{n^{\frac{1}{2}}}\right) - F(x)\right\} dF(x)\right)^2$$

which is the noncentrality parameter $\lambda^H(H_n^P)$ of the Kruskal-Wallis H test. (See Andrews [1], p. 726.)

In many situations, the noncentrality parameter $\lambda^{\mathcal{L}}$ can be computed easily with the aid of the following lemma which, though stated in a form appropriate to our purpose, is due to Hodges and Lehmann [11].

LEMMA 7.2 (*Hodges-Lehmann*). *If*

(i) *F is a continuous cumulative distribution function, differentiable in each of the open intervals $(-\infty, a_1)$, (a_1, a_2), $\cdots$, (a_{s-1}, a_s), (a_s, ∞) and the derivative of F is bounded in each of these intervals and*

(ii) *the function* $(d/dx)J[F(x)]$ *is bounded as* $x \to \pm\infty$ *then*

$$\text{(7.9)}\quad \begin{aligned} \lim_{n\to\infty} n^{\frac{1}{2}} \int_{-\infty}^{+\infty} \left[J\left\{ \sum_{\alpha=1}^{c} \lambda_\alpha F\left(x + \frac{\theta_\alpha - \theta_j}{n^{\frac{1}{2}}}\right)\right\} - J\{F(x)\} \right] dF(x) \\ = \left(1 \Big/ \sum_{i=1}^{c} s_i\right) \sum_{\alpha=1}^{c} s_\alpha(\theta_\alpha - \theta_j) \int_{-\infty}^{+\infty} \frac{d}{dx} J\{F(x)\}\, dF(x). \end{aligned}$$

The proof of this lemma follows by the methods used in Section 3 and 4 of Hodges-Lehmann (1961).

In case the conditions of Lemma 7.2 are satisfied, then

$$\text{(7.10)}\quad \lambda^{\mathfrak{L}}(H_n^P) = \sum_{\alpha=1}^{c} s_\alpha(\theta_\alpha - \bar{\theta})^2 \left(\int_{-\infty}^{+\infty} \frac{d}{dx} J[F(x)] f(x)\, dx\right)^2 \Big/ A^2$$

where

$$\text{(7.11)}\quad \bar{\theta} = \sum_{\alpha=1}^{c} s_\alpha \theta_\alpha \Big/ \sum_{\alpha=1}^{c} s_\alpha$$

and A^2 is defined in (7.2).

8. Asymptotic relative effciency. The concept of asymptotic relative efficiency of one test with respect to another is due to Pitman. An exposition of his work, together with some extensions is presented by Noether [23a].

THEOREM 8.1. *If* $m_i = n \cdot s_i$ *and if the distribution function* F *is such that*

$$\text{(1)}\quad \lim_{n\to\infty} n^{\frac{1}{2}} \int_{-\infty}^{+\infty} \left[F\left(x + \frac{\theta}{n^{\frac{1}{2}}}\right) - F(x) \right] dF(x)$$

exists

$$\text{(2)}\quad \lim_{n\to\infty} n^{\frac{1}{2}} \int_{-\infty}^{\infty} \left[J\left\{ \left(1 \Big/ \sum_{i=1}^{c} s_i\right) \sum_{\alpha=1}^{c} s_\alpha F\left(x + \frac{\theta_\alpha - \theta_j}{n^{\frac{1}{2}}}\right)\right\} - J[F(x)] \right] dF(x) \Big/ A$$

exists then the asymptotic relative efficiency of the H *test with respect to an arbitrary* $\mathfrak{L}$ *test for testing the hypothesis* H_0 *against* H_n^P *is given by*

$$\text{(8.1)}\quad e_{H,\mathfrak{L}}^P(F(x)) = \frac{12 \sum_{\alpha=1}^{c} s_\alpha \left\{ \sum_{i=1}^{c} s_i \lim_{n\to\infty} \int_{-\infty}^{+\infty} n^{\frac{1}{2}} \cdot \left[F\left(x + \frac{\theta_i - \theta_\alpha}{n^{\frac{1}{2}}}\right) - F(x) \right] dF(x) \right\}^2 A^2}{\left(\sum_{j=1}^{c} s_j\right)^2 \sum_{\alpha=1}^{c} s_\alpha \left(\lim_{n\to\infty} \int_{-\infty}^{+\infty} n^{\frac{1}{2}} \left[J\left\{ \left(1 \Big/ \sum_{j=1}^{c} s_j\right) \sum_{i=1}^{c} \cdot s_i F\left(x + \frac{\theta_i - \theta_\alpha}{n^{\frac{1}{2}}}\right)\right\} - J\{F(x)\} \right] dF(x) \right)^2}.$$

The proof of the above theorem follows by taking the ratio of the two noncentrality factors after the alternatives have been equated. The details are omitted since similar considerations have been given in several other papers, e.g., Andrews [1], Hannan [9].

COROLLARY 8.1. *If in addition to the hypotheses of Theorem* 8.1, *the hypotheses of Lemma* 7.2 *are satisfied, then*

$$(8.2)\qquad e^P_{H,\mathfrak{L}}(F(x)) = 12A^2\left(\int_{-\infty}^{+\infty} f^2(x)\,dx \Big/ \int_{-\infty}^{+\infty} \frac{d}{dx}\{J[F(x)]\}f(x)\,dx\right)^2$$

where f is the density of F.

Here $e^P_{H,\mathfrak{L}}$ does not depend upon c, α, β, and is a function of F only.

It may be remarked that (8.2) agrees with the results found by Chernoff-Savage [2] and Hodges-Lehmann [11] for the two-sample case, and hence the results of this paper as well as those of [2] apply directly to the c-sample problem.

The asymptotic relative efficiency of the classical $\mathfrak{F}$ test with respect to an arbitrary $\mathfrak{L}$ test is contained in the following

THEOREM 8.2. *If*

(i) *for all i, $\lim_{n\to\infty} m_i(n)/n = s_i$ exists and is positive,*

(ii) *the distribution function $F(x)$ satisfies the assumptions of Lemma* 7.2, *and*

$$\text{(iii)}\qquad \int_{-\infty}^{+\infty} x^2\,dF(x) - \left(\int_{-\infty}^{+\infty} x\,dF(x)\right)^2 = \sigma^2$$

exists, then, the asymptotic relative efficiency of the classical $\mathfrak{F}$ test with respect to an arbitrary $\mathfrak{L}$ test for testing the hypothesis H_0 against H^P_n is

$$(8.3)\qquad e^P_{\mathfrak{F},\mathfrak{L}}(F(x)) = \frac{A^2}{\sigma^2}\left(1 \Big/ \int_{-\infty}^{+\infty} \frac{d}{dx} J[F(x)]\,dF(x)\right)^2.$$

PROOF. The $\mathfrak{F}$ statistic is defined as

$$\mathfrak{F} = \frac{1}{c-1}\sum_{i=1}^{c} m_i(X_{i\cdot} - \bar{X})^2 \Big/ \frac{1}{N-c}\sum_{i=1}^{c}\sum_{j=1}^{m_i}(X_{ij} - X_i)^2$$

where $X_{i\cdot} = \sum_{j=1}^{m_i} X_{ij}/m_i$ and $\bar{X} = \sum_{i=1}^{c}\sum_{j=1}^{m_i} X_{ij}/N$. It has been shown by Andrews [1] that under the hypothesis H^P_n, this has a limiting noncentral chi-square distribution with $c - 1$ degrees of freedom and noncentrality parameter $\lambda^{\mathfrak{F}}(H^P_n)$ given by

$$(8.5)\qquad \lambda^{\mathfrak{F}}(H^P_n) = \sum_{i=1}^{c} s_i[(\theta_i - \bar{\theta})/\sigma]^2.$$

Now proceeding by standard arguments, the proof follows.

In particular, when $J = \Phi^{-1}$, where Φ is the standard cumulative normal distribution function having the density ϕ,

$$(8.6)\qquad e^P_{\mathfrak{L},\mathfrak{F}}(F(x)) = \sigma^2\left(\int_{-\infty}^{+\infty} \frac{f^2(x)\,dx}{\phi\{\Phi^{-1}[F(x)]\}}\right)^2$$

which is known to be the asymptotic efficiency of the two sample normal scores test with respect to the student's t-test and is always $\geqq 1$. When $F(x)$ is a normal distribution function, this is 1. See in this connection Chernoff-Savage [2] and Hodges-Lehmann [11].

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APPENDIX

10. Higher order terms. Before we prove that the C terms of Lemma 5.1 are uniformly of higher order, we state the following elementary results which are used repeatedly. (Also in this connection see Chernoff and Savage [2].)

10A. *Elementary results.*

1. $H \geqq \lambda_i F^{(i)} \geqq \lambda_0 F^{(i)}$; $\quad i = 1, \cdots, c$.
2. $1 - F^{(i)} \leqq (1 - H)/\lambda_i \leqq (1 - H)/\lambda_0$. $\quad i = 1, \cdots, c$.
3. $F^{(i)}(1 - F^{(i)}) \leqq H(1 - H)/\lambda_i^2 \leqq H(1 - H)/\lambda_0^2$; $\quad i = 1, \cdots, c$.
4. $dH \geqq \lambda_i \, dF^{(i)} \geqq \lambda_0 \, dF^{(i)}$; $\quad i = 1, \cdots, c$.
5. Let (a_N, b_N) be the interval S_{N_ϵ} where

$$S_{N_\epsilon} = \{x: H(1 - H) > \eta_\epsilon \lambda_0/N\}, \tag{10.1}$$

when η_ϵ can be chosen independent of $F^{(i)}$ and λ_i; $i = 1, \cdots, c$, such that

$$P\{X_{ij} \,\varepsilon\, S_{N_\epsilon}; i = 1, \cdots, c; j = 1, \cdots, m_i\} \geqq 1 - \epsilon. \tag{10.2}$$

10B. *Detailed consideration of the C-terms of Lemma* 5.1. First, let us consider

$$\begin{aligned} C_{iN} &= \lambda_i \int_{0<H<1} [S_{m_i}^{(i)}(x) - F^{(i)}(x)] J'[H(x)] \, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)]; \\ &\qquad\qquad i = 1, \cdots, j-1, j+1, \cdots, c \\ &= \lambda_i [C_{1N}^{(i)} + C_{2N}^{(i)}]; \qquad i = 1, \cdots, c; i \neq j, \end{aligned} \tag{10.3}$$

where

$$C_{1N}^{(i)} = \int_{S_{N_\epsilon}} [S_{m_i}^{(i)}(x) - F^{(i)}(x)] J'[H(x)] \, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)]; \qquad i = 1, \cdots, c; i \neq j, \tag{10.4}$$

and

$$C_{2N}^{(i)} = \int_{\bar{S}_{N_\epsilon}} [S_{m_i}^{(i)}(x) - F^{(i)}(x)] J'[H(x)] \, d[S_{m_j}^{(j)}(x) - F^{(j)}(x)]; \qquad i = 1, \cdots, c; i \neq j. \tag{10.5}$$

First note that

$$E(C_{1N}^{(i)}) = E\{E(C_{1N}^{(i)} \mid X_{j1}, \cdots, X_{jm_j})\} = 0; i = 1, \cdots, c; i \neq j. \tag{10.6}$$

Next,

$$[C_{1N}^{(i)}]^2 = 2 \iint_{x,y \varepsilon S_{N_\epsilon}, x<y} [S_{m_i}^{(i)}(x) - F^{(i)}(x)][S_{m_i}^{(i)}(y) - F^{(i)}(y)]J'[H(x)]J'[H(y)] \cdot d[S_{m_j}^{(j)}(x) - F^{(j)}(x)]\, d[S_{m_j}^{(j)}(y) - F^{(j)}(y)]$$

$$+ \frac{1}{m_j} \int_{x \varepsilon S_{N_\epsilon}} [S_{m_i}^{(i)}(x) - F^{(i)}(x)]^2 [J'[H(x)]]^2\, dS_{m_j}^{(j)}(x); \qquad i = 1, \cdots, c; i \neq j.$$

Therefore,

$$E(C_{1N}^{(i)})^2 = E[E\{(C_{1N}^{(i)})^2 \mid X_{j1}, \cdots, X_{jm_j}\}]$$

$$= -\frac{2}{m_i m_j} \iint_{x,y \varepsilon S_{N_\epsilon}, x<y} F^{(i)}(x)[1 - F^{(i)}(y)] \cdot J'[H(x)]J'[H(y)]\, dF^{(j)}(x)\, dF^{(j)}(y)$$

$$+ \frac{1}{m_i m_j} \int_{x \varepsilon S_{N_\epsilon}} F^{(i)}(x)[1 - F^{(i)}(x)]\{J'[H(x)]\}^2\, dF^{(j)}(x); \qquad i = 1, \cdots, c; i \neq j$$

(10.7)

$$\leqq \frac{K}{N^2} \iint_{x,y \varepsilon S_{N_\epsilon}, x<y} H(x)[1 - H(y)][H(x)(1 - H(x))]^{-(\frac{3}{2})+\delta} \cdot [H(y)(1 - H(y))]^{-(\frac{3}{2})+\delta}\, dH(x)\, dH(y)$$

$$+ \frac{K}{N^2} \int_{x \varepsilon S_{N_\epsilon}} H(x)[1 - H(x)][H(x)(1 - H(x)]^{(-3+2\delta)}\, dH(x)$$

$$\leqq \frac{K}{N^2} + \frac{K\eta_\epsilon^{-1+2\delta}}{N^{1+2\delta}} = o\left(\frac{1}{N}\right); \qquad [K \text{ is generic}].$$

Hence from (10.6) and (10.7), we obtain, using Markoff inequality,

$$|C_{1N}^{(i)}| = o_p(N^{-(\frac{1}{2})}). \tag{10.8}$$

We now consider $C_{2N}^{(i)}$. Let $H_1 = H(a_N)$, $H_2 = H(b_N)$. Then from (10.1) $H_1 = 1 - H_2 < K/N$. With probability greater than $1 - \epsilon$, there are no observations in $\bar{S}_{N_\epsilon}$ and

$$|C_{2N}^{(i)}| \leqq \int_0^{H_1} F^{(i)}(x)|J'[H(x)]|\, dF^{(j)}(x)$$

$$+ \int_{H_2}^1 (1 - F^{(i)}(x)|J'[(x)]|\, dF^{(j)}(x); \qquad i = 1, \cdots, c; i \neq j$$

(10.9)

$$\leqq K \int_0^{H_1} \frac{H\, dH}{[H(1 - H)]^{(\frac{3}{2})-\delta}} + \int_{H_2}^1 \frac{(1 - H)\, dH}{[H(1 - H)]^{(\frac{3}{2})-\delta}}$$

$$\leqq K \int_0^{H_1} H^{-(\frac{1}{2})+\delta}\, dH \leqq K \frac{1}{N^{(\frac{1}{2})+\delta}}.$$

Hence

$$(10.10) \qquad |C_{2N}^{(i)}| = o_p(N^{-(\frac{1}{2})}); \qquad i = 1, \cdots, c; i \neq j.$$

Consequently,

$$(10.11) \qquad C_{iN} = \lambda_i[C_{1N}^{(i)} + C_{2N}^{(i)}] = o_p(N^{-(\frac{1}{2})}); \quad i = 1, \cdots, c; i \neq j.$$

The proof of $C_{jN} = o_p(N^{-(\frac{1}{2})})$ follows by first showing that

$$C_{jN} = -\tfrac{1}{2}\lambda_j[C_{11N} + C_{12N} - C_{13N}]$$

where

$$\text{(a)} \qquad C_{11N} = \int_{S_{N_\epsilon}} [S_{m_j}^{(j)}(x) - F^{(j)}(x)]^2 J''[H(x)]\, dH(x),$$

$$\text{(b)} \qquad C_{12N} = \int_{\bar{S}_{N_\epsilon}} [S_{m_j}^{(j)}(x) - F^{(j)}(x)]^2 J''[H(x)]\, dH(x),$$

$$\text{(c)} \qquad C_{13N} = \frac{1}{m_j} \int J'[H(x)]\, dS_{m_j}^{(j)}(x)$$

and then showing that each C_{1KN} is $o_p(N^{-(\frac{1}{2})})$; $k = 1, 2, 3$. The proofs of the above statement are omitted since they are essentially contained in the work of Chernoff and Savage [2].

Next consider

$$C_{c+1,N} = \int_{I_N} [H_N(x) - H(x)]^2 J''[\theta H_N(x) + (1 - \theta)H(x)]\, dS_{m_j}^{(j)}(x),$$
$$0 < \theta < 1.$$

With probability $>1 - \epsilon$, the interval I_N can be replaced by S_{N_ϵ} without changing $C_{c+1,N}$. Furthermore since

$$\mathrm{Sup}_{H_N>0} |H(x)/H_N(x)| = O_p(1)$$

and

$$\mathrm{Sup}_{H_N<1} |[1 - H(x)]/[1 - H_N(x)]| = O_p(1),$$

for each $\epsilon > 0$, there exists an $\eta_\epsilon^* > 0$ such that with probability greater than $1 - \epsilon$, we have for $\{x: 0 < H_N(x) < 1\}$,

$$[\theta H_N(x) + (1 - \theta)H(x)][1 - \{\theta H_N(x) + (1 - \theta)H(x)\}] > \eta_\epsilon^* H(x)[1 - H(x)].$$

Then

$$|C_{c+1,N}| \leqq K(\eta_\epsilon^*)^{-(\frac{3}{2})+\delta} C_{\alpha N}$$

where

$$C_{\alpha N} = \int_{S_{N_\epsilon}} [H_N(x) - H(x)]^2 \{H(x)[1 - H(x)]\}^{-(\frac{3}{2})+\delta}\, dS_{m_j}^{(j)}(x)$$

and

$$E(C_{\alpha N}) = E[E(C_{\alpha N} \mid X_{j1}, \cdots, X_{jm_j})]$$

$$= \frac{1}{N} \int_{S_{N_\epsilon}} \sum_{i=1}^{c} \lambda_i F^{(i)}(1 - F^{(i)})[H(1 - H)]^{-(\frac{3}{2})+\delta} \, dF^{(j)}(x)$$

$$+ \frac{1}{N^2} \int_{S_{N_\epsilon}} (1 - F^{(j)})(1 - 2F^{(j)})[H(1 - H)]^{-(\frac{3}{2})+\delta} \, dF^{(j)}(x)$$

$$\leqq \frac{K}{N} \int_{S_{N_\epsilon}} [H(1 - H)]^{-(\frac{3}{2})+\delta} \, dH + \frac{K}{N^2} \int_{S_{N_\epsilon}} [H(1 - H)]^{-(\frac{3}{2})+\delta} \, dH$$

$$\leqq \frac{K}{N^{(\frac{1}{2})+\delta}}.$$

Consequently $C_{c+1,N} = o_p(N^{-(\frac{1}{2})})$.

The negligibility of $C_{c+2,N}$ and $C_{c+3,N}$ follows from Assumptions 2 and 3 of Lemma 5.1 and the proof of the negligibility of $C_{c+4,N}$ proceeds in the same manner as given by Chernoff and Savage for the term C_{4N} and therefore is not given here.

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