NOTES

ON A PROBLEM IN NON-LINEAR PREDICTION THEORY

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In [2] Masani and Wiener proved that the best least squares non-linear predictor $E(f_n \mid B_0)$ can, in theory, be calculated from the time series $\{f_n, n \leqq 0\}$ if one is given that $\{f_n, -\infty < n < \infty\}$ is a bounded, ergodic, strictly stationary stochastic process such that the spectrum of the joint distributions $F_{n_1, \dots, n_k}$ for distinct $n_1, \cdots, n_k$ has positive (k-dimensional) Lebesgue measure. In this note we prove that $E(f_n \mid B_0)$ is determined by the time series $\{f_n, n \leqq 0\}$ (and is free of the possibly unknown probability measure) if one is only given that $\{f_n, -\infty < n < \infty\}$ is an integrable strictly stationary stochastic process.

Our result is not really a generalization of the result of Masani and Wiener. They actually give a procedure, albeit impractical, for calculating $E(f_n \mid B_0)$, whereas our method is nonconstructive.

We shall use the following definitions from [2]:

R = real numbers;

Z = integers;

$\Omega = R^Z$ = the set of all functions from Z to R;

$f_n : \Omega \to R$ defined by $f_n(g) = g(n)$ for all g in Ω;

B_n = "present and past of f_n" = the σ-field generated by $\{f_k, k \leqq n\}$;

$B_{-\infty}$ = "remote past of $\{f_n\}_{-\infty}^{\infty}$" = $\bigcap_n B_n$;

B_∞ = "space spanned by $\{f_n\}_{-\infty}^{\infty}$" = the σ-field generated by $\bigcup_n B_n$;

$T : \Omega \to \Omega$ defined by $T(g)(n) = g(n+1)$ for all g in Ω and for all n in Z.

Suppose $\{\tilde{f}_n(\omega)\}_{-\infty}^{\infty}$ is a time series where $\{\tilde{f}_n\}_{-\infty}^{\infty}$ is a strictly stationary stochastic process on a probability space $(\tilde{\Omega}, \tilde{B}, P)$. Then by considering the mapping $\varphi : \omega \to \{\tilde{f}_n(\omega)\}_{-\infty}^{\infty}$ from $\tilde{\Omega}$ to Ω, we can assume that the time series $\{\tilde{f}_n(\omega)\}_{-\infty}^{\infty}$ came from the process $\{f_n\}_{-\infty}^{\infty}$ on (Ω, B_∞, π) where $\pi = P \circ \varphi^{-1}$ is an invariant probability measure for T (i.e. $\pi(A) = \pi(T(A))$ for all A in B_∞). Thus we shall assume that the strictly stationary stochastic process is $\{f_n\}_{-\infty}^{\infty}$ defined on (Ω, B_∞, π) for some π in the set S of all probability measures on B_∞ which are invariant under T and for which f_0 is integrable.

Consider those π in S for which f_n is square integrable. A main problem in non-linear prediction theory is for fixed n to calculate the B_0-measurable function $\hat{f}_n$ for which $\int_\Omega |f_n - \hat{f}_n|^2 \, d\pi = \min \{\int_\Omega |f_n - g|^2 \, d\pi, g \text{ in } L_2(\Omega, B_0, \pi)\}$. It is well known that $\hat{f}_n = E_\pi(f_n \mid B_0)$ a.e. $[\pi]$ where $E_\pi(f_n \mid B_0)$ is the π conditional expectation of f_n given B_0. Since π is the only unknown quantity, n order for $E_\pi(f_n \mid B_0)$ to be determined by $\{f_n, n \leqq 0\}$ there must exist a B_0-measurable

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function $\hat{f}_n$, independent of π, such that $\hat{f}_n = E_\pi(f_n \mid B_0)$ a.e. $[\pi]$ for all π in S. This will follow if B_0 is a sufficient subfield for (Ω, B_n, S).

THEOREM. *B_n is a sufficient subfield for* (Ω, B_∞, S), $-\infty \leqq n \leqq \infty$.

PROOF. Let $I = \{A \text{ in } B_\infty : T(A) = A\}$. We shall first show that $I \cap B_{-\infty}$ is a sufficient subfield for (Ω, B_∞, S). Let A be in B_m, m finite, and let 1_A be its indicator function. Define

$$\begin{aligned} g_A &= \lim_{n\to\infty} n^{-1} \textstyle\sum_{k=1}^n 1_A \circ T^{-k} \text{ whenever the limit exists} \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then g_A is I-measurable and by the ergodic theorem $g_A = E_\pi(1_A \mid I)$ a.e. $[\pi]$ for all π in S. Since g_A is B_m-measurable and $g_A = g_A \circ T^{-k}$, g_A is $I \cap B_{m-k}$-measurable for all k. Hence g_A is $I \cap B_{-\infty}$-measurable and

$$g_A = E_\pi(1_A \mid I \cap B_{-\infty}) \text{ a.e. } [\pi] \text{ for all } \pi \text{ in } S. \tag{1}$$

Let M be the set of all A in B_∞ for which there exists a $I \cap B_{-\infty}$-measurable function g_A satisfying (1). It follows that if $A \subseteq B$, then $g_A \leqq g_B$ a.e. $[\pi]$ for all π in S. Hence, by the monotone convergence theorem, M is a monotone class. We have shown that M contains the field $\bigcup_n B_n$, and therefore $M = B_\infty$. Thus $I \cap B_{-\infty}$ is a sufficient subfield for (Ω, B_∞, S).

B_n, for n finite, is a separable σ-field since it is generated by a countable number of real valued functions, f_k, $k \leqq n$. Therefore, since B_n contains the sufficient subfield $I \cap B_{-\infty}$, B_n is itself a sufficient subfield for (Ω, B_∞, S), cf. Burkholder ([1], Theorem 5). Since $B_{-\infty} = \bigcap_n B_n$ and $B_n \subseteq B_{n+1}$ for all n, $B_{-\infty}$ is also a sufficient subfield for (Ω, B_∞, S), cf. Burkholder ([1], Theorem 3 (i)). Q.E.D.

Let $g(\cdots, f_{-1}, f_0) = E_\pi(f_n\ B_0)$ a.e. $[\pi]$ for all π in S. It should be remarked that g is independent of π and must agree with various known results. Thus if $\{f_n, -\infty < n < \infty\}$ is bounded and ergodic, $g(\cdots, f_{-1}, f_0)$ must agree *almost surely* with the procedure given by Masani and Wiener in [2]. If $\{f_n, -\infty < n < \infty$ is normally distributed with zero mean, then $g(\cdots, f_{-1}, f_0)$ must agree *almost surely* with the value given by linear prediction. Finally if $\{f_n, -\infty < n < \infty\}$ is independent identically distributed, then $g(\cdots, f_{-1}, f_0) = lim_{n\to\infty}, n^{-1} \sum_{k=0}^{n=1} f_{-k}$ *almost surely*!

REFERENCES

[1] BURKHOLDER, D. L. (1961). Sufficiency in the undominated case. *Ann. Math. Statist.* **32** 1191–1200.

[2] MASANI, P. and WIENER, N. (1959). Non-linear prediction. *Probability and Statistics The Harald Cramér Volume*, Wiley, New York. 190–212.