ON THE SAMPLE SIZE AND SIMPLIFICATION OF A CLASS OF SEQUENTIAL PROBABILITY RATIO TESTS[1]

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Summary. T is the sequential probability ratio test (SPRT) based on the sequence $\{X_n\}$ whose family of distributions $\{P_\theta, \theta \varepsilon \Theta\}$ satisfies certain sufficiency, monotone likelihood ratio, and consistency assumptions. Sufficiency reduces the criterion of T to $q_{\theta_2 n}/q_{\theta_1 n}$, where $q_{\theta n}$ is the density of X_n, θ_1 and θ_2 are the values of θ specified by the hypothesis and alternative respectively, and $\theta_1 < \theta_2$. It is assumed that $q_{\theta n}(x)$ has the asymptotic form: $q_{\theta n}(x) \sim f_{\theta n}(x) \equiv K(n)C(\theta, x)e^{nh(\theta,x)}$ as $n \to \infty$. The Simplified SPRT T^* is proposed where T^* uses the criterion $e^{nh(\theta_2,\cdot)}/e^{nh(\theta_1,\cdot)}$. The following conditions are relevant: Condition C states that $h(\theta, x)$ has a unique maximum at $x = \theta$ and $h(\theta, \theta)$ is free of θ; Condition D(i) requires $\Delta_{\theta n}(x) \equiv q_{\theta n}(x)/f_{\theta n}(x)$ to be bounded for all θ, x, and n; and Condition D(ii) states that for each θ, $\Delta_{\theta n}(x) \to 1$ uniformly in x for x in a neighborhood of $x = \theta$. Let N and N^* be the sample sizes of T and T^* respectively, and let θ_0 be the solution of $h(\theta_2, x) = h(\theta_1, x)$. It is shown in Section 3 that Conditions C and D imply: $P_\theta(N^* > n) < \gamma\delta^n/n^{\frac{1}{2}}$, where $0 < \delta < 1, \gamma < \infty$ and $\theta \neq \theta_0$. The same is true for N. Thus, the moment generating functions of N and N^* exist, and inequalities for the expected values of N and N^* are readily obtained with respect to any P_θ, $\theta \neq \theta_0$. The following monotonicity properties of $E_\theta N$ and $E_\theta N^*$ are established under an additional condition in Section 4: the expected values increase for θ less than a certain interval containing θ_0 and decrease for θ greater than this interval. Several examples are discussed in Section 5, and the conditions are checked in Section 6.

0. Introduction. $\{X_n, n \geqq n_0\}$ is a sequence of random variables defined on the probability space $(\Omega, \mathcal{A}, \mathcal{P})$, where $\mathcal{P} \equiv \{P_\theta, \theta \varepsilon \Theta\}$ is a family of distributions and θ is real; $\mathcal{A}_n$ is the Borel field generated by $\{X_j, j \leqq n\}$; and $P_{\theta n}$ is the density of P_θ on $\mathcal{A}_n$ with respect to some σ-finite measure. It is desired to test the hypothesis $H_1 : \theta = \theta_1$ against the alternative $H_2 : \theta = \theta_2$, where $\theta_1 < \theta_2$. Let $R_n \equiv P_{\theta_2 n}/P_{\theta_1 n}$ and let T denote the sequential probability ratio test (SPRT) ([7], [4]) based on $\{X_n\}$ with B and A as stopping bounds, where $0 < B < 1 < A < \infty$. Finally, let $q_{\theta n}(x)$ be the density of P_θ on the Borel field generated by X_n.

We say that $\{X_n\}$ is of the S-MLR case whenever the following assumptions hold: (i) X_n is sufficient (S) for $\mathcal{P}$ on $\mathcal{A}_n$ for each n; (ii) $\mathcal{P}$ is a monotone likelihood ratio family (MLR) on $\mathcal{A}_n$ for each n; (iii) $X_n \to \theta$ a.e. P_θ (consistency) for each $\theta \varepsilon \Theta$; (iv) $q_{\theta n}(x) > 0$ if and only if $x \varepsilon \Theta$; and (v) Θ is an interval.

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Several aspects of the S-MLR case were treated by Wirjosudirdjo [9], the author [3], Berk [2] and Wijsman (8). The assumptions given above are as stated in [3]. The results of this paper resemble the corresponding well-known results about the IID case, where the X_i are identically and independently distributed.

We remark that the S-MLR case covers many examples of the IID case (e.g., when the distribution of the latter belongs to the exponential family). An interesting example, where the converse holds, is given in Subsection 5.1. This result was found by C. Stein and M. A. Girshick [2] independently of each other as remarked in [7], p. 133.

1. Conditions. Throughout this paper limits are taken as $n \to \infty$, and $a_n \sim b_n$ means that $a_n/b_n \to 1$. The first four of the following conditions are as stated in [3].

CONDITION B. (i) *There exist functions* K, $C > 0$ *and* h *such that* $q_{\theta n}(x) \sim f_{\theta n}(x) \equiv K(n)C(\theta, x)e^{nh(\theta,x)}$ *for each* θ, $x \varepsilon \Theta$, *and, thus,* $n^{-1} \ln r_n(x) \to g(x) \equiv h(\theta_2, x) - h(\theta_1, x)$; (ii) $g(x)$ *is strictly increasing and continuous for any* $\theta_1 < \theta_2$.

CONDITION A_2. *For each fixed* x, $h(\theta, x)$ *has a unique maximum at* $\theta = x$.

CONDITION A_3. *If* $g(\theta_0) = 0$, *then* $r_n(\theta_0 + (c/n)) \to \alpha e^{c\beta}$ *for some* α, β *and all* $c \neq 0$.

CONDITION A_4. (i) *If* $g(\theta_0) = 0$, *then* $n^{\frac{1}{2}}(X_n - \theta_0)$ *has under* P_{θ_0} *a limiting distribution (not necessarily a probability distribution)* Q, *say, which is continuous at the origin;* (ii) $0 < Q([0, \infty)) < 1$.

CONDITION C. *For each fixed* θ, $h(\theta, x)$ *has a unique maximum at* $x = \theta$; *and* $h(\theta, \theta)$ *is a constant free of* θ. *In view of Condition* B, *it may be assumed without loss of generality that* $h(\theta, \theta) = 0$ *for all* $\theta \varepsilon \Theta$.

Let $\Delta_{\theta n}(x) = q_{\theta n}(x)/f_{\theta n}(x)$. By Condition B, $\Delta_{\theta n}(x) \to 1$ for all θ, $x \varepsilon \Theta$.

CONDITION D. (i) *There exist constants* k *and* k' *such that* $k \leqq \Delta_{\theta n}(x) \leqq k'$ *for all* θ, $x \varepsilon \Theta$ *and all* n ($k' \geqq 1$ *necessarily*); (ii) *for each* $\theta \varepsilon \Theta$, *there exists a neighborhood* $U(\theta) \subset \Theta$, *such that* $\Delta_{\theta n}(x) \to 1$ *uniformly in* x *for* $x \varepsilon U(\theta)$.

CONDITION E. *For each* $x \varepsilon \Theta$, *there exist* ϵ_1 *and* $\epsilon_2 \geqq 0$ *such that the* $q_{\theta n}(x)$ *are increasing or decreasing in* θ, *for all* n, *according as* $\theta \leqq x - \epsilon_1$ *or* $x \geqq x + \epsilon_2$.

2. The Simplified SPRT. The sufficiency assumption implies that $R_n = r_n(X_n)$, where $r_n = q_{\theta_2 n}/q_{\theta_1 n}$.

CONCLUSION 2.1. *Let* $\{X_n\}$ *be of the* S-MLR *case and let* b_n *and* a_n *be the solutions of* $r_n(x) = B$ *and* $r_n(x) = A$ *respectively. Then, at the* nth *stage,* T *rejects* H_1 *if* $X_n \geqq a_n$, *accepts* H_1 *if* $X_n \leqq b_n$, *and resumes sampling if* $b_n < X_n < a_n$. *Moreover,* b_n, $a_n \to \theta_0$ *if Condition* B *holds.*

PROOF. The first statement is immediate from the MLR property. By Condition B, if $\epsilon > 0$ then $r_n(\theta_0 \pm \epsilon) \to \pm \infty$ and, hence, the second statement.

DEFINITION 2.1. *Let* $\{X_n\}$ *be of the* S-MLR *case and let Condition* B *hold. Let* $R_n^* \equiv r_n^*(X_n)$, *where* $r_n^*(\cdot) \equiv e^{nh(\theta_2,\cdot)}/e^{nh(\theta_1,\cdot)} \equiv e^{ng(\cdot)}$. *The test* T^* *is called the Simplified* SPRT *based on* $\{X_n\}$ *if* T^* *uses the criterion* R_n^*: *at the* nth *stage* T^* *rejects* H_1 *if* $X_n \geqq a_n^*$, *accepts* H_1 *if* $X_n \leqq b_n^*$, *and resumes sampling if*

$b_n^* < X_n < a_n^*$, *where* $b_n^* = g^{-1}(n^{-1} \ln B) < \theta_0 < g^{-1}(n^{-1} \ln A) = a_n^*$ *and* (*therefore*) $b_n^*, a_n^* \to \theta_0$.

Since Lemma 2.1 of [3] requires only the monotonicity of $r_n(x)$ we conclude that the limiting behavior of R_n^* is identical to that of R_n and is governed by $g(\theta)$ under the hypothesis of Theorem 2.3 of [3]: $R_n \to 0$ or ∞ a.e. P_θ according as $g(\theta) <$ or > 0, whereas $\liminf R_n = 0$ and $\limsup R_n = \infty$ a.e. P_{θ_0} where $g(\theta_0) = 0$. Thus, the above-mentioned hypotheses (essentially Conditions B, A_3 and A_4) imply that T^*, like T, terminates almost surely with respect to any P_θ.

3. The distribution of the sample sizes. We assume in this section that densities are taken with respect to Lebesgue measure.

LEMMA 3.1. *Let* $\{X_n\}$ *be of the* S-MLR *case and let Conditions* B, C *and* D *hold. Assume that* $C(\theta, x)$ *is a continuous and* $h(\theta, x)$ *is a twice continuously differentiable function of* x. *Let* $\lambda = \lim K(n)/n^{\frac{1}{2}}$. *Then, Condition* A_4 *implies that* $0 < \lambda < \infty$.

PROOF. By Conditions B and D(ii), given ϵ and $y > 0$, there exists m such that

$$(3.1) \qquad (1 - \epsilon)q_{\theta n}(x) \leqq K(n)C(\theta, x)e^{nh(\theta,x)} \leqq (1 + \epsilon)q_{\theta n}(x)$$

for all $x \in I_n \equiv (\theta, \theta + y/n^{\frac{1}{2}})$ and $n \geqq m$. Let $Y_n = n^{\frac{1}{2}}(X_n - \theta)$. Since $\int_{I_n} q_{\theta n}(x)\, dx = P(0 < Y_n < y)$, Condition C, the hypothesis, and (3.1) imply

$$(3.2) \qquad (1 - \epsilon)P_\theta(0 < Y_n < y) \leqq (K(n)/n^{\frac{1}{2}})yC(\theta, y_1),$$

$$(3.3) \qquad (K(n)/n^{\frac{1}{2}})e^{nh(\theta,\theta+y/n^{\frac{1}{2}})}yC(\theta, y_2) \leqq (1 + \epsilon)P_\theta(0 < Y_n < y)$$

for some y_1, y_2 in I_n. In view of Condition A_4, y can be chosen such that $0 < Q([0, y)) < 1$. By applying Taylor's theorem to $h(\theta, \theta + y/n^{\frac{1}{2}})$, using Condition C in (3.3), and letting $n \to \infty$ in (3.2) and (3.3), we conclude the proof.

COROLLARY 3. *Let the assumptions of Lemma* 3.1 *hold, and let* $q^*_{\theta n}(y)$ *denote the density of* $Y_n = n^{\frac{1}{2}}(X_n - \theta)$. *Then,* $q^*_{\theta n}(y) \to \lambda C(\theta, \theta)e^{h''(\theta,\theta)y^2/2}$. *Thus, if* $\lim q^*_{\theta n}(y)$ *is a density, then it is necessarily the density of a* normal $(0, \sigma^2(\theta))$ *variable, where* $\sigma^2(\theta) \equiv -1/h''(\theta, \theta)$. *Moreover,* $C(\theta, \theta) = [-h''(\theta, \theta)]^{\frac{1}{2}}/[(2\pi)^{\frac{1}{2}}\lambda]$.

PROOF. Notice that $q^*_{\theta n}(y) = (1/n^{\frac{1}{2}})q_{\theta n}(\theta + y/n^{\frac{1}{2}})$. We start with

$$(3.4) \qquad (1 - \epsilon)n^{-\frac{1}{2}}f_{\theta n}(\theta + y/n^{\frac{1}{2}}) \leqq q^*_{\theta n}(y) \leqq (1 + \epsilon)n^{-\frac{1}{2}}f_{\theta n}(\theta + y/n^{\frac{1}{2}})$$

for all x between θ and $\theta + y/n^{\frac{1}{2}}$. The steps of the preceding proof give the conclusion when $y \neq 0$. The case $y = 0$ is treated by continuity.

LEMMA 3.2. *Let* $\{X_n\}$ *be of the* S-MLR *case and let Conditions* B, C, D, *and* A_4 *hold. Assume that* $C(\theta, x)$ *is a continuous and* $h(\theta, x)$ *is a continuously differentiable function of* x. *Let* t_1 *and* t_2 *be any two real numbers with* $t_1 < t_2$. *Then, for any* $\theta \neq \theta_0$, *there exists* $\gamma > 0$ *such that*

$$(3.5) \qquad P_\theta(t_1/n < g(X_n) < t_2/n) \leqq [(t_2 - t_1)/n^{\frac{1}{2}}]\gamma\delta^n$$

where $0 < \delta \equiv e^{h(\theta,\theta_0)} < 1$.

PROOF. Since $y = g(x)$ is a strictly increasing differentiable function (Condition B and the hypothesis), the same is true of its inverse $x = u(y)$. Let G_n and π_n denote the event in question and its probability. By Condition D(i),

$$\begin{aligned} \pi_n &\leqq k'K(n) \int_{G_n} C(\theta, x)e^{nh(\theta,x)}\,dx \\ &= k'K(n) \int_{t_1/n}^{t_2/n} C(\theta, u(y))e^{nh(\theta,u(y))}u'(y)\,dy \\ &= k'(t_2 - t_1)(K(n)/n)C(\theta, u(y_n))e^{nh(\theta,u(y_n))}u'(y_n) \end{aligned} \tag{3.6}$$

for some y_n, $t_1/n < y_n < t_2/n$, and each n. Let $t = \max(|t_1|, |t_2|)$. Since $u(0) = \theta_0$, the mean value theorem implies

$$\begin{aligned} nh(\theta, u(y_n)) &= nh(\theta, \theta_0) + ny_n u'(\bar{y}_n)h'(\theta, u(\bar{y}_n)) \\ &\leqq nh(\theta, \theta_0) + tu'(\bar{y}_n)h'(\theta, u(\bar{y}_n)) \end{aligned} \tag{3.7}$$

where $-t/n < \bar{y}_n < t/n$, and $h' = \partial h/\partial x$. Notice that the continuity of the following functions enables us to bound them: $C(\theta, u(y_n))$, $u'(y_n)$, $u'(\bar{y}_n)$, $h'(\theta, u(\bar{y}_n))$. The constant k' in (3.6) may be replaced by $\max \Delta_{\theta n}(u(y))$, where the maximum is taken as y ranges over $(t_1/n, t_2/n)$. Thus, by (3.6), (3.7), and the preceding comments

$$\pi_n \leqq (K(n)/n)(t_2 - t_1)\gamma' e^{nh(\theta,\theta_0)} \tag{3.8}$$

for some γ', $0 < \gamma' < \infty$. Lemma 3.1 and (3.8) imply (3.5).

THEOREM 3.1. *Let $\{X_n, n \geqq n_0\}$ be of the* S-MLR *case and let T and T^* be the* SPRT *and Simplified* SPRT *based on $\{X_n\}$ with B and A as stopping bounds. Assume that Conditions* B, C, D *and* A_4 *hold. Assume further that for each $\theta \varepsilon \Theta$, $q_{\theta n}(x)$ and $C(\theta, x)$ are continuous while $h(\theta, x)$ is twice continuously differentiable functions of x. Then, for $\theta \neq \theta_0$, there exists $\gamma > 0$ such that*

$$P_\theta(N > n) \leqq (1/n^{\frac{1}{2}}) \ln (A/B)\gamma\delta^n \tag{3.9}$$

where $0 < \delta = e^{h(\theta,\theta_0)} < 1$ and $n \geqq n_0$. The same holds for N^*.

PROOF. Lemma 3.2 gives the result for N^*. T continues sampling at the nth stage if X_n is observed to be x and if

$$\begin{aligned} n^{-1} \ln B &< n^{-1} \ln \{[\Delta_{\theta_2 n}(x)/\Delta_{\theta_1 n}(x)]\cdot[C(\theta_2, x)/C(\theta_1, x)]\} + g(x) \\ &< n^{-1} \ln A. \end{aligned} \tag{3.10}$$

Let $V_n(x)$ stand for the expression between brackets in (3.10). Conclusion 2.1 and the continuity of $V_n(x)$ guarantee the existence of V', V_n', V_n'' and V'' such that $0 < V' \leqq V_n' \leqq V_n(x) \leqq V_n'' \leqq V'' < \infty$ for all $x \varepsilon (b_n, a_n)$. Moreover, $V_n', V_n'' \rightarrow C(\theta_2, \theta_0)/C(\theta_1, \theta_0) \equiv V(\theta_0)$. Thus, $N > n$ implies

$$n^{-1} \ln (B/V'') < g(X_n) < n^{-1}(A/V'). \tag{3.11}$$

Notice that V' and V'' in (3.11) may be replaced by quantities which are arbitrarily close to $V(\theta_0)$. Absorbing the effect of the replacement in the γ of Lemma 3.2, we arrive at the result for N.

COROLLARY 3.2. *Under the assumptions of Theorem* 3.1, $E_\theta e^{tN}$ *and* $E_\theta e^{tN^*} < \infty$

for $t < -h(\theta, \theta_0)$ and $\theta \neq \theta_0$. Thus, N and N^* possess moments of all orders with respect to P_θ, $\theta \neq \theta_0$.

PROOF. From (3.9), we conclude that for some $J > 0$, $E_\theta e^{tN} \leqq J \sum (\delta e^t)^n < \infty$ for $t < \ln \delta^{-1} = -h(\theta, \theta_0)$ and $\theta \neq \theta_0$ ($h(\theta, \theta_0) < h(\theta, \theta) = 0$ by Condition C).

COROLLARY 3.3. *Under the assumptions of Theorem 3.1, there exists $\gamma > 0$ such that*

$$E_\theta N \leqq 1 + (n_0^{-\frac{1}{2}}) \ln (A/B)\gamma\delta^{n_0}/(1 - \delta). \tag{3.12}$$

The same holds for N^*.

PROOF. $E_\theta N = 1 + \sum_n P_\theta(N > n)$.

4. Monotonicity properties of the ASN function of T and T^*. The following lemma describes a property of any MLR family of distributions.

LEMMA 4.1. *Let $\{X_n\}$ be a sequence of random variables whose densities $\{q_{\theta n}(x)\}$ constitute a* MLR *family. Let $c_n(\theta_1, \theta_2)$ be the solution of $q_{\theta_1 n}(x) = q_{\theta_2 n}(x)$, $\theta_1 < \theta_2$. Then, $P_{\theta_2}(b < X_n < a) <$ or $> P_{\theta_1}(b < X_n < a)$ according as $b < a < c_n$ or $c_n < b < a$.*

PROOF. If $b < a < c_n$, then $q_{\theta_2 n}(x) < q_{\theta_1 n}(x)$ for all $x \,\varepsilon\, (b, a)$. Thus, the first part of the conclusion. The second part is similar.

Usually, Condition E is intuitively clear and may be checked directly or by means of the following lemma.

LEMMA 4.2. *Let $\{X_n\}$ be a sequence of random variables whose densities satisfy Conditions* B *and* A_2. *Assume that for each $x \,\varepsilon\, \Theta$, $h(\theta, x)$, $C(\theta, x)$ and the $\Delta_{\theta n}(x)$ are continuously differentiable functions of θ at $\theta = x$, while $q_{\theta n}(x)$ has a unique maximum at $\theta = M_n(x)$, say. Then, $M_n(x) \to x$, and Condition* E *holds with $x - \epsilon_1 = \inf_n M_n(x)$ and $x + \epsilon_2 = \sup_n M_n(x)$.*

PROOF. Notice that $n^{-1} \ln q_{\theta n}(x)$ has a unique maximum at $\theta = M_n(x)$, and that $n^{-1}(\partial/\partial\theta) \ln q_{xn}(x) \to (\partial/\partial\theta)h(x, x) = 0$. Thus, $M_n(x) \to x$. The last statement of the lemma is obvious.

THEOREM 4.1. *Let $\{X_n, n \geqq n_0\}$ be of the* S-MLR *case and let T and T^* be the* SPRT *and Simplified* SPRT *based on $\{X_n\}$ with B and A as stopping bounds. Assume that Conditions* B *and* E *hold. Then, there exist η_1 and $\eta_2 \geqq 0$ such that $P_\theta(N > n)$ is increasing or decreasing according as $\theta \leqq \theta_0 - \eta_1$ or $\geqq \theta_0 + \eta_2$, for all n. The same holds for N^*.*

PROOF. We prove the conclusion for N. Let $a = \sup_n (a_n(\theta_1, \theta_2))$ and $b = \inf_n(b_n(\theta_1, \theta_2))$, where the a_n and b_n are as defined in Conclusion 2.1. By Conclusion 2.1, $a, b \,\varepsilon\, \Theta$, and $b \leqq \theta_0 \leqq a$. In Condition E, take $x = a$ and b respectively. Define η_1 and η_2 through $\theta_0 + \eta_2 = a + \epsilon_2(a)$ and $\theta_0 - \eta_1 = b - \epsilon_1(b)$. If $\theta_0 + \eta_2 < \theta' < \theta''$, then $q_{\theta'' n}(a) < q_{\theta' n}(a)$ for all n and thus, by the MLR property, $a_n(\theta_1, \theta_2) < a < c_n(\theta', \theta'')$, where the last quantity is defined in Lemma 4.1. Using the sufficiency of X_n for $\mathcal{P}$ on $\mathcal{A}_n$, we obtain

$$\begin{aligned} P_\theta(N > n) &= P_\theta(b_j(\theta_1, \theta_2) < X_j < a_j(\theta_1, \theta_2), j = n_0, \cdots, n) \\ &= \int p_{\theta n}(x_{n_0}, \cdots, x_n)\, d\mu(x_{n_0}, \cdots, x_n) \\ &= \int q_{\theta n}(x_n) u(x_{n_0}, \cdots, x_n)\, d\mu(x_{n_0}, \cdots, x_n) \end{aligned} \tag{4.1}$$

where the integrals are taken over the obvious region, and u is free of θ. According to Lemma 4.1, $q_{\theta''n}(x) < q_{\theta'n}(x)$ for $b_n(\theta_1, \theta_2) < x < a_n(\theta_1, \theta_2)$ and all u. By (4.1), $P_{\theta'}(N > n) > P_{\theta''}(N > n)$ for all n. The conclusion for $\theta' < \theta'' < \theta_0 - \eta_1$ is similar.

COROLLARY 4.1. *Under the assumptions of Theorem* 4.1, *there exist* η_1 *and* $\eta_2 \geqq 0$ *such that* $E_\theta N$ *is increasing or decreasing according as* $\theta \leqq \theta_0 - \eta_1$ *or* $\geqq \theta_0 + \eta_2$. *The same holds for* N^*.

5. Examples. Most of the conditions of the preceding sections were checked in [3] for the examples given below. Conditions C, D, and E are discussed in Section 6. In each example, $\{X_n, n \geqq n_0\}$ is obtained through reduction by invariance under some group of transformations; $q_{\theta n}(x)$ is the density of X_n and $f_{\theta n}(x) \equiv K(n)C(\theta, x)e^{nh(\theta,x)}$ is its asymptotic equivalent. The $C(\theta, x)$ will not always be exhibited but are obtainable from Section 3 of [3]. B and A are the stopping bounds. The $h(\theta, x)$ given here may differ from those given in [3] by constants. This is done to satisfy Condition C ($h(\theta, \theta) = 0$).

5.1. *The sequential central* χ^2*-test.* The Z_j are IID Normal (ζ, θ) and $X_n = \sum_{i=1}^n (Z_i - \bar{Z}_n)^2/(n-1)$ for $n \geqq 2$.

$$q_{\theta n}(x) \equiv f_{\theta n}(x) \equiv K(n)x^{-1}e^{-h(\theta,x)}e^{nh(\theta,x)}, \tag{5.1.1}$$

$$h(\theta, x) = \tfrac{1}{2}[-(x/\theta) + \ln (x/\theta) + 1]. \tag{5.1.2}$$

Thus, T continues sampling at the nth stage if

$$\begin{aligned}(5.1.3)\quad [\theta_1\theta_2/(\theta_2 - \theta_1)][\ln (\theta_2/\theta_1) + 2 \ln B/(n-1)]& \\ < X_n < [\theta_1\theta_2/(\theta_2 - \theta_1)][\ln (\theta_2/\theta_1) + 2 \ln A/(n-1)&\end{aligned}$$

which is in agreement with [7] (p. 133). Upon replacing $(n-1)$ by n in (5.1.3) we obtain the corresponding result for T^*.

5.2 *The sequential t-test.* The Z_j are IID Normal (ζ, σ^2); $\theta = \zeta/\sigma$; and $X_n = \bar{Z}_n/S_n$, where $\bar{Z}_n$ and S_n are the sample mean and standard deviation respectively; and $n \geqq 2$. We conclude from [3] that

$$h(\theta, x) = \tfrac{1}{2}[\alpha^2(\xi\theta) + \ln \alpha^2(\xi\theta) - \theta^2 + \ln (1 - \xi^2) - 1], \tag{5.2.1}$$

$$\xi = x(1 + x^2)^{\frac{1}{2}}, \qquad \alpha(z) = [z + (z^2 + 4)^{\frac{1}{2}}]/2. \tag{5.2.2}$$

Thus, T^* continues sampling at the nth stage if $X_n/(1 + X_n^2)^{\frac{1}{2}} = \xi$ and

$$\begin{aligned}(5.2.3)\quad (\theta_2^2 - \theta_1^2) + (2/n) \ln B < \exp [2 \sinh^{-1} (\xi\theta_2/2)] + 2 \sinh^{-1} (\xi\theta_2/2)& \\ - \exp [2 \sinh^{-1}(\xi\theta_1/2] - 2 \sinh^{-1}(\xi\theta_1/2) < (\theta_2^2 - \theta_1^2) + (2/n) \ln A.&\end{aligned}$$

5.3. *The sequential* χ^2*-test.* The Z_j are IID where $Z_j = (Z_{j1}, \cdots, Z_{jq})$; the Z_{ji} are independently Normal $(\zeta_i, 1)$; $\theta = \sum_{i=1}^q \zeta_i^2$; and $X_n = \sum_{i=1}^q Z_{\cdot i}^2$. It follows from [3] that

$$f_{\theta n}(x) \equiv K(n)x^{-\frac{1}{2}}(x/\theta)^{(q-1)/4} \exp [-(n/2)(x^{\frac{1}{2}} - \theta^{\frac{1}{2}})^2]. \tag{5.3.1}$$

Thus, T^* continues sampling at the nth stage if

$$(5.3.2)\quad (\theta_1^{\frac{1}{2}} + \theta_2^{\frac{1}{2}})/2 + n^{-1}(\theta_2^{\frac{1}{2}} - \theta_1^{\frac{1}{2}})^{-1} \ln B < X_n^{\frac{1}{2}} < (\theta_1^{\frac{1}{2}} + \theta_2^{\frac{1}{2}})/2 + n^{-1}(\theta_2^{\frac{1}{2}} - \theta_1^{\frac{1}{2}})^{-1} \ln A.$$

5.4. *The sequential F-test.* The Z_j are IID where $Z_j = (Z_{j1}, \cdots, Z_{jk})$; the Z_{ji} are independently Normal (ζ_i, σ^2) and $\zeta_{s+1} = \cdots = \zeta_k = 0$ for some $s < k$; $\theta = \sum_{i=1}^{q} \zeta_i^2/k\sigma^2$ for some q, $1 \leqq q \leqq s$ [4]. For $n \geqq 2$, $X_n = n\sum_{i=1}^{q} Z_{\cdot i}^2 / \sum_{j=1}^{n}[Z_{i=1}^{s}(Z_{ji} - Z_{\cdot i})^2 + \sum_{i=s+1}^{k} Z_{ji}^2]$. It follows from [3] that $kh(\theta, x)$ is the same as h of (5.2.1) evaluated at $(\theta^{\frac{1}{2}}, x^{\frac{1}{2}})$. Thus, T^* continues sampling at the nth stage if $\xi \equiv x_n/(1 + x_n)$:

$$(5.4.1)\quad (\theta_2 - \theta_1) + (2/kn) \ln B < \exp[2 \sinh^{-1}((\xi\theta_2)^{\frac{1}{2}}/2)] + 2\sinh^{-1}(\xi\theta_2)^{\frac{1}{2}} - \exp[2\sinh^{-1}((\xi\theta_1)^{\frac{1}{2}}/2)] - 2\sinh^{-1}(\xi\theta_1)^{\frac{1}{2}} < (\theta_2 - \theta_1) + (2/kn) \ln A.$$

5.5 *The sequential ordinary correlation coefficient test.* The Z_i are IID where $Z_j = (U_j, V_j)$ is bivariate Normal $(\mu, \eta; \sigma^2, \tau^2; \theta)$; and $X_n = \sum_1^n (U_i - \bar{U}_n)\cdot(V_i - \bar{V}_n)/[\sum_1^n(U_i - \bar{U}_n)^2 \sum_1^n (V_i - \bar{V}_n)^2]^{\frac{1}{2}}$. We conclude from [3] (after some obvious transformation) that T^* continues sampling at the nth stage if

$$(5.5.1)\quad [(1 - \theta_1^2)/(1 - Q_2^2)]^{\frac{1}{2}} B^{1/n} < (1 - \theta_1 X_n)/(1 - \theta_2 X_n) < [(1 - \theta_1^2)/(1 - \theta_2^2)]^{\frac{1}{2}} A^{1/n}.$$

The sequential multiple correlation coefficient test is similar to Example 5.5; the sequential T^2-test is similar to Example 5.4; and the sequential Model II analysis of variance test is similar to Example 5.1 in the sense that their h functions are related [3].

6. Establishing some conditions. Condition C can be easily checked directly, and in most of the examples of Section 5 it is equivalent to Condition A_2 due to the symmetry of h in its two arguments. In Example 5.1, Condition D is trivially true while Condition E is actually equivalent to Condition A_2 and holds with $\epsilon_1 = \epsilon_2 = 0$ for all $x > 0$.

(1) *Condition* D *in Example* 5.3. It follows from [3] and (5.3.1) that

$$(6.1)\quad \begin{aligned} \Delta_{\theta n}(x) &= 1 + e^{-2n(\theta x)^{\frac{1}{2}}} && \text{for } q = 1 \\ &= 1 - e^{-2n(\theta x)^{\frac{1}{2}}} && \text{for } q = 3 \\ &= [n(\theta x)^{\frac{1}{2}}]^{(q-1)/2}/2^{(q-3)/2}\Gamma[(q-1)/2] \int_0^1 [y(2-y)]^{(q-3)/2} \\ &\quad \cdot[1 + \exp(-2n(\theta x)^{\frac{1}{2}}(1-y))] \exp(-n(\theta x)^{\frac{1}{2}}y)\, dy, && \text{otherwise.} \end{aligned}$$

Condition D is immediate for $q = 1$ or 3. Otherwise, let $I(\lambda) \equiv \int_0^\lambda z^{(q-3)/2} e^{-z}\, dz$ (the Incomplete Gamma function of $(q - 1)/2$). By (6.1)

$$(6.2)\quad 0 \leqq \Delta_{\theta n}(x) \leqq 2I[n(\theta x)^{\frac{1}{2}}]/\Gamma((q-1)/2) \leqq 2, \qquad \text{if } q > 3,$$

$$(6.3)\quad 0 \leqq \Delta_{\theta n}(x) \leqq 2(2)^{\frac{1}{2}}, \qquad \text{if } q = 2$$

for all θ, $x \varepsilon [0, \infty)$ and all $n \geqq 2$. Thus, D(i) is established. Condition D(ii) holds trivially when $\theta = 0$. Otherwise, let $U(\theta) = (\theta - \epsilon, \infty)$, where $0 < \epsilon < \theta$. Let $0 < \delta < 1$, write the integral in (6.1) as $\int_0^\delta + \int_\delta^1$, and notice that $\int_0^\delta < \Delta_{\theta n}(x) < \int_0^\delta + \int_\delta^1$. Bounding the proper parts of the integrand, we obtain

$$\begin{aligned}(6.4)\quad &(1 - \delta/2)^{(q-3)/2}[\Gamma((q-1)/2)]^{-1}(1 + e^{-2n(\theta x)^{\frac{1}{2}}})I(n(\theta x)^{\frac{1}{2}}\delta) \\ &< \Delta_{\theta n}(x) < [\Gamma((q-1)/2)]^{-1}\{(1 + e^{-2n(\theta x)^{\frac{1}{2}}(1-\delta)})I(n(\theta x)^{\frac{1}{2}}\delta) \\ &\qquad + 2(1-\delta)(1-\delta/2)^{(q-3)/2}(n(\theta x)^{\frac{1}{2}})^{(q-1)/2}e^{-n(\theta x)^{\frac{1}{2}}\delta}\}.\end{aligned}$$

Since the continuity of the limit function of a monotone sequence guarantees the uniformity of convergence (Dini's theorem); and since the sequences in (6.4) are monotone for all n, except $(n(\theta x)^{\frac{1}{2}})^{(q-1)/2}e^{-n(\theta x)^{\frac{1}{2}}\delta}$ which is monotone for $n \geqq (q-1)/2\delta(\theta(\theta - \epsilon))^{\frac{1}{2}}$, we conclude that

$$(6.5)\quad (1 - \delta/2)^{(q-3)/2}(1 - \delta) < \Delta_{\theta n}(x) < (1 + \delta) + (1 - \delta/2)^{(q-3)/2}(1 - \delta)\delta$$

for all $x \varepsilon U(\theta)$ and large enough n. Since δ is arbitrary, Condition D(ii) is established. The argument for $q = 2$ is similar and requires slightly different bounds in (6.4).

(2) *Condition* D *in Example* 5.2. It follows from (5.2.1), and [3] that

$$(6.6)\quad \Delta_{\theta n}(x) = (n/2\pi)^{\frac{1}{2}}(-\psi''(v_0))^{\frac{1}{2}} \int_0^\infty (v_0/v) \exp\left[\tfrac{1}{2}(v^2 - v_0^2)\right] \cdot \exp\{n[\psi(v) - \psi(v_0)]\}\, dv,$$

where

$$(6.7)\quad \psi(v) = -\tfrac{1}{2}(1 + x^2)v^2 + \theta x v - \tfrac{1}{2}\theta^2 + \ln v,$$

$$(6.8)\quad v_0 = \alpha(\xi\theta)/(1 + x^2)^{\frac{1}{2}},$$

and $\alpha(\cdot)$ and ξ are as given in (5.2.2). Notice that v_0 is the point where ψ achieves its maximum. Let $0 < \delta < v_0$ and consider the interval $I_v \equiv (v_0 - \delta, v_0 + \delta)$. Also let I_x be the interval $(\theta - \epsilon, \theta + \epsilon)$ for $\epsilon > 0$. Taylor's theorem implies that $\psi(v) - \psi(v_0) = \frac{1}{2}\psi''(v_0 + t\delta)(v - v_0)^2$, where $-1 \leqq t \leqq 1$. By (6.7), we have $\psi''(v) = -(1 + x^2) - v^{-2}$. The continuity of $\psi''(v)$ in (v, x), and the continuity of v_0 in x imply that for some $\eta > 0$ ($\eta \to 0$ as $\delta, \epsilon \to 0$)

$$(6.9)\quad \tfrac{1}{2}(\psi''(v_0) - \eta)(v - v_0)^2 < \psi(v) - \psi(v_0) < \tfrac{1}{2}(\psi''(v_0) + \eta)(v - v_0)^2$$

and all $v \varepsilon I_v$ and $x \varepsilon I_x$. Similarly,

$$(6.10)\quad 1 - \eta < (v_0/v) \exp\left[\tfrac{1}{2}(v^2 - v_0)^2\right] < 1 + \eta$$

for all $v \varepsilon I_v$ and $x \varepsilon I_x$. We choose δ small enough to guarantee that the right hand side of (6.9) is negative. Splitting the integral in (6.6) into $\int_{I_v} + \int_{\bar{I}_v}$, it is easy to see (proof of Theorem 3.1 in [3]) that for each θ, x, there exist $L_\theta(x)$, $\rho_\theta(x) > 0$ such that $\int_{\bar{I}_v} \leqq n^{\frac{1}{2}}L_\theta(x) \exp[-n\rho_\theta(x)]$. Let $y = n^{\frac{1}{2}}(v - v_0)$ and conclude from (6.6)–(6.10)

$$(1-\eta)(-\psi''(v_0)/2\pi)^{\frac{1}{2}}\int_{-\delta n^{\frac{1}{2}}}^{\delta n^{\frac{1}{2}}}\exp\left[\tfrac{1}{2}(\psi''(v_0)-\eta)y^2\right]dy$$
$$< \Delta_{\theta n}(x) < (1+\eta)(-\psi''(v_0)/2\pi)^{\frac{1}{2}}\int_{-\delta n^{\frac{1}{2}}}^{\delta n^{\frac{1}{2}}}\exp\left[\tfrac{1}{2}\psi''(v_0)+\eta)y^2\right]dy \tag{6.11}$$
$$+L_\theta(x)n^{\frac{1}{2}}\exp\left[-n\rho_\theta(x)\right].$$

The integrals in (6.11) converge to 1 uniformly in x for $x \varepsilon I_x$ (Dini's theorem). Since $\Delta_{\theta n}(x) \to 1$ for all $x \varepsilon \Theta$, there exist L_θ and ρ_θ such that $L_\theta(x) \leqq L_\theta < \infty$ and $\rho_\theta(x) \geqq \rho_\theta > 0$ for all $x \varepsilon I_x$. Thus, the extreme right term of (6.11) converges to 0 uniformly in x for $x \varepsilon I_x$. Since η is arbitrary, we have Condition D(ii) and also D(i).

(3) *Note on Condition* D. We remark that the preceding proof can be easily generalized to establish Condition D for densities which admit (apart from a constant $K(n)$) the integral representation of Theorem 3.2 of [3]. This is the case in the examples of Section 5 (other than 5.1), where ψ possesses a unique maximum at $z^0(\theta, x)$; and (after a translation to z^0 and possibly a reflection of some of the axes) Condition (b) of [3] holds. Referring to Theorem 3.2 of [3] we claim that Condition D obtains provided: (i) the $\psi_i'(z^0) < 0$, $i = r_0 + 1, \cdots, r$; $[\psi_{ij}''(z^0)]$, $i, j = 1, \cdots, r_0$ is negative definite; f and ψ are continuous in the pair (z, x) for each fixed θ; (ii) the $k_n(z; \theta, x) \to k(z; \theta, x)$ as $n \to \infty$ uniformly in (z, x) for (z, x) in a neighborhood of (z^0, θ) and each fixed θ. We remark that most of the preceding additional restrictions were already checked for our examples in the form of what was called Condition (c) in [3]. It is interesting to notice that the power of n in $\Delta_{\theta n}(x)$, (as furnished by Theorem 3.2 of [3]) enables one to give a proof which combines the multidimensional analogues of (1) and (2).

(4) *Condition* E *in Example* 5.2. For some $K(n)$,

$$q_{\theta n}(x) = K(n)\int_0^\infty v^{-1}e^{v^2/2}e^{n\psi(v)}\,dv \tag{6.12}$$

where ψ is given in (6.7). Differentiating under the integral sign we have

$$dq_{\theta n}(x)/d\theta = nq_{\theta n}(x)\left[x\left\{\int_0^\infty e^{v^2/2}e^{n\psi(v)}\,dv\Big/\int_0^\infty v^{-1}e^{v^2/2}e^{n\psi(v)}\,dv\right\}-\theta\right]. \tag{6.13}$$

It follows from Theorem 3.2 of [3] and (6.8) that

$$n^{-1}\,d\ln q_{\theta n}(x)/d\theta \to x\alpha(\xi\theta)/(1+x^2)^{\frac{1}{2}}-\theta. \tag{6.14}$$

The right hand side vanishes at $\theta = x$. In view of Lemma 4.2, Condition E is established.

(5) *Condition* E *in Example* 5.3. From (5.3.1) and (6.1), Condition E is easy to establish for $q = 1$ or 3. Otherwise, Theorem 3.2 of [3] gives

$$n^{-1}\,d\ln q_{\theta n}(x)/d\theta \to [(x/\theta)^{\frac{1}{2}}-1)/2. \tag{6.15}$$

By Lemma 4.2 and the fact that the right hand side of (6.15) vanishes at $\theta = x$, we establish Condition E.

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