OPTIMAL STOPPING WHEN THE FUTURE IS DISCOUNTED[1]

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1. Introduction. Let (β, X), (β_1, X_1), (β_2, X_2), $\cdots$ be a sequence of independent, identically distributed, two-dimensional random variables with $0 < \beta < 1$ and $0 < E|X| < \infty$. (No assumption about the independence of β and X is necessary.) Let S_n be the partial sum $X_1 + \cdots + X_n$, β^n be the partial product $\beta_1 \cdots \beta_n$, and adopt the conventions that $S_0 = 0$ and $\beta^0 = 1$. This note determines

$$U(x) = \sup E(\beta^t(x + S_t)) \tag{1}$$

as t ranges over the set T of stopping times for $X_1, X_2, \cdots$ and finds a (typically unique) *optimal t for x*, that is a t such that the sup in (1) is attained.

In contrast to the problem invented and studied by Chow and Robbins in [3], and its generalizations [6] and [10], the mere existence of an optimal stopping time for our problem is simple to demonstrate, as was noted in [3].

2. An optimal stopping time. Each *stopping time t* for $X_1, X_2, \cdots$ has the nonnegative integers with $+\infty$ adjoined as its range; the event $\{t = n\}$ depends only on $(X_1, \cdots, X_n)$ and $(\beta_1 \cdots \beta_n)$; and $P\{t = 0\}$ is either 0 or 1. It is natural in this problem to adopt the convention that $\beta^t = 0$ whenever $t = \infty$, and the convention that multiplication by 0 always yields 0; so $\beta^t S_t$ is 0 if $t = \infty$, though S_∞ is not defined.

Lemma 1. *There is a random variable G with a finite expectation such that, for each t,*

$$|\beta^t S_t| \leqq G \quad \textit{almost certainly}. \tag{2}$$

Proof. Let $G = \sum_k |\beta^k S_k|$.

Plainly, Lemma 1 implies that U is well defined.

Lemma 2. *U is convex and nondecreasing. There is a unique number s, necessarily greater than 0, such that*

$$x < U(x) < s \qquad \textit{for } x < s. \tag{3}$$

and

$$U(x) = x \qquad \textit{for } x \geqq s. \tag{4}$$

Proof. The lemma is immediate from these three observations: (i) According to (1), U is the sup of linear functions of x with nonnegative slopes. (ii) For

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$t = 0$, the linear function evaluated at x is x; except for this t, the slope of the linear function does not exceed $E(\beta) < 1$; (iii) $U(0) > 0$.

Here is the formulation of "the principle of optimality" [1] appropriate to the problem of this note:

LEMMA 3. *For all* x,

$$U(x) \geqq E(\beta U(x + X)). \tag{5}$$

PROOF. The proof of (5) is straightforward if X is discrete (for example, as in the proof of Theorem 2.14.1 in [5]). Since U is nondecreasing and continuous, there is no difficulty in modifying the proof for discrete X so as to apply to any X.

Here is an immediate consequence of Lemma 3.

LEMMA 4. *For all* x,

$$U(x), \beta^1 U(x + S_1), \beta^2 U(x + S_2), \cdots \tag{6}$$

is a nonnegative, expectation decreasing semimartingale. Consequently,

$$U(x) \geqq E(\beta^1 U(x + S_1)) \geqq E(\beta^t(x + S_t)), \tag{7}$$

for all stopping times $t \geqq 1$.

LEMMA 5. *For* $x \leqq s$, $U(x) = E(\beta U(x + X))$.

PROOF. If $x < s$, then, by (3), $x < U(x)$. Consequently, $U(x)$ is the sup of $E(\beta^t(x + S_t))$ over stop times $t \geqq 1$. So (7) implies

$$U(x) = E(\beta^1 U(x + S_1)) = E(\beta U(x + X)). \tag{8}$$

By continuity, equality holds for $x = s$ also.

Intuitively, Lemma 5 implies that if the process $U(x)$, $\beta^1 U(x + S_1)$, $\beta^2 U(x + S_2), \cdots$ is observed only until the first n such that $x + S_n > s$, then the observed process is a martingale. More formally, for each $y \geqq 0$, let $\tau(y)$ be the first n, if any, such that $S_n > y$. For $x \leqq s$, let $t(n, x)$ be the minimum of n and $\tau(s - x)$.

LEMMA 6. *For each* $x \leqq s$,

$$\{\beta^{t(n,x)} U(x + S_{t(n,x)}), n = 0, 1, \cdots\} \tag{9}$$

is a uniformly integrable martingale.

PROOF. Since each $t(n, x)$ is a bounded stopping time, Lemma 4 implies that each term in (9) has an expectation. Lemma 5 implies that the process is a martingale. To see that it is uniformly integrable notice that, for $x > 0$, $U(x)$ is majorized by a linear function of x, as (3) and (4) imply. So Lemma 1 implies that for each x there is a single random variable with a finite expectation that majorizes every term of (9). This certainly implies uniform integrability.

THEOREM 1. *For each* $x \leqq s$, $\tau(s - x)$ *is optimal for* x. *Equivalently,*

$$U(x) = E(\beta^{\tau(s-x)}(x + S_{\tau(s-x)})). \tag{10}$$

PROOF. Since $t(n, x)$ converges to $\tau(s - x)$, and U is continuous, the martingale in Lemma 6 certainly converges to $\beta^{\tau(s-x)} U(x + S_{\tau(s-x)})$, call it Y, wherever

$\tau(s - x) < \infty$, and even where $\tau(s - x) = \infty$. The uniform integrability of the martingale implies that its first term, namely the constant $U(x)$, is the expected value of Y. Moreover, since $x + S_{\tau(s-x)} > s$, $U(x + S_{\tau(s-x)})$ equals $x + S_{\tau(s-x)}$. This completes the proof.

If the entire sequence $\beta_1, X_1, \beta_2, X_2, \cdots$ are independent, then in (10), 'β' may be replaced by '$\bar{\beta}$', where $\bar{\beta}$ is the mean value of β.

The two special cases of Theorem 1 in which $x = s$ and $x = 0$ yield

$$s = E(\beta^{\tau(0)}(s + S_{\tau(0)})), \tag{11}$$

and

$$U(0) = E(\beta^{\tau(s)} S_{\tau(s)}). \tag{12}$$

Of course, (11) is linear in s and reduces the evaluation of s to the determination of the joint distribution of $\beta^{\tau(0)}$ and $S_{\tau(0)}$. (For information about the joint distribution of the ladder variables $\tau(0)$ and $S_{\tau(0)}$ see, for example, [7].)

Since s has been evaluated by (11) and $\tau(s - x)$ is optimal for x, according to Theorem 1, the main purpose of this note has been achieved.

3. Other optimal stopping times.

THEOREM 2. *Let $x < s$. The conjunction of these two conditions is necessary and sufficient for t to be optimal at x:*

$$P[x + S_t < s] = 0; \tag{13}$$

and

$$P[(\exists n)(n < t, \text{ and } x + S_n > s)] = 0. \tag{14}$$

PROOF. That (13) together with (14) is sufficient for the optimality of t is a simple consequence of Theorem 1 together with the fact that $s = E(\beta U(s + X))$.

That (13) is necessary follows from the implication: $x < s$ implies $x < U(x)$.

That (14) is also necessary is a relatively routine consequence of this lemma.

LEMMA 7. *For all $x > s$, $x > E(\beta U(x + X))$.*

PROOF OF LEMMA: Plainly, for $x > s$,

$$x - s \geqq U(x + X) - U(s + X), \tag{15}$$

so

$$\begin{aligned} x - s &> E(\beta U(x + X)) - E(\beta U(s + X)) \\ &= E(\beta U(x + X)) - U(s) \\ &= E(\beta U(x + X)) - s. \end{aligned} \tag{16}$$

COROLLARY. *Suppose x is less than s. Then, unless there is positive probability that $S_n = s - x$ for some n, $\tau(s - x)$ is the only stopping time that is optimal for x.*

4. The elementary case. This section shows that Theorem 1 has a particularly simple form if β is a constant and X is *elementary*, that is, X is almost surely an integer less than or equal to 1. Let R be the generating function of $-X$. Of course,

$$R(y) = E(y^{-X}), \tag{17}$$

and, as is easily verified and well known, R is convex, continuous, and strictly decreasing for $0 < y \leqq 1$.

THEOREM 3. *If X is elementary and β is constant,* $(0<\beta<1)$, *then s satisfies:*

$$\beta R(s/(s+1)) = 1. \tag{18}$$

The proof of Theorem 3 depends upon a lemma in which it is convenient to abbreviate $E(\beta^{\tau(0)})$ to $\varphi(\beta)$.

LEMMA 8. *If the X's have an elementary distribution, and β is a positive constant less than* 1, *then*

$$\beta R(\varphi(\beta)) = 1. \tag{19}$$

PROOF. The conditional distribution of $\tau(0)$ given $X_1 = -k$ is the same as the distribution of $1 + \tau_1 + \cdots + \tau_{k+1}$ where the τ_i are independent and each τ_i has the same distribution that $\tau(0)$ has. Consequently,

$$E[\beta^{\tau(0)} \mid X_1] = \beta\varphi^{-X_1+1}(\beta). \tag{20}$$

Taking expectations of both sides of (20), gives

$$\begin{aligned} \varphi(\beta) &= E(\beta^{t(0)}) \\ &= \beta\varphi(\beta)E(\varphi^{-X_1}(\beta)) \\ &= \beta\varphi(\beta)R(\varphi(\beta)). \end{aligned} \tag{21}$$

Lemma 8, which is a version of Wald's fundamental identity, permits the calculation of φ implicitly in terms of R. For the demonstration given, we are indebted to David Blackwell, David Gilat, Soren Johansen, and Roger Purves. We are indebted to J. Kemperman for pointing out to us that, in different notation, it was established in [8], p. 82; the appendix of [2]; and in [9], p. 47.

PROOF OF THEOREM 3. Under the hypotheses of Theorem 3, (11) implies

$$s = (s + 1)\varphi(\beta). \tag{22}$$

Lemma 8 now applies.

Theorem 3 permits an explicit calculation of $\tau(s)$ from R, as can be seen thus. $\tau(s)$ is the same as $\tau(j - 1)$ where j is the integer determined by $j - 1 \leqq s < j$. Since R is decreasing for $0 < y \leqq 1$ and

$$(j - 1)/j \leqq s/(s + 1) < j/(j + 1), \tag{23}$$

$$R((j - 1)/j) \geqq R(s/(s + 1)) > R(j/(j + 1)). \tag{24}$$

Consequently, in view of Theorem 3, to find j, it is only necessary to calculate $R(\frac{1}{2})$, $R(\frac{2}{3})$, $\cdots$ successively until the first j such that

$$R(j/(j + 1)) < 1/\beta. \tag{25}$$

As is easily seen, $\tau(j - 1)$ has the same distribution as the sum of j independent copies of $\tau(0)$. Consequently, $E(\beta^{\tau(j-1)}) = \varphi^j(\beta)$. Since $\tau(j - 1) = \tau(s)$ and $S_{\tau(j-1)} = j$, (12) implies

$$(26) \qquad U(0) = j\varphi^j(\beta).$$

In view of (22), (23) and (26),

$$(27) \qquad j((j - 1)/j)^j \leqq U(0) < j(j/(j + 1))^j.$$

Of course, (27) implies that for large j, the optimal return $U(0)$ depends mainly on j, and otherwise very little on β and even very little on the distribution of X.

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