A POTENTIAL THEORETIC PROOF OF A THEOREM OF DERMAN AND VEINOTT

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1. Introduction and summary. The purpose of this note is to provide alternative proofs of results of Derman and Veinott [1]. The method of proof uses the potential theory for Markov chains developed by Kemeny and Snell in [2], [3], [4], [5].

In [2], Kemeny and Snell treat transient and recurrent chains separately, whereas Derman and Veinott consider chains having one positive recurrent class, C, $(0 \varepsilon C)$, and a set of transient states, T, with $T \cup C = \{0, 1, 2, \cdots\}$. This difference does not cause any great difficulty. Kemeny and Snell also assume in [2] that their recurrent chains are noncyclic. The recurrent class, C, will therefore be assumed noncyclic in this note, the extension to the cyclic case being handled in the usual way.

By relabelling the states, one may write the transition matrix, P, as

$$P = \begin{pmatrix} P^1 & 0 \\ R & Q \end{pmatrix} \tag{1.1}$$

where P^1 is the transition matrix of the recurrent class and Q is the (substochastic) transition matrix of the set of transient states.

In this note, the notation will be that of Derman and Veinott [1] and all references to the work of Kemeny and Snell will be to [2]. In matrix notation, the Derman-Veinott equation is

$$(I - P)v = w^* \tag{1.2}$$

where $w^* = w - g\mathbf{1}$, $w = (w_0, w_1, w_2, \cdots)^T$ is a known vector, $\mathbf{1} = (1, 1, 1, \cdots)^T$, $v = (v_0, v_1, v_2, \cdots)^T$ is an unknown vector and g is an unknown constant.

2. Proof of the existence and uniqueness of a solution. Under the assumption that $m_{00} < +\infty$ and $m_{i0} < +\infty (i \varepsilon T)$, it follows first that the chain reaches C with probability one from any starting state, and second that there is a unique solution to the equation $\alpha P = \alpha$ with $\alpha\mathbf{1} = 1$ where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots)$. This solution is given by $\alpha = (\alpha^1, 0)$ where $\alpha^1 P^1 = \alpha^1$ and $\alpha^1\mathbf{1} = 1$. α_i may be written explicitly as $\alpha_i = {}_0P^*_{0i}/m_{00}$, $(i = 0, 1, 2, \cdots)$. Write $|w| = (|w_0|, |w_1|, |w_2|, \cdots)^T$, and ${}_0P^* = ({}_0P^*_{ij})$.

Theorem 2.1. (*Existence*). *For a given w and P, assume that $m_{i0} = \sum_j {}_0P^*_{ij} < +\infty$ and $\sum_j {}_0P^*_{ij} |w_j| < +\infty$ for $i \varepsilon \{0\} \cup T$. Let $g = m_{00}^{-1} \sum_j {}_0P^*_{0j} w_j = \sum_j \alpha_j w_j$. Then* (i) $\alpha w^* = 0$, (ii) ${}_0P^* |w^*| < +\infty$, ${}_0P^*\mathbf{1} < +\infty$, *and* (iii) $(I - P)\, {}_0P^* w^* = w^*$. *Conclusion* (iii) *states that $v = {}_0P^* w^* = {}_0P^* w - {}_0P^* g\mathbf{1}$ is a solution to* (1.2).

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PROOF. (i) follows directly by computation. To prove (ii), it suffices to show that $\sum_j {}_0P_{ij} |w_j^*| < +\infty$ and $\sum_j {}_0P^*_{ij} < +\infty$ for $i \varepsilon C$ since it is true by assumption for $i \varepsilon T$. The second inequality is a well known result for positive recurrent classes and the first follows from (i) and Lemma 3, p. 229. (iii) follows from part of the proof of Theorem 10, p. 231–232 or directly since $[(I - P)\, {}_0P^*]_{ij} = \delta_{ij} - p_{i0}\, {}_0P^*_{0j}$ and $\sum_j w_j^*\, {}_0P^*_{0j} = 0$.

That this is the unique solution (up to an additive constant) follows from

THEOREM 2.2. (*Uniqueness*). *Assume that* $m_{i0} = \sum_j {}_0P^*_{ij} < +\infty$ *and* $\sum_j {}_0P^*_{ij} |w_j| < +\infty$ *for* $i \varepsilon \{0\} \cup T$. *Let* g *be a constant and* v *a vector such that* $\alpha|v| < +\infty$ *and* $(I - P)v = w^*$. *Then* (i) $g = m_{00}^{-1} \sum_j {}_0P^*_{0j} w_j = \alpha w$, *and* (ii) $v = {}_0P^* w^* + c\mathbf{1}$ *where* c *is a constant.*

PROOF. (i) follows directly since $\alpha|v| < +\infty$ implies $\alpha w^* = \alpha(I - P)v = 0$ which implies that $\alpha w - g\alpha\mathbf{1} = 0$ so that $g = \alpha w$.

To prove (ii), write (1.2) as

$$\begin{pmatrix} I - P^1 & 0 \\ -R & I - Q \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} w^{*1} \\ w^{*2} \end{pmatrix} \tag{2.1}$$

and set

$$ {}_0P^* = \begin{pmatrix} {}_0P^{*1} & 0 \\ {}_0P^{*2} & N \end{pmatrix}. \tag{2.2}$$

$\alpha|v| < +\infty$ implies $\alpha^1|v^1| < +\infty$ and this, along with $(I - P^1)v^1 = w^{*1}$ implies that $v^1 = (\alpha^1 v^1)\mathbf{1} - G^1 w^{*1}$ by Corollary 1, p. 242 of [2] where G^1 is as defined in [2]. The existence of $v^* = -G^1 w^{*1}$ for a positive recurrent chain implies that v^* is a potential with charge w^{*1} by the remark after Corollary 2, p. 241. Hence $v^* = {}_0P^{*1} w^{*1} + v_0^* \mathbf{1}$ by Theorem 11, p. 232 of [2]. It follows that

$$v^1 = {}_0P^{*1} w^{*1} + c\mathbf{1}. \tag{2.3}$$

Equation (2.1) also gives $-Rv^1 + (I - Q)v^2 = w^{*2}$. But since Q is the transition matrix of a transient chain, $(I - Q)^{-1}$ exists and, further $(I - Q)^{-1} = N$. Thus

$$v^2 = Nw^{*2} + NRv^1. \tag{2.4}$$

Letting ${}^CH_{ij}$ $(i \varepsilon T, j \varepsilon C)$ be the probability of hitting C at j, starting from i, it is easy to see that $NR = {}^CH$ and that ${}^CH\mathbf{1} = 1$. Further ${}^CH_0P^{*1} = {}_0P^{*2}$. Putting these together with (2.3) and (2.4) yields

$$\begin{aligned} v^2 &= Nw^{*2} + {}^CHv^1 \\ &= Nw^{*2} + {}^CH({}_0P^{*1} w^{*1} + c\mathbf{1}) \\ &= Nw^{*2} + {}_0P^{*2} w^{*1} + c\mathbf{1}. \end{aligned} \tag{2.5}$$

Putting (2.3) and (2.5) together gives $v = {}_0P^* w^* + c\mathbf{1}$ and the proof is complete.

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