SOME SHARP MULTIVARIATE TCHEBYCHEFF INEQUALITIES

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1. Introduction and summary. If X is a random variable with $EX = 0$, var $(X) = \sigma^2$, then the inequalities

$$P[|X| \geqq \epsilon] \leqq \sigma^2/\epsilon^2, \tag{1.1}$$

$$P[X \geqq \epsilon] \leqq \sigma^2/(\epsilon^2 + \sigma^2), \tag{1.2}$$

where $\epsilon > 0$, are known as, respectively, the Tchebycheff and the one-sided Tchebycheff inequalities. It may be noted that under the stated conditions (1.2) can always be attained and (1.1) can be attained if $\sigma^2 \leqq \epsilon^2$. Let $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ be a random vector with $E\mathbf{X} = \mathbf{0}$, $E\mathbf{X}'\mathbf{X} = \mathbf{\Sigma}$, T_+ be a closed convex region in R^n, and $T = T_+ \cup \{\mathbf{X} \mid -\mathbf{X} \varepsilon T_+\}$. In [3], Marshall and Olkin have obtained sharp upper bounds on $P[\mathbf{X} \varepsilon T]$ and $P[\mathbf{X} \varepsilon T_+]$ as multivariate generalizations of (1.1) and (1.2). They have used these bounds to obtain explicit sharp bounds on $P[\min X_i \geqq 1]$, $P[\min |X_i| \geqq 1]$, $P[|\prod X_i| \geqq 1]$, $P[\prod X_i \geqq 1, X_i > 0, i = 1, 2, \cdots, n]$ etc., when only the variances σ_i^2 of X_i, $i = 1, 2, \cdots, n$, are known.

Let $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$ be a vector of n nonnegative random variables with $E\mathbf{Y} = \boldsymbol{\mu}$ and $\varphi \geqq 0$ be a homogeneous concave function on the nonnegative orthant R_+^n of R^n. In Section 2 we have proved the main inequality of this paper,

$$P[\varphi(\mathbf{Y}) \geqq \epsilon] \leqq \varphi(\boldsymbol{\mu})/\epsilon, \qquad \epsilon > 0,$$

which is attainably sharp if $\varphi(\boldsymbol{\mu}) \leqq \epsilon$. In Section 3 this result has been used to obtain various generalizations of (1.1) and (1.2) to n jointly distributed random variables or random vectors, which are also generalizations of the inequalities due to Marshall and Olkin for the case where only variances are known. In Section 4 we have obtained some sharp probability inequalities for some functions of symmetric psd (positive semidefinite) random matrices and have discussed their relation to inequalities due to Mudholkar [4].

2. The main inequality. We shall obtain the main inequality and establish its sharpness through the following two lemmas.

Lemma 2.1. *Let* $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$ *be a random vector with* $E\mathbf{Y} = \boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_n)$, *and* $f \geqq 0$ *be a concave function defined on* R^n. *Then for* $\epsilon > 0$,

$$P[f(\mathbf{Y}) \geqq \epsilon] \leqq f(\boldsymbol{\mu})/\epsilon.$$

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PROOF. By the standard argument for the Tchebycheff-type inequalities we have

$$P[f(\mathbf{Y}) \geqq \epsilon] \leqq Ef(\mathbf{Y})/\epsilon \leqq f(E\mathbf{Y})/\epsilon.$$

The latter inequality follows from Jensen's inequality since f is concave. It is obvious from the proof that the conclusion of the lemma is valid for any $f \geqq 0$ such that $Ef(\mathbf{Y}) \leqq f(\boldsymbol{\mu})$.

LEMMA 2.2. *For every homogeneous function $g \geqq 0$ on R^n and $\epsilon > 0$ there exists a random vector $\mathbf{Y}_0$ such that $E\mathbf{Y}_0 = \boldsymbol{\mu}$ and*

$$P[g(\mathbf{Y}_0) \geqq \epsilon] = g(\boldsymbol{\mu})/\epsilon,$$

provided $g(\boldsymbol{\mu})/\epsilon \leqq 1$.

PROOF. The distribution of the random vector $\mathbf{Y}_0$ is given by

$$P(\mathbf{Y}_0 = (\epsilon/g(\boldsymbol{\mu}))\cdot\boldsymbol{\mu}] = g(\boldsymbol{\mu})/\epsilon,$$
$$P[\mathbf{Y}_0 = 0] = 1 - g(\boldsymbol{\mu})/\epsilon.$$

Then $E\mathbf{Y}_0 = \boldsymbol{\mu}$, and because of the homogeneity of g,

$$P[g(\mathbf{Y}_0) \geqq \epsilon] = P[\mathbf{Y}_0 = (\epsilon/g(\boldsymbol{\mu}))\cdot\boldsymbol{\mu}] = g(\boldsymbol{\mu})/\epsilon.$$

The main inequality, and its attainability, contained in Theorem 2.1, is an immediate consequence of the above two lemmas.

THEOREM 2.1. *Let $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$ be a random vector of jointly distributed nonnegative random variables Y_i with $E\mathbf{Y} = \boldsymbol{\mu}$. Then for any nonnegative, concave, homogeneous function φ on the nonnegative orthant R_+^n of R_n, and $\epsilon > 0$, we have*

$$P[\varphi(\mathbf{Y}) \geqq \epsilon] \leqq \varphi(\boldsymbol{\mu})/\epsilon, \tag{2.1}$$

and the equality can be attained if $\varphi(\boldsymbol{\mu})/\epsilon \leqq 1$.

DEFINITION. An inequality $P(Y \varepsilon T) \leqq K$, valid for all the random variables in a family $\mathcal{Y}$ of random variables is said to be sharp if it cannot be improved. That is, for any $\delta > 0$ there exists a random variable $Y_0 \varepsilon \mathcal{Y}$ such that $P(Y_0 \varepsilon T) \leqq K - \delta$ is violated, i.e., $P(\mathbf{Y}_0 \varepsilon T) > K - \delta$. If the sharp inequality can be attained we shall call it attainably sharp.

THEOREM 2.2. *Under the conditions of Theorem 2.1 the inequality*

$$P[\varphi(\mathbf{Y}) > \epsilon] \leqq \varphi(\boldsymbol{\mu})/\epsilon \tag{2.2}$$

is sharp.

PROOF. Since $\{\mathbf{Y} \mid \varphi(\mathbf{Y}) > \epsilon\} \subset \{\mathbf{Y} \mid \varphi(\mathbf{Y}) \geqq \epsilon\}$ the inequality is true. If it is not sharp, it can be improved. That is, there exists a number $\epsilon_0 > \epsilon$ such that

$$P[\varphi(\mathbf{Y}) > \epsilon] \leqq \varphi(\boldsymbol{\mu})/\epsilon_0$$

for all random vectors $\mathbf{Y}$ with $E\mathbf{Y} = \boldsymbol{\mu}$. Let ϵ_1 be a real number such that $\epsilon_0 > \epsilon_1 > \epsilon$. Then by Theorem 2.1, there is a random vector $\mathbf{Y}_1$, $E\mathbf{Y}_1 = \boldsymbol{\mu}$, such that

$$\varphi(\mathbf{u})/\epsilon_1 = P[\varphi(\mathbf{Y}_1) \geqq \epsilon_1]$$
$$\leqq P[\varphi(\mathbf{Y}_1) > \epsilon] \leqq \varphi(\mathbf{u})/\epsilon_0 .$$

This is a contradiction since $\varphi(\mathbf{u})/\epsilon_0 < \varphi(\mathbf{u})/\epsilon_1$. Hence (2.2) is sharp.

Theorem 2.1 and Theorem 2.2 may be used to generate new probability inequalities, by taking for $\mathbf{Y}$ various nonnegative random variables and for φ different nonnegative homogeneous concave functions on R_+^n. In Section 3 we shall obtain various generalizations of the Tchebycheff inequality and one-sided Tchebycheff inequality in this manner. At this stage it will be convenient to discuss some relevant examples of nonnegative homogeneous concave functions on R_+^n.

EXAMPLES OF NONNEGATIVE, HOMOGENEOUS, CONCAVE FUNCTIONS ON R_+^n. Let $\mathbf{t} = (t_1, t_2, \cdots, t_n)$ be an n-tuple of nonnegative real numbers and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be n nonnegative real numbers such that $\sum_1^n \alpha_i = 1$. Then

$$\varphi_r(\mathbf{t}) = (\textstyle\sum_{i=1}^n \alpha_i t_i^r)^{1/r}, \qquad r \leqq 1, \tag{2.3}$$

known as a Hölder-Minkowski function [2], is a nonnegative, homogeneous concave function. Some interesting particular cases of $\varphi_r(\mathbf{t})$, where $\mathbf{t}$ is an n-tuple of nonnegative reals, are

$$\varphi_{-\infty}(\mathbf{t}) = \lim_{r\to-\infty} \varphi_r(t) = \min(t_1, t_2, \cdots, t_n), \tag{2.4}$$

$$\varphi_{-1}(\mathbf{t}) = (\textstyle\sum_1^n \alpha_i t_i^{-1})^{-1}, \tag{2.5}$$

$$\varphi_0(\mathbf{t}) = \lim_{r\to 0} \varphi_r(t) = (\textstyle\prod_1^n t_i^{\alpha_i}), \tag{2.6}$$

$$\varphi_1(\mathbf{t}) = \textstyle\sum_1^n \alpha_i t_i . \tag{2.7}$$

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1/n$ then $\varphi_{-1}(\mathbf{t})$, $\varphi_0(\mathbf{t})$ and $\varphi_1(\mathbf{t})$ are, respectively, the harmonic mean, the geometric mean and the arithmetic mean of the numbers $t_1, t_2, \cdots, t_n$. The well known inequalities between the three means are particular cases of the inequality

$$\varphi_r(\mathbf{t}) \leqq \varphi_s(\mathbf{t}) \quad \text{if} \quad r < s, \tag{2.8}$$

for the Hölder-Minkowski functions.

Let $t_{(1)} \leqq t_{(2)} \leqq \cdots \leqq t_{(n)}$ denote the ordered values and $E_k(\mathbf{t})$, $k = 1, 2, \cdots, n$, the kth elementary symmetric function of the nonnegative numbers $t_1, t_2, \cdots, t_n$. Then

$$\varphi(\mathbf{t}) = \textstyle\sum_1^n c_i t_{(i)}, \qquad c_1 \geqq c_2 \geqq \cdots \geqq c_n \geqq 0, \tag{2.9}$$

$$\varphi(\mathbf{t}) = E_k^{1/k}(\mathbf{t}), \qquad k = 1, 2, \cdots, n, \tag{2.10}$$

$$\varphi(\mathbf{t}) = E_k(\mathbf{t})/E_{k-1}(\mathbf{t}), \qquad k = 2, 3, \cdots, n, \tag{2.11}$$

are nonnegative, homogeneous, concave functions of $t_1, t_2, \cdots, t_n$.

Let

$$T_{(r,k)}(\mathbf{t}) = \textstyle\sum_{i_1+i_2+\cdots+i_n=r} (\prod_{j=1}^n \delta_{i_j} t_j^{i_j}), \qquad i_j \geqq 0, j = 1, 2, \cdots, n, k > 0,$$
$$\text{and } k > n-1 \text{ if } k \text{ is not an integer}, \ \delta_{i_j} = \tbinom{k}{i_j}.$$

Then

$$\varphi(\mathbf{t}) = T^{1/r}_{(r,k)}(\mathbf{t}) \tag{2.12}$$

is a nonnegative, homogeneous and concave function of $t_1, t_2, \cdots, t_n$. If $k = 1$ then $T_{(r,k)}(\mathbf{t}) = E_r(\mathbf{t})$.

3. Generalizations of the Tchebycheff and the one-sided Tchebycheff inequalities. Theorem 3.1 provides a generalization of the Tchebycheff inequality to n jointly distributed random variables.

THEOREM 3.1. *Let $X_1, X_2, \cdots, X_n$ be n jointly distributed random variables with $EX_i^2 = \sigma_i^2$, $i = 1, 2, \cdots, n$, and $\varphi \geqq 0$ be a homogeneous concave function on R_+^n. Then for $\epsilon > 0$,*

$$P[\varphi(X_1^2, X_2^2, \cdots, X_n^2) \geqq \epsilon] \leqq \varphi(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2)/\epsilon, \tag{3.1}$$

and the equality can be attained if $\varphi(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2) \leqq \epsilon$. The inequality remains attainably sharp even if the random variables are required to satisfy an additional condition $EX_i = 0$.

PROOF. The first part of this theorem follows from Theorem 2.1 with $Y_i = X_i^2$, $i = 1, 2, \cdots, n$. The inequality remains trivially valid even if X_i's are required to satisfy some additional conditions. Its attainability follows from the random variables X_{0i}, $i = 1, 2, \cdots, n$, with joint distribution given by $P[X_{0i} = \pm\sigma_i/\beta^{\frac{1}{2}}, i = 1, 2, \cdots, n] = \beta/2$, $P[X_{0i} = 0, i = 1, 2, \cdots, n] = 1 - \beta$, where $\beta = \varphi(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2)/\epsilon \leqq 1$.

EXAMPLES. By taking for φ different nonnegative homogeneous functions discussed in Section 2 one can obtain various explicit generalizations of (1.1). The following are four of these attainably sharp inequalities:

$$P[\textstyle\sum_1^n \alpha_i |X_i|^{2r} \geqq \epsilon] \leqq (\sum_1^n \alpha_i \sigma_i^{2r})^{1/r}/\epsilon^{1/r}, \tag{3.2}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are positive, $\sum \alpha_i = 1$, $r \leqq 1$;

$$P[\textstyle\sum_1^n a_i X_{(i)}^2 \geqq \epsilon] \leqq \sum_1^n a_i \sigma_{(i)}^2/\epsilon, \tag{3.3}$$

where $X_{(1)}^2 \leqq X_{(2)}^2 \leqq \cdots \leqq X_{(n)}^2$ and $\sigma_{(1)}^2 \leqq \sigma_{(2)}^2 \leqq \cdots \leqq \sigma_{(n)}^2$ denote, respectively, the ordered values of $X_1^2, X_2^2, \cdots, X_n^2$ and $\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2$, and $a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0$;

$$P[E_k(X_1^2, X_2^2, \cdots, X_n^2) \geqq \epsilon] \leqq E_k^{1/k}(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2)/\epsilon^{1/k}, \tag{3.4}$$

where $E_k(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2)$ is the kth elementary symmetric function of $\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2$, $k = 1, 2, \cdots, n$, and

$$\begin{aligned} P[E_k(X_1^2, X_2^2, \cdots, X_n^2) &\geqq \epsilon E_{k-1}(X_1^2, X_2^2, \cdots, X_n^2)] \\ &\leqq E_k(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2)/\epsilon E_{k-1}(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2), \qquad k = 1, 2, \cdots, n. \end{aligned} \tag{3.5}$$

REMARK 1. It is clear from Theorem 2.2 that the statement $\varphi(X_1^2, X_2^2, \cdots, X_n^2) \geqq \epsilon$ in (3.1) may be replaced by $\varphi(X_1^2, X_2^2, \cdots, X_n^2) > \epsilon$ without affecting the sharpness of (3.1). Also, in view of the example of the random variables X_{0i}, $i = 1, 2, \cdots, n$, for which (3.1) is attained it is evident that the inequality

$$(3.6)\quad P[\varphi(X_1^2, X_2^2, \cdots, X_n^2) \geqq \epsilon, \mathbf{X} > \mathbf{0} \text{ or } \mathbf{X} < \mathbf{0}] \leqq \varphi(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2)/\epsilon,$$

where $\mathbf{X} > \mathbf{0}$ denotes $X_i > 0$, $i = 1, 2, \cdots, n$, is attainably sharp. Using this fact and specializing inequalities (3.2), (3.3), (3.4), and (3.5) we can get the inequalities (5.1), (5.3), (5.5), (5.6) and (5.7) of Marshall and Olkin [3].

We shall need the following result due to Marshall and Olkin [3] to obtain a generalization of the one-sided Tchebycheff inequality when only the variances of n jointly distributed random variables are known.

THEOREM 3.2 (Marshall and Olkin). *Let* $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ *be a random vector with* $E\mathbf{X} = \mathbf{0}$, $E\mathbf{X}'\mathbf{X} = \mathbf{\Sigma}$. Let $T = T_+ \cup \{\mathbf{X} \mid -\mathbf{X} \varepsilon T_+\}$, *where* $T_+ \subseteq R^n$ *is a closed convex set. If* $\mathcal{A} = \{\mathbf{a} \mid \mathbf{a} \varepsilon R^n, \mathbf{a}\mathbf{X}' \geqq 1$ *for all* $\mathbf{X} \varepsilon T_+\}$, *then*

$$(3.7)\qquad P(\mathbf{X} \varepsilon T) \leqq \inf_{\mathbf{a}\varepsilon\mathcal{A}} \mathbf{a}\mathbf{\Sigma}\mathbf{a}',$$

$$(3.8)\qquad P[\mathbf{X} \varepsilon T_+] \leqq \inf_{\mathbf{a}\varepsilon\mathcal{A}} \{\mathbf{a}\mathbf{\Sigma}\mathbf{a}'/(1 + \mathbf{a}\mathbf{\Sigma}\mathbf{a}')\}.$$

The equality can be attained in (3.7) whenever $\inf_{\mathbf{a}\varepsilon\mathcal{A}} \mathbf{a}\mathbf{\Sigma}\mathbf{a}' \leqq 1$; the equality can always be attained in (3.8).

REMARK 2. If only variances σ_i^2 of X_i, $i = 1, 2, \cdots, n$, are known we can write $\mathbf{\Sigma} = \sigma_1^2\sigma_2^2 \cdots \sigma_n^2\mathbf{P}$, where $\mathbf{P} \varepsilon \mathcal{P}$ is the unknown correlation matrix, $\mathcal{P}$ being the set of all $(n \times n)$ correlation matrices. In this situation it is clear that (3.7) and (3.8) can be modified as, respectively,

$$(3.9)\qquad P[\mathbf{X} \varepsilon T] \leqq \sup_{\mathcal{P}} \inf_{\mathcal{A}} \mathbf{a}\mathbf{\Sigma}\mathbf{a}',$$

$$(3.10)\qquad P[\mathbf{X} \varepsilon T_+] \leqq \sup_{\mathcal{P}} \inf_{\mathcal{A}} \{(\mathbf{a}\mathbf{\Sigma}\mathbf{a}')/(1 + \mathbf{a}\mathbf{\Sigma}\mathbf{a}')\}.$$

Both (3.9) and (3.10) are sharp.

THEOREM 3.3. *Let* $X_1, X_2, \cdots, X_n$ *be* n *jointly distributed random variables with* $EX_i = 0$ *and* $EX_i^2 = \sigma_i^2$, $i = 1, 2, \cdots, n$. *Then for nonnegative numbers* $\alpha_1, \alpha_2, \cdots, \alpha_n$, $\sum \alpha_i = 1$ *and* $r \leqq \frac{1}{2}$,

$$(3.11)\quad P[\textstyle\sum \alpha_i |X_i|^{2r} \geqq \epsilon, \mathbf{X} > \mathbf{0}] \leqq (\sum \alpha_i\sigma_i^{2r})^{1/r}/(\epsilon^{1/r} + (\sum \alpha_i\sigma_i^{2r})^{1/r}),$$

and the equality can always be attained in (3.11).

PROOF. If $r \leqq \frac{1}{2}$, $(\sum \alpha_i |X_i^2|^r)^{1/2r}$ is a concave function of $(X_1, X_2, \cdots, X_n)$. Hence $T_+ = \{\mathbf{X} \mid (\sum | X_i^2|^r)^{1/2r} \geqq 1, \mathbf{X} \geqq \mathbf{0}\}$ is a closed convex set in R^n. Therefore, by Remark 2 the bound

$$P(\mathbf{X} \varepsilon T) = P[\textstyle\sum_{i=1}^n \alpha_i |X_i|^{2r} \geqq 1, \mathbf{X} > \mathbf{0} \quad \text{or} \quad \mathbf{X} < \mathbf{0}] \leqq \sup_{\mathcal{P}} \inf_{\mathcal{A}} \mathbf{a}\mathbf{\Sigma}\mathbf{a}'$$

is sharp. But $\varphi(X_1^2, X_2^2, \cdots, X_n^2) = (\sum \alpha_i(X_i^2)^r)^{1/r}$ is nonnegative, homogeneous and concave. Therefore using the inequality (3.6) we get

$$(3.12)\qquad \sup_{\mathcal{P}} \inf_{\mathcal{A}} \mathbf{a}\mathbf{\Sigma}\mathbf{a}' = (\textstyle\sum \alpha_i\sigma_i^{2r})^{1/r}.$$

Again, using (3.10) and Theorem 3.2, and noting that $\mathbf{a\Sigma a}' \geqq 0$, we get

$$P[\sum \alpha_i X_i^{2r} \geqq 1, \mathbf{X} > \mathbf{0}] \leqq \sup_{\mathcal{P}} \inf_{\mathfrak{a}} \{\mathbf{a\Sigma a}'/(1 + \mathbf{a\Sigma a}')\},$$
$$\leqq (\sum \alpha_i \sigma_i^{2r})^{1/r}/[1 + (\sum \alpha_i \sigma_i^{2r})^{1/r}].$$

Therefore, using a simple transformation we get the bound

$$P[\sum \alpha_i |X_i|^{2r} \geqq \epsilon, \mathbf{X} > \mathbf{0}] \leqq (\sum \alpha_i \sigma_i^{2r})^{1/r}/[\epsilon^{1/r} + (\sum \alpha_i \sigma_i^{2r})^{1/r}].$$

The equality in this bound is attained for the random variables X_{oi}, $i = 1, 2, \cdots, n$, with joint distribution given by

$$P[X_{0i} = \sigma_i/\beta^{\frac{1}{2}}, i = 1, 2, \cdots, n] = \beta/(1 + \beta),$$
$$P[X_{0i} = -\sigma_i/\beta^{\frac{1}{2}}, i = 1, 2, \cdots, n] = 1/(1 + \beta),$$

where $\beta = (\sum \sigma_i^{2r})^{1/r}/\epsilon^{1/r}$. Hence (3.11) is attainably sharp.

Remark 3. The above proof is valid if $(\sum |X_i|^{2r})^{1/r}$ is replaced by any nonnegative, homogeneous, concave function $\varphi(X_1^2, X_2^2, \cdots, X_n^2)$ of $X_1^2, X_2^2, \cdots, X_n^2$ which is also a nonnegative power of a concave function of $(X_1, X_2, \cdots, X_n)$.

Examples. Using the properties of the Hölder-Minkowski function, discussed in Section 2, one can obtain the following sharp bounds which may be considered as generalizations of one-sided Tchebycheff inequality:

$$P[\min X_i \geqq \epsilon] \leqq \min \sigma_i^2/(\epsilon^2 + \min \sigma_i^2), \tag{3.13}$$

$$P[(\sum \alpha_i X_i^{-1})^{-1} \geqq \epsilon, \mathbf{X} > \mathbf{0}] \leqq (\sum \alpha_i \sigma_i^{-1})^{-\frac{1}{2}}/(\epsilon^{\frac{1}{2}} + (\sum \alpha_i \sigma_i^{-1})^{-\frac{1}{2}}) \tag{3.14}$$

$$P[\prod X_j^{\alpha_i} \geqq \epsilon, \mathbf{X} > \mathbf{0}] \leqq \prod \sigma_i^{2\alpha_i}/(\epsilon^2 + \prod \sigma_i^{2\alpha_i}), \tag{3.15}$$

$$P[\sum \alpha_i X_i \geqq \epsilon, \mathbf{X} > \mathbf{0}] \leqq (\sum \alpha_i \sigma_i)^{\frac{1}{2}}/(\epsilon^{\frac{1}{2}} + (\sum \alpha_i \sigma_i)^{\frac{1}{2}}). \tag{3.16}$$

Now let $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$ be n jointly distributed $(1 \times p)$ random vectors with $E\mathbf{X}_i = \mathbf{0}$, $E\mathbf{X}_i'\mathbf{X}_i = \mathbf{\Sigma}_i$, $i = 1, 2, \cdots, n$. Let $s_i^2 = \mathbf{X}_i\mathbf{X}_i'$ and $\operatorname{tr} \mathbf{\Sigma}_i = \sigma_i^2$, $\mathbf{\Sigma}_i$ positive definite, $i = 1, 2, \cdots, n$. Then $Es_i^2 = \sigma_i^2$, $i = 1, 2, \cdots, n$, and we have the following generalization of Theorem 3.1 to jointly distributed random vectors.

Theorem 3.4. *For any concave, homogeneous function* $\varphi \geqq 0$ *on* R_+^n, *the inequality*

$$P[\varphi(s_1^2, s_2^2, \cdots, s_n^2) \geqq \epsilon] \leqq \varphi(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2)/\epsilon, \qquad \epsilon > 0, \tag{3.17}$$

can be attained if $\varphi(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2) \leqq \epsilon$.

Proof. The validity of the inequality is an immediate consequence of Theorem 2.1. To show that it can be attained we note that there exist orthogonal matrices $\mathbf{T}_i$ such that $\mathbf{T}_i\mathbf{\Sigma}_i\mathbf{T}_i' = \mathbf{D}(\sigma_{ij}^2)$, where σ_{ij}^2, $j = 1, 2, \cdots, p$, are the characteristic roots of $\mathbf{\Sigma}_i$, and $\mathbf{D}(\sigma_{ij}^2)$ is the diagonal matrix with elements $\sigma_{i1}^2, \sigma_{i2}^2, \cdots, \sigma_{ip}^2$, $i = 1, 2, \cdots, p$. Consider n jointly distributed vectors $\mathbf{Y}_1, \mathbf{Y}_2, \cdots, \mathbf{Y}_n$ such that

$$P(\mathbf{Y}_i\mathbf{Y}_i' = \sigma_i^2, i = 1, 2, \cdots, n] = \beta,$$
$$P(\mathbf{Y}_i\mathbf{Y}_i' = 0, i = 1, 2, \cdots, n] = 1 - \beta,$$

where $\beta = \varphi(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2)/\epsilon$, $E\mathbf{Y}_i = \mathbf{0}$, $E\mathbf{Y}_i'\mathbf{Y}_i = \mathbf{D}(\sigma_{ij}^2)$, $i = 1, 2, \cdots, n$. Such marginally uncorrelated random vectors may be constructed by spreading the probability mass symmetrically about the origin and the coordinates planes. Now let $\mathbf{X}_{0i} = \mathbf{Y}_i\mathbf{T}_i$, $i = 1, 2, \cdots, n$. Then $s_{0i}^2 = \mathbf{X}_{0i}\mathbf{X}_{0i}' = \mathbf{Y}_i\mathbf{Y}_i'$, $E\mathbf{X}_{0i} = \mathbf{0}$, $E\mathbf{X}_{0i}'\mathbf{X}_{0i} = \mathbf{\Sigma}_i$, $i = 1, 2, \cdots, n$, and for the jointly distributed random vectors $\mathbf{X}_{01}, \mathbf{X}_{02}, \cdots, \mathbf{X}_{0n}$ equality (3.17) is attained.

Explicit inequalities for the jointly distributed random vectors may be obtained from Theorem 3.4 by using various explicit forms of φ as before.

4. Some sharp inequalities for random matrices. Let $\mathbf{Z}(p \times p)$ be a symmetric psd random matrix with $E\mathbf{Z} = \mathbf{\Sigma}$. Let $g(\mathbf{Z}) \geqq 0$ be a homogeneous concave function on the space of $(p \times p)$ symmetric psd matrices. Then we have, for $\epsilon > 0$,

$$P[g(\mathbf{Z}) \geqq \epsilon] \leqq Eg(\mathbf{Z})/\epsilon \leqq g(\mathbf{\Sigma})/\epsilon,$$

by arguments similar to the proof of Lemma 2.1 (for a matrix-version of Jensen's inequality see [1]). Suppose that $g(\mathbf{\Sigma}) \leqq \epsilon$ and consider random matrix $\mathbf{Z}_0$ with distribution,

$$P[\mathbf{Z}_0 = (\epsilon/g(\mathbf{\Sigma}))\mathbf{\Sigma}] = g(\mathbf{\Sigma})/\epsilon,$$
$$P[\mathbf{Z}_0 = \mathbf{0}] = 1 - g(\mathbf{\Sigma})/\epsilon.$$

It is easy to see that $E\mathbf{Z}_0 = \mathbf{\Sigma}$ and $P[g(\mathbf{Z}_0) \geqq \epsilon] = g(\mathbf{\Sigma})/\epsilon$. Therefore we have the following:

THEOREM 4.1. *Let* $\mathbf{Z}$ *be a* $(p \times p)$ *symmetric* psd *random matrix with* $E\mathbf{Z} = \mathbf{\Sigma}$, *and* $g(\mathbf{Z}) \geqq 0$ *be a homogeneous, concave function of* $\mathbf{Z}$. *Then, the inequality*

$$P[g(\mathbf{Z}) \geqq \epsilon] \leqq g(\mathbf{\Sigma})/\epsilon, \tag{4.1}$$

$\epsilon > 0$, *can be attained if* $g(\mathbf{\Sigma}) \leqq \epsilon$.

EXAMPLES. Let $z_1 \leqq z_2 \leqq \cdots \leqq z_p$ and $\sigma_1 \leqq \sigma_2 \leqq \cdots \leqq \sigma_p$ denote the ordered characteristic roots of, respectively, $\mathbf{Z}$ and $\mathbf{\Sigma}$. Then the following sharp inequalities can be obtained from Theorem 4.1 as examples:

$$P[\det(\mathbf{Z}) \geqq \epsilon] \leqq \det^{1/p}(\mathbf{\Sigma})/\epsilon^{1/p}, \tag{4.2}$$

$$P[\textstyle\sum_{i=1}^p c_i z_i \geqq \epsilon] \leqq \sum_{i=1}^p c_i\sigma_i/\epsilon, \tag{4.3}$$

where $c_1 \geqq c_2 \geqq \cdots \geqq c_p \geqq 0$,

$$\begin{aligned} P[E_k(z_1, z_2, \cdots, z_p) \geqq \epsilon] \\ \leqq E_k^{1/k}(\sigma_1, \sigma_2, \cdots, \sigma_p)/\epsilon^{1/p}, \qquad k = 1, 2, \cdots, p; \end{aligned} \tag{4.4}$$

$$\begin{aligned} P[E_k(z_1, z_2, \cdots, z_p) \geqq \epsilon E_{k-1}(z_1, z_2, \cdots, z_r)] \\ \leqq E_k(\sigma_1, \sigma_2, \cdots, \sigma_p)/\{\epsilon E_{k-1}(\sigma_1, \sigma_2, \cdots, \sigma_p)\}, \quad k = 2, 3, \cdots, p; \end{aligned} \tag{4.5}$$

$$\begin{aligned} P[\textstyle\sum_{1=i}^p z_i^r \geqq \epsilon] \\ \leqq (\textstyle\sum \sigma_i^r)^{1/r}/\epsilon^{1/r}; \end{aligned} \tag{4.6}$$

$$(4.7)\quad P[E_k^{1/k}(z_1, z_2, \cdots, z_j) \geqq \epsilon] \leqq E_k^{1/k}(\sigma_1, \sigma_2, \cdots, \sigma_j)/\epsilon, \qquad 1 \leqq k \leqq j \leqq p;$$

$$(4.8)\quad P[E_k(z_1, z_2, \cdots, z_j) \geqq \epsilon E_{k-1}(z_1, z_2, \cdots, z_j)] \leqq E_k(\sigma_1, \sigma_2, \cdots, \sigma_j)/\{\epsilon E_{k-1}(\sigma_1, \sigma_2, \cdots, \sigma_j), \quad 1 \leqq k \leqq j \leqq p.$$

For a reference to the concavity of functions $E_k^{1/k}(z_1, z_2, \cdots, z_j)$ and $E_k(z_1, z_2, \cdots, z_j)/E_{k-1}(z_1, z_2, \cdots, z_j)$, $1 \leqq k \leqq j \leqq p$ see [1].

Let $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$ be independently distributed random vectors with $E\mathbf{X}_i = \mathbf{0}$, $E\mathbf{X}_i'\mathbf{X}_i = \mathbf{\Sigma}_i$, $i = 1, 2, \cdots, p$. Let $\mathbf{X}'(p \times n) = [\mathbf{X}_1'\mathbf{X}_2', \cdots, \mathbf{X}_n']$ and $\mathbf{Z} = \mathbf{X}'\mathbf{X}$. Then $\mathbf{Z}$ is a $(p \times p)$ symmetric psd random matrix with $E\mathbf{Z} = \mathbf{\Sigma}_1 + \mathbf{\Sigma}_2 + \cdots + \mathbf{\Sigma}_n = \mathbf{\Sigma}$, say. Let $z_1 \leqq z_2 \leqq \cdots \leqq z_p$ and $\sigma_1 \leqq \sigma_2 \leqq \cdots \leqq \sigma_p$ denote, as before, the characteristic roots of $\mathbf{Z}$ and $\mathbf{\Sigma}$ respectively and let $f \geqq \mathbf{0}$ be a symmetric concave function on R_+^n. Then Mudholkar [4] proved the inequality

$$(4.9)\qquad P[f(z_1, z_2, \cdots, z_p) \geqq \epsilon] \leqq f(\sigma_1, \sigma_2, \cdots, \sigma_p)/\epsilon,$$

and illustrated it by the inequalities (4.2) to (4.6) above. Mudholkar [4] conjectured that many of these inequalities are sharp. It can be seen that, even though inequalities (4.2) to (4.6) are sharp in the context of Theorem 4.1, they may or may not be sharp in the context of [4]. It may be interesting to know whether inequality (4.9) is a particular case of inequality (4.1).

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