LOCALLY MINIMAX TESTS[1]

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1. Introduction. This paper represents an extension of the local minimax results contained in Section 2 of Giri and Kiefer (1964), hereafter G-K (1964). Other sections of G-K (1964) deal with topics other than local minimaxity, with which this paper is not concerned.

In G-K (1964) the property of local minimaxity is defined and Lemma 1 states conditions under which a given test is locally minimax. These conditions on the statistical problem and on the given test are then verified for the settings in which Hotelling's T^2-test and the test based on the squared multiple correlation coefficient, R^2, are customarily employed, showing that the T^2- and R^2-tests are locally minimax.

The present paper deals with the generalizations of the T^2-and R^2-problems, namely, the MANOVA problem and the problem of testing the independence of sets of variates. Whereas both the T^2- and R^2-tests are best fully invariant tests, in both the general MANOVA and independence problems there is a large class of fully invariant admissible tests (Schwartz (1966a, b, c)). Of course different tests within this class may have different contours of constant power. Since the definition of local minimaxity is relative to a family of contours approaching the null hypothesis, it seemed possible at the outset that different fully invariant tests might be locally minimax for different families of contours.

However, examination of the local behavior of the probability ratio of the maximal invariant (under all linear-affine transformations which leave the problem invariant) reveals that in both the MANOVA problem and in testing the independence of *two* sets of variates there is a unique locally best test in the class of fully invariant tests. (These results are given in Theorem 1 and 3 respectively where the meaning of locally best is made clear.) Hence, in both of these problems if any fully invariant test is to be locally minimax it must be the one which is locally best invariant.

Once the locally minimax test has been guessed the verification that it satisfies the conditions of Lemma 1 of G-K (1964) follows very closely the verifications given in G-K (1964) for the T^2- and R^2-tests. The computations are slightly more complicated in the more general settings considered in this paper.

In addition the actual results can have more complicated statements in the more general settings because of the variety of different families of contours which may be considered. Detailed consideration of different families of contours is given only in the MANOVA problem and not in the independence problem. In

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the MANOVA problem it is found that the locally best fully invariant test is locally minimax for certain families of contours but not for others, and that the families for which it is locally minimax depend on the sample size.

A further generalization of the independence problem is to consider more than two sets of variates. In this case there is no locally best fully invariant test. Different fully invariant tests will be locally minimax on different families of contours, but the delineation of families of contours and corresponding locally minimax tests appears complicated. Some local results concerning fully invariant tests are obtained for the general setting but the local minimax problem is treated only in the special case in which all sets contain the same number of variates.

At the time this work was done the papers of Constantine (1963) and James (1964), giving expansions of the probability density of the maximal invariant in terms of zonal polynomials, had not yet appeared. The first term of these expansions give the locally best fully invariant test. (It seems likely that this fact is known to both of the authors cited and perhaps to earlier authors of papers on the distribution theory.) The proof of this fact depends on showing that the series beyond the first term is uniformly of smaller order as the parameter approaches the null hypothesis. The necessary estimate seems more easily obtained by examining the terms of the series before they have been evaluated in terms of zonal polynomials. For this and other reasons indicated below, the zonal polynomial expansions will not be used. Instead, Stein's representation of the probability ratio of the maximal invariant (Stein (1956), Schwartz (1966c), Wijsman (1966)) is used to derive both the first term and the requisite estimate of the remainder.

Two other reasons why this approach is better adapted to the purposes of the present work are the following: Firstly, the proof of local minimaxity requires the local behavior of the probability ratio not only under all linear transformations leaving the problem invariant, but also, under the subgroup of lower triangular matrices. The two computations are quite similar and, in part, can be done simultaneously. Secondly, in the independence problem the same approach will work for any number of sets of variates whereas Constantine and James consider only two sets of variates.

2. Notation and definition. Let X be a space with associated σ-field, which along with other obvious measurability considerations, we shall not mention in what follows. Let $\{P_\theta\ ; \theta \varepsilon \Theta\}$ be a family of probability measures on X. We shall be interested in testing the hypothesis $H_0 : \theta \varepsilon \Theta_0 \subset \Theta$ and for each α, $0 < \alpha < 1$, the class of tests of level α will be denoted by Q_α. We adopt the usual convention that if $\psi \varepsilon Q_\alpha$, $E(\psi \mid \theta)$ is the probability of rejecting H_0 under θ.

Let Ω be a subset of the positive real numbers having zero as a limit point (any completely ordered set of the same cardinality, and an associated limit point would serve). For each $\lambda \varepsilon \Omega$ let Φ_λ be a non-empty subset of $\Theta - \Theta_0$.

DEFINITION 1. $\{\Phi_\lambda\ ; \lambda \varepsilon \Omega\}$ is a local family of subsets of Θ if for each α,

$0 < \alpha < 1$,

(1) $$\lim_{\lambda \to 0} \sup_{\psi_\lambda \varepsilon Q_\alpha} \inf_{\Phi_\lambda} E(\psi_\lambda \mid \theta) = \alpha.$$

DEFINITION 2. ψ^* is locally minimax of level α with respect to the local family $\{\Phi_\lambda\}$ if

(2) $$\lim_{\lambda \to 0} [(\inf_{\Phi_\lambda} E(\psi^* \mid \theta) - \alpha)(\sup_{\psi_\lambda \varepsilon Q_\alpha} \inf_{\Phi_\lambda} E(\psi_\lambda \mid \theta) - \alpha)^{-1}] = 1,$$

with the ratio in (2) assumed not to equal 0/0 for any $\lambda > 0$.

In the examples considered below Φ_λ will always be a fully invariant subset of the parameter space.

The following notation will be used throughout this paper: If A is any matrix then A_{ij} will denote the entry in the ith row and jth column of A. If A is a square matrix, A', $|A|$ and etr A will denote the transpose, determinant and the exponential of the trace of A, respectively. The $j \times j$ identity matrix will be denoted by I_j. A matrix all of whose entries are zero will be denoted by $\mathbf{0}$.

Also, let $GL(j)$ denote the full linear group of $j \times j$ non-singular real matrices and let $G_T(j)$ be the subgroup of lower triangular matrices (i.e., all matrices whose entries above the main diagonal are zero). Let $G_T^+(j)$ be the subgroup of $G_T(j)$ consisting of matrices all of whose diagonal elements are positive. Let $O(j)$ be the group of $j \times j$ orthogonal matrices. Let $\mu_{L(j)}$, $\mu_{T(j)}$ and $\mu_{O(j)}$ denote (left) Haar measure on $GL(j)$, $G_T(j)$ and $O(j)$, respectively, with $\mu_{O(j)}$ normalized. We shall usually omit the j, writing μ_L, μ_T and μ_O, in a context where the dimensions of the relevant groups are fixed.

Finally, R^j will denote the j-dimensional real translation group.

3. The MANOVA problem and invariance considerations. This and the next two sections deal with the MANOVA problem. In this section the relevant probability ratios are derived and as one consequence the locally best test among fully invariant tests is obtained in Theorem 1. The next section discusses local minimax properties of this test.

In the canonical form of the MANOVA problem $W = (Y, U, Z)$ is $p \times (r + n + m)$ and it will always be assumed that $r + m > p$ (so that $YY' + ZZ'$ will be non-singular with probability one). The columns of W are independent, normally distributed p-vectors with common, unknown, non-singular covariance matrix Σ. Also $EY = \xi(p \times r)$, $EU = \gamma(p \times n)$ and $EZ = O(p \times m)$. The problem is to test $\mathrm{H}_0 : \xi = 0$.

In formulating local minimax properties we must consider different families of alternative hypotheses. If the alternative hypothesis is $H_1 : \xi \neq 0$, then the MANOVA problem remains invariant under $GL(p) \times O(r)$ acting on W by $(A, H)W = (AYH, AU, AZ)$ and also under R^{pn} acting by translation on U. We shall always assume that the alternative, which is a subset of $\{(\xi, \Sigma) \mid \xi \neq 0\}$, is invariant under the groups just mentioned.

Convenient choices of a maximal invariant in the sample space of a sufficient statistic and in the parameter space are, respectively, L, the set of (ordered) latent roots $l_1 \geqq l_2 \geqq \cdots \geqq l_r \geqq 0$ of $Y'(YY' + ZZ')^{-1}Y$, and $\bar{\delta}$, the set of (ordered) latent roots $\bar{\delta}_1 \geqq \bar{\delta}_2 \geqq \cdots \geqq \bar{\delta}_r \geqq 0$ of $\xi'\Sigma^{-1}\xi$.

In G-K (1964) the first step in verifying the local minimaxity of the T^2-test is to reduce the original problem using the Hunt-Stein theorem. For the MANOVA problem the groups R^{pn} and $G_T(p)$ satisfy the Hunt-Stein theorem and therefore so does their semi-direct product. Hence if Φ_λ is a fully invariant subset of $H_1 : \xi \neq 0$ (and *a fortiori* invariant under $G_T(p)$ and R^{pn}) there is, for each α, a level α test which is $G_T(\mathrm{p})$- and R^{pn}-invariant and which maximizes, among all tests, the minimum power on Φ_λ. In terms of the definition of local minimaxity it is sufficient to consider in the denominator of (2) only that subset of Q_α which consists of $G_T(p)$- and R^{pn}-invariant tests.

A maximal invariant under R^{pn} is (Y, Z), and throughout this section we consider the MANOVA problem in terms of (Y, Z), the matrix U having been eliminated by invariance under R^{pn}. We shall require the probability ratio of the maximal invariant both under $GL(p) \times O(r)$ and also under $G_T(p)$. Since the initial parts of both derivations are the same we shall temporarily let G stand for either $GL(p) \times (O(r))$ or $G_T(p)$; also let μ_G denote (left) Haar measure on G.

G acts transitively on H_0 so that under H_0 the maximal invariant has a single probability distribution. There exists a unique $A \,\varepsilon\, G_T^{+}(p)$ such that $A\Sigma A' = I_p$, and if $\xi^* = A\xi$ then the probability density of the maximal invariant under (ξ, Σ) is the same as under (ξ^*, I_p). The density wrt Lebesgue measure Q, of (Y, Z) under (ξ^*, I_p) is

$$c \operatorname{etr} \{ -\tfrac{1}{2}(yy' + zz') + \xi^{*\prime}y\} \operatorname{etr} \{ -\tfrac{1}{2}\xi^{*\prime}\xi^*\}$$

where c is a constant. The measure with volume element $|yy' + zz'|^{-(m+r)/2}\, dQ$ is invariant under all linear transformations. With respect to this measure the joint density of (Y, Z) under (ξ^*, I_p) is

$$f_{(\xi^*, I_p)}(y, z) = c \operatorname{etr} \{ -\tfrac{1}{2}\xi^{*\prime}\xi^*\} |yy' + zz'|^{(m+r)/2} \operatorname{etr} \{ -\tfrac{1}{2}(yy' + zz') + \xi^{*\prime}y\}.$$

By Stein's representation the probability ratio of the maximal invariant under G is given by

$$(3) \qquad \int_G f_{(\xi^*, I_p)}(g(y, z))\, d\mu_G(g) \left(\int_G f_{(0, I_p)}(g(y, z))\, d\mu_G(g)\right)^{-1}.$$

Since $r + m > p$ by assumption, $(YY' + ZZ')$ is non-singular with probability one under all (ξ, Σ), so that it suffices to consider (3) only on $\{(y, z) \mid |yy' + zz'| \neq 0\}$. There exists a unique $g_0 \,\varepsilon\, G_T^{+}(p)$ such that $g_0(yy' + zz')g_0' = I_p$. Multiplying g on the right by g_0 in the integrands of (3) shows that (3) can be rewritten as

$$(4) \quad \operatorname{etr} \{ -\tfrac{1}{2}\xi^{*\prime}\xi^*\} \int_G |gg'|^{(m+r)/2} \operatorname{etr} \{ -\tfrac{1}{2}gg' + \xi^* g g_0 y\}\, d\mu_G(g) \cdot \left(\int_G |gg'|^{(m+r)/2} \operatorname{etr} \{ -\tfrac{1}{2}gg'\}\, d\mu_G(g)\right)^{-1}.$$

It is known (e.g., Stein (1956)) that $\mu_{L(p)}$ and $\mu_{T(p)}$ can be chosen such that for every function F integrable wrt $\mu_{L(p)}$

$$(5) \qquad \int_{GL(p)} F(g)\, d\mu_L(g) = \int_{G_T(p)} \int_{O(p)} F(gh)\, d\mu_O(h)\, d\mu_T(g).$$

If μ_L and μ_T are chosen to satisfy (5), it follows that the denominator of (4) is the same constant, D (say), whether $G = G_T(p)$ or $G = GL(p) \times O(r)$.

Taking $G = GL(p) \times O(r)$ in (4) and using (5) to rewrite the integral over $GL(p)$ yields

$$(6) \quad D^{-1} \operatorname{etr}\{-\tfrac{1}{2} \operatorname{tr} \xi^{*\prime}\xi^{*}\} \int_{G_T(p)} \int_{O(p)} \int_{O(r)} |gg'|^{(m+r)/2} \cdot \operatorname{etr}\{-\tfrac{1}{2}gg' + \xi^{*\prime}gh_1g_0yh\}\, d\mu_{O(r)}(h)\, d\mu_{O(p)}(h_1)\, d\mu_T(g)$$

for the probability ratio of the maximal invariant under $GL(p) \times O(r)$.

Similarly from (4), the probability ratio of the maximal invariant under $G_T(p)$ is given by

$$(7) \quad D^{-1} \operatorname{etr}\{-\tfrac{1}{2}\xi^{*\prime}\xi^{*}\} \int_{G_T(p)} |gg'|^{(m+r)/2} \operatorname{etr}\{-\tfrac{1}{2}gg' + \xi^{*\prime}gg_0y\}\, d\mu_T(g).$$

We next derive an estimate of (6) for $\operatorname{tr} \xi^{*\prime}\xi^{*}$ near zero, after which we shall return to an evaluation of (7). Note, first, that in (6) ξ^{*} can be multiplied on the left by any member of $O(p)$ and on the right by any member of $O(r)$ without changing the value of the integral. Hence, (6) depends on ξ^{*} only through the latent roots of $\xi^{*\prime}\xi^{*}$. Hence, in evaluating (6) we may assume that $\xi^{*\prime}\xi^{*}$ is diagonal with diagonal entries $\delta_1, \cdots, \delta_r$, and (6) must then be a symmetric function of $\delta_1, \cdots, \delta_r$ since it only depends on the latent roots. Secondly,

$$(8) \qquad \operatorname{tr} g_0y(g_0y)' = \operatorname{tr} yy'(yy' + zz')^{-1},$$

from which it follows that the entries of g_0y are uniformly bounded.

Considering the innermost integral in (6) and expanding the exponential,

$$\begin{aligned}\int_{O(r)} \operatorname{etr}\{\xi^{*\prime}gh_1g_0yh\}\, d\mu_{O(r)}(h) &= \int_{O(r)} \textstyle\sum_{f=0}^{\infty} f!^{-1}[\operatorname{tr}(\xi^{*\prime}gh_1g_0yh)]^f\, d\mu_{O(r)}(h)\\ &= \textstyle\sum_{f=0}^{\infty} \int_{O(r)} (2f)!^{-1}[\operatorname{tr}(\xi^{*\prime}gh_1g_0yh)]^{2f}\, d\mu_{O(r)}(h),\end{aligned}$$

since integrals of odd powers in the expansion are zero, James (1961), p. 876.

From relation (9) of James (1961) and the fact that the first zonal polynomial Z_1 is simply the trace, James (1960), we find that the last expression is equal to

$$(9) \quad 1 + r^{-1} \operatorname{tr}(\xi^{*\prime}\, gh_1g_0yy'g_0'h_1'g'\xi^{*}) + \textstyle\sum_{f=2}^{\infty} (2f)!^{-1} \int_{O(r)} [\operatorname{tr}(\xi^{*\prime}gh_1g_0yh)]^{2f}\, d\mu_{O(r)}(h).$$

From the relation $|\operatorname{tr} AB| \leqq [\operatorname{tr} AA' \operatorname{tr} BB']^{\frac{1}{2}}$ we have, for $\operatorname{tr} \xi^{*\prime}\xi^{*}$ sufficiently small and $f \geqq 2$,

$$\begin{aligned}(10) \quad [\operatorname{tr}(\xi^{*\prime}gh_1g_0yh)]^{2f} &\leqq (\operatorname{tr} gg')^f[\operatorname{tr} h'y'g_0'g_0yh\xi^{*\prime}\xi^{*}]^f\\ &\leqq (\operatorname{tr} gg')^f \operatorname{tr}((h'y'g_0'g_0yh)^2) \operatorname{tr}((\xi^{*\prime}\xi^{*})^2),\end{aligned}$$

since from (8) and the fact that $h \varepsilon O(r)$ we have $\operatorname{tr}(h'y'g_0'g_0yh\xi^{*\prime}\xi^{*}) < 1$ for all g_0, y and h when $\operatorname{tr} \xi^{*\prime}\xi^{*}$ is sufficiently small. Also, since $y'g_0'g_0y = y'(yy' + zz')^{-1}y$

$$(11) \qquad \operatorname{tr}(h'y'g_0'g_0yh)^2 \leqq \operatorname{tr}((I_r)^2) = r.$$

From (10), (11) and the inequality $\operatorname{tr}((\xi^{*\prime}\xi^{*})^2) \leqq (\operatorname{tr}(\xi^{*\prime}\xi^{*}))^2$, we have, for

$f \geqq 2$ and $\mathrm{tr}\, \xi^{*\prime}\xi^*$ small,

$$[\mathrm{tr}\,(\xi^{*\prime}gh_1g_0yh)]^{2f} \leqq r(\mathrm{tr}\,(\xi^{*\prime}\xi^*))^2(\mathrm{tr}\, gg')^f,$$

and therefore

$$(12)\quad \textstyle\sum_{f=2}^{\infty}(2f)!^{-1}\int_{O(r)}[\mathrm{tr}\,\xi^*gh_1g_0yh]^{2f}\,d\mu_{O(r)}(h)$$
$$\leqq r(\mathrm{tr}\,(\xi^{*\prime}\xi^*)^2)\textstyle\sum_{f=2}^{\infty}(2f)!^{-1}(\mathrm{tr}\, gg')^f$$
$$\leqq r(\mathrm{tr}\,(\xi^{*\prime}\xi^*))^2\,\mathrm{etr}\,\{\tfrac{1}{4}gg'\}.$$

Applying (12) to (9) and the result thereby obtained to (6), we find that the probability ratio of the maximal invariant under $GL(p) \times O(r)$ has the form

$$(13)\quad D^{-1}\,\mathrm{etr}\,\{-\tfrac{1}{2}\xi^{*\prime}\xi^*\}\int_{G_T(p)}\int_{O(p)}|gg'|^{(m+r)/2}\{\mathrm{etr} - \tfrac{1}{2}gg'\}$$
$$\cdot(1 + r^{-1}\,\mathrm{tr}\,(\xi^{*\prime}gh_1g_0yy'g_0'h_1g'\xi^*))\,d\mu_{O(p)}(h_1)\,d\mu_T(g) + o(\mathrm{tr}\,\xi^{*\prime}\xi^*)$$

where the last term is $o(\mathrm{tr}\,\xi^{*\prime}\xi^*)$ uniformly in (y, z).

Evaluating the integral over $O(p)$ in (13) according to relation (11) of James (1961) gives

$$(14)\quad \mathrm{etr}\,\{-\tfrac{1}{2}\xi^{*\prime}\xi^*\}[1 + (Drp)^{-1}\,\mathrm{tr}\,(y'g_0'g_0y)\int_{G_T(p)}|gg'|^{(m+r)/2}$$
$$\cdot\,\mathrm{etr}\,\{-\tfrac{1}{2}gg'\}\,\mathrm{tr}\,(g'\xi^*\xi^{*\prime}g)\,d\mu_T(g)] + o(\mathrm{tr}\,\xi^{*\prime}\xi^*).$$

Since the integrand in (14) is linear in $\delta_1, \cdots, \delta_r$, the diagonal entries of $\xi^{*\prime}\xi^*$, and since, as noted earlier, the probability ratio is symmetric in $\delta_1, \cdots, \delta_r$, the integral over $G_T(p)$ in (14) must be a multiple of $\sum_{i=1}^{r}\delta_i = \mathrm{tr}\,\xi^{*\prime}\xi^*$. We conclude that the probability ratio of the maximal invariant under $GL(p) \times O(r)$ has the form

$$(15)\quad (\mathrm{etr}\,\{-\tfrac{1}{2}\xi^{*\prime}\xi^*\})(1 + K\,\mathrm{tr}\,(y'(yy' + zz')^{-1}y)\,\mathrm{tr}\,(\xi^{*\prime}\xi^{*\prime})) + o(\mathrm{tr}\,\xi^{*\prime}\xi^*),$$
$$= 1 + \mathrm{tr}\,\xi^{*\prime}\xi^*(K\,\mathrm{tr}\,(y'(yy' + zz')^{-1}y - \tfrac{1}{2})) + o(\mathrm{tr}\,\xi^{*\prime}\xi^{*\prime})$$

where K is a positive constant and the last term is $o(\mathrm{tr}\,\xi^{*\prime}\xi^*)$ uniformly in (y, z).

THEOREM 1. *Let $\psi \in Q_\alpha$ be a fully invariant test. Then the power function of ψ has the form*

$$(16)\qquad E(\psi \mid (\xi, \Sigma)) = \alpha + B(\psi)\,\mathrm{tr}\,\xi'\Sigma^{-1}\xi + o(\mathrm{tr}\,\xi'\Sigma^{-1}\xi),$$

uniformly in (ξ, Σ). The test ψ^*, with acceptance region $\mathrm{tr}\, Y'(YY' + ZZ')Y \leqq C_\alpha$ *is the essentially unique test which maximizes $B(\psi)$ among all fully invariant tests of level α.*

PROOF. Since $\xi^{*\prime}\xi^* = \xi'\Sigma^{-1}\xi$ we have, from (15),

$$E(\psi \mid \xi, \Sigma) = E(\psi \mid O, I_p)$$
$$+ (\mathrm{tr}\,\xi'\Sigma^{-1}\xi)E(\psi(K\,\mathrm{tr}\,Y'(YY' + ZZ')^{-1}Y - \tfrac{1}{2}) \mid O, I_p)$$
$$+ o(\mathrm{tr}\,\xi'\Sigma^{-1}\xi)$$

which proves (16) with

$$B(\psi) = E(\psi(K \operatorname{tr} Y'(YY' + ZZ')^{-1}Y - \tfrac{1}{2}) \mid O, I_p).$$

The proof of the final statement, which is essentially a repetition of the proof of the Neyman-Pearson lemma using the local probability ratio (15), will be omitted.

We remark that if ψ is an unbiased fully invariant test, then $B(\psi) \geqq 0$. Since the test $\psi \equiv \alpha$ is essentially different from ψ^*, we must have $B(\psi^*) > 0$.

We return now to an evaluation of (7), the probability ratio of the maximal invariant under $G_T(p)$, which will be needed in the next section. In the integrand of (7), let $v = g_0 y$. A left invariant measure on the group $G_T(p)$ is given by

$$d\mu_T(g) = \textstyle\prod_{i=1}^{p} |g_{ii}|^{-i} \prod_{i \geqq j} dg_{ij}.$$

Hence (7) is equal to

$$(17) \quad D^{-1} \operatorname{etr} \{-\tfrac{1}{2}\xi^{*\prime}\xi^*\} \int_{G_T(p)} \textstyle\prod_{i=1}^{p} |g_{ii}|^{m+r-i} \cdot \exp \{-\tfrac{1}{2}\sum_{i \geqq j} g_{ij}^2 + (\sum_{k=1}^{r} \xi_{ik}^* v_{jk}) g_{ij}\} \prod_{i \geqq j} dg_{ij}.$$

The integral in (17) is separable in all of the variables individually. For $i > j$ integration with respect to g_{ij} yields $\exp \{\frac{1}{2}(\sum_{k=1}^{r} \xi_{ik}^* v_{jk})^2\}$. For $i = j$ integration with respect to g_{ii} yields a factor

$$\exp \{\tfrac{1}{2}(\textstyle\sum_{k=1}^{r} \xi_{ik}^* v_{ik})^2\} E[\chi^2((\sum_{k=1}^{r} \xi_{ik}^* v_{ik})^2)]^{(m+r-i)/2} = \varphi(\tfrac{1}{2}(m + r - i), \tfrac{1}{2}; \tfrac{1}{2}(\sum_{k=1}^{r} \xi_{ik} v_{ik})^2).$$

In this last equation $E[\chi^2(s)]^t$ denotes the expectation of the tth power of a non-central chi-square random variable with non-centrality parameter $E\chi^2(s) - 1 = s$, and φ is the confluent hypergeometric function (sometimes denoted by ${}_1F_1$),

$$(18) \qquad \varphi(a, b; \chi) = \textstyle\sum_{j=0}^{\infty} [\Gamma(a + j)\Gamma(b)/\Gamma(a)\Gamma(b + j)j!]\chi^j.$$

Hence we find that (7) is equal to

$$(19) \quad \operatorname{etr} \{-\tfrac{1}{2}\, \xi^{*\prime}\xi^*\} \exp \{\tfrac{1}{2}\textstyle\sum_{i>j} (\sum_{k=1}^{r} \xi_{ik}^* v_{jk})^2\} \cdot \prod_{j=1}^{p} \varphi(\tfrac{1}{2}(m + r - j + 1), \tfrac{1}{2}; \tfrac{1}{2}(\sum_{k=1}^{r} \xi_{jk}^* v_{jk})^2).$$

From (18) and (19) and the fact that the v_{jk} are bounded, we obtain for the local probability ratio

$$(20) \quad \operatorname{etr} \{-\tfrac{1}{2}\xi^{*\prime}\xi^*\}[1 \cdot + \tfrac{1}{2}\textstyle\sum_{i>j} (\sum_{k=1}^{r} \xi_{ik}^* v_{jk})^2 + \sum_{j=1}^{p} \tfrac{1}{2}(m + r - j + 1)(\sum_{k=1}^{r} \xi_{ik}^* v_{jk})^2] + R$$

where $R = o(\operatorname{tr} \xi^{*\prime}\xi^*)$ uniformly in the ξ_{ij}^* and v_{ij}. (In contrast to Theorem 1, there is, of course, no locally best test in the class of $G_T(p)$- (or $G_T(p) \times O(r)$-) invariant tests.)

4. Local minimax properties of the test ψ^* based on tr $Y'(YY' + ZZ')^{-1}Y$. As noted earlier, the first step in the proof is to reduce the original MANOVA problem by invariance under R^{pn} and $G_T(p)$. It is unnecessary to compute explicitly the maximal invariants under the group generated by R^{pn} and $G_T(p)$. From Section 3 we know that any R^{pn}- and $G_T(p)$-invariant test is a function of $V = g_0 Y$, where $g_0(YY' + ZZ')g_0' = I_p$, and that its distribution under (ξ, Σ) depends only on $\xi^* = A\xi$, where $A\Sigma A' = I_p$, with g_0, $A \varepsilon G_T^{\ +}(p)$. Even though ξ^*(and V) are not $G_T(p)$-invariant but only $G_T^{\ +}(p)$-invariant it will make the development simpler to construct *a priori* measures on $\{\xi^*\}$; any such measure induces a corresponding measure on the space of a $G_T(p)$-maximal invariant. We shall regard ξ^* as a random variable having various probability measures γ_λ and $E_{\gamma_\lambda}[f(\xi^*)]$ will denote the expectation under γ_λ of $f(\xi^*)$. For Φ a subset of the parameter space let $\Phi^* = \{\xi^* \mid (\xi, \Sigma) \varepsilon \Phi\}$.

The following lemma is an adaptation to the present setting of Lemma 1 of G-K (1964):

LEMMA 1. *Let $\{\Phi_\lambda ; \lambda \varepsilon \Omega\}$ be a family of fully invariant subsets of $\{(\xi, \Sigma) \mid \xi \neq 0\}$. Suppose that for each λ there exists S_λ, a fully invariant subset of the closure of Φ_λ with* tr $\xi'\Sigma^{-1}\xi$ *constant on S_λ, and probability measures γ_λ on S_λ^* such that*

(i) $\lim_{\lambda\to 0} \sup_{S_\lambda^*} (\operatorname{tr} \xi^{*\prime}\xi^*) = 0$,

(ii) $\inf_{S_\lambda^*} (\operatorname{tr} \xi^*\xi^{*\prime}) = \inf_{\Phi_\lambda^*} (\operatorname{tr} \xi^{*\prime}\xi^*) > 0$,

(iii) $E_{\gamma_\lambda}[\sum_{i>j} (\sum_{k=1}^r \xi_{ik}^* v_{jk})^2 + \sum_{j=1}^p (m + r - j + 1)(\sum_{k=1}^r \xi_{jk}^* v_{jk})^2]$
$$= d_\lambda \sum_{j=1}^p \sum_{k=1}^r v_{jk}^2 .$$

Then, $\{\Phi_\lambda\}$ is a local family and for each α, the test ψ^* with acceptance region tr $Y'(YY' + ZZ')^{-1}Y \leqq C_\alpha$ *is locally minimax with respect to $\{\Phi_\lambda\}$ as $\lambda \to 0$.*

PROOF. We show first that ψ^* is locally minimax wrt $\{S_\lambda\}$ which, from (i) and (ii) and (21) below, is a local family of subsets of $\{(\xi, \Sigma)\}$. By virtue of the Hunt-Stein theorem it suffices to show that ψ^* is locally minimax wrt $\{S_\lambda^*\}$ for the MANOVA problem reduced by invariance under the group generated by R^{pn} and $G_T(p)$.

From (16) we have that

$$E(\psi^* \mid \xi^*) = \alpha + B(\psi^*)(\operatorname{tr} \xi^{*\prime}\xi^*) + o(\operatorname{tr} \xi^{*\prime}\xi^*) \tag{21}$$

where $B(\psi^*) > 0$. Hence (2.1) of G-K (1964) is satisfied. (The regularity conditions preceding (2.1) are clearly satisfied.) From (20) and (iii) and the fact that

$$\sum_{j=1}^p \sum_{k=1}^r v_{jk}^2 = \operatorname{tr} g_0 yy' g_0' = \operatorname{tr} y'(yy' + zz')^{-1}y$$

it follows immediately that (2.2) of G-K (1964) is satisfied for the reduced problem.

Hence all of the assumptions of Lemma 1 of G-K (1964) are satisfied for the reduced problem and we conclude that

$$\lim_{\lambda>0} (\inf_{S_\lambda^*} E(\psi^* \mid \xi^*) - \alpha)(\sup_{\psi_\lambda \varepsilon Q_\alpha \cap I} \inf_{S_\lambda^*} E(\psi_\lambda \mid \xi^*) - \alpha)^{-1} = 1, \tag{22}$$

where I denotes the class of R^{pn}- and $G_T(p)$-invariant tests. By the Hunt-Stein theorem (22) implies

$$\text{(23)} \quad \lim_{\lambda>0} (\inf_{S_\lambda} E(\psi^* \mid (\xi, \Sigma)) - \alpha)(\sup_{\psi_\lambda \varepsilon Q_\alpha} \inf_{S_\lambda} E(\psi_\lambda \mid (\xi, \Sigma)) - \alpha)^{-1} = 1.$$

Finally, from the results of Das Gupta, Anderson, and Mudholkar (1964), it follows that ψ^* has a power function which is strictly increasing in each of the latent roots $\bar{\delta}_1 \geqq \bar{\delta}_2 \geqq \cdots \geqq \bar{\delta}_r$ of $\xi'\Sigma^{-1}\xi(= \xi^{*\prime}\xi^*)$. Hence from (21) and (ii)

$$\lim_{\lambda>0} \inf_{S_\lambda} E(\psi^* \mid \xi, \Sigma)(\inf_{\Phi_\lambda} E(\psi^* \mid \xi, \Sigma))^{-1} = 1;$$

and, since $\inf_{S_\lambda} (\psi \mid \xi, \Sigma) \geqq \inf_{\Phi_\lambda} (\psi \mid \xi, \Sigma)$ for every $\psi \varepsilon Q_\alpha$, we conclude that (23) holds with S_λ replaced by Φ_λ.

LEMMA 2. *Let $\epsilon > 0$ and for each i, $1 \leqq i \leqq p$, let η_i be a fixed $1 \times r$ matrix such that $\eta_i\eta_i' = \epsilon(m + r - i)^{-1}(m + r - i + 1)^{-1}p^{-1}(m + r)(m + r - p)$. Let η be the $p \times r$ matrix whose ith row is η_i. Let H be uniformly distributed over $O(r)$ and let $\zeta = \eta H$. Then $\zeta'\zeta = \eta'\eta$ and for every $p \times r$ matrix v*

$$E[\textstyle\sum_{i>j} (\sum_{k=1}^r \zeta_{ik}v_{jk})^2 + \sum_{j=1}^p (m + r - j + 1)(\sum_{k=1}^r \zeta_{jk}v_{jk})^2] = \epsilon p^{-1}(m + r) \operatorname{tr} v'v,$$

where E denotes expectation wrt ζ.

PROOF. From relation (9) of James (1961),

$$\text{(24)} \qquad \int_{O(r)} (\operatorname{tr} Ah)^2 \, d\mu_O(h) = \operatorname{tr} AA'$$

for all $k \times r$ matrices A with $k \leqq r$. Since $E(\sum_{k=1}^r \zeta_{ik}v_{jk})^2$ has the form (24) with $A' = \eta_i'(v_{j1}, v_{j2}, \cdots, v_{jr})$ we obtain

$$\begin{aligned} &E[\textstyle\sum_{i>j} (\sum_{k=1}^r \zeta_{ik}v_{ik})^2 + \sum_{j=1}^p (m + r - j + 1)(\sum_{k=1}^p \zeta_{jk}v_{jk})^2] \\ &\quad = \textstyle\sum_{i>j} (\eta_i\eta_i') \sum_{k=1}^r v_{jk}^2 + \sum_{j=1}^p (m + r - j + 1)(\eta_j\eta_j') \sum_{k=1}^r v_{jk}^2 \\ \text{(25)} \qquad &\quad = \epsilon p^{-1}(m + r)(m + r - p) \textstyle\sum_{k=1}^r \sum_{j=1}^p v_{jk}^2 \\ &\qquad \cdot [\textstyle\sum_{i>j} (m + r - i + 1)^{-1}(m + r - i)^{-1} + (m + r - j)^{-1}] \\ &\quad = \epsilon p^{-1}(m + r) \operatorname{tr} v'v. \end{aligned}$$

Here we have used the simple identities

$$\textstyle\sum_{i=1}^N (M - i)^{-1}(M - i + 1)^{-1} = NM^{-1}(M - N)^{-1} \quad \text{for} \quad M > N$$

and

$$(M - N_0)^{-1} + \textstyle\sum_{i=N_0}^N (M - i)^{-1}(M - i + 1)^{-1} = (M - N)^{-1}.$$

Note that η as defined in Lemma 2 satisfies

$$\begin{aligned} \operatorname{tr} \eta'\eta &= \textstyle\sum_{i=1}^p \eta_i\eta_i' \\ &= \epsilon(m + r)(m + r - p)p^{-1}\textstyle\sum_{i=1}^p (m + r - i)^{-1}(m + r - i + 1)^{-1} \\ &= \epsilon. \end{aligned}$$

THEOREM 2. *Let $\{\Phi_\lambda ; \lambda \varepsilon \Omega\}$ be a local family of fully invariant subsets of*

$\{(\xi, \Sigma) \mid \xi \neq 0\}$ *and let* $\epsilon_\lambda = \inf \{\mathrm{tr}\ \xi'\Sigma^{-1}\xi \mid (\xi, \Sigma)\ \varepsilon\ \Phi_\lambda\}$. *Suppose there exists* $\lambda_0 > 0$ *such that* $\lambda < \lambda_0$ *implies the existence of* $(\eta^{(\lambda)}, I_p)$ *belonging to the closure of* Φ *for which* $\eta_i^{(\lambda)}$, *the* ith *row of* η, *satisfies*

$$(26)\qquad \eta_i^{(\lambda)}\eta_i^{(\lambda)\prime} = \epsilon_\lambda(m + r = i)^{-1}(m + r - i + 1)^{-1}p^{-1}(m + r)(m + r - p).$$

Then, for each α, *the test* ψ^* *with acceptance region* $\mathrm{tr}\ Y'(YY' + ZZ')^{-1} \leqq C_\alpha$ *is locally minimax* wrt $\{\Phi_\lambda\}$ *as* $\lambda \to 0$.

PROOF. Let S_λ be the intersection of the closure of Φ_λ and $\{(\xi, \Sigma) \mid \mathrm{tr}\ \xi'\Sigma^{-1}\xi = \epsilon_\lambda\}$. Since S_λ is fully invariant, $(\eta^{(\lambda)}H, I_p)\ \varepsilon\ S_\lambda$ for all $H\ \varepsilon\ O(r)$. Let γ_λ be the probability measure on S_λ^* for which with probability one $\xi^* = \eta^{(\lambda)}H$ and H is uniformly distributed on $O(r)$.

Conditions (i) and (ii) of Lemma 1 are clearly satisfied for $\{\Phi_\lambda \mid \lambda < \lambda_0\}$ and by Lemma 2 the measure γ_λ satisfies (iii) of Lemma 1.

Theorem 2 is the main result on local minimax properties. The condition (26) imposes a constraint on the family $\{\Phi_\lambda\}$ which is dicussed below. However it is clear that if, for each $\lambda < \lambda_0$ $\{(\xi, \Sigma) \mid \mathrm{tr}\ \xi'\Sigma^{-1}\xi = \epsilon_\lambda = \bar{\delta}_1\}$ is contained in the closure of Φ_λ, then the conditions of Theorem 2 can be satisfied for all m, r and p by making all columns of $\eta^{(\lambda)}$ except the first equal to zero. In particular we have

COROLLARY 1. *For every* α, m, r *and* p, $\mathrm{tr}\ Y'(YY' + ZZ')^{-1}Y \leqq C_\alpha$ *is locally minimax* wrt $\{\{(\xi, \Sigma) \mid \mathrm{tr}\ \xi'\Sigma^{-1}\xi = \bar{\delta}_1 = \lambda\}\}$ *and* wrt $\{\{(\xi, \Sigma) \mid \mathrm{tr}\ \xi'\Sigma^{-1}\xi = \lambda\}\}$

5. Further discussion of the MANOVA problem. (a) In order to indicate the nature of the constraint imposed by (26), suppose $p = r = 2$. Then (26) becomes

$$(27)\qquad \begin{aligned} \eta_1^{(\lambda)}\eta_1^{(\lambda)\prime} &= \tfrac{1}{2}\epsilon_\lambda m(m + 1)^{-1}, \\ \eta_2^{(\lambda)}\eta_2^{(\lambda)\prime} &= \tfrac{1}{2}\epsilon_\lambda (m + 2)(m + 1)^{-1}. \end{aligned}$$

It is easily checked that if $\eta^{(\lambda)}$ satisfies (27), the ratio of the smallest latent root of $\eta^{(\lambda)}\eta^{(\lambda)\prime}$ to the largest latent root is at most

$$(\eta_1^{(\lambda)}\eta_1^{(\lambda)\prime})(\eta_2^{(\lambda)}\eta_2^{(\lambda)})^{-1} = m(m + 2)^{-1}.$$

Conversely, if there exists (ξ, Σ) belonging to the closure of Φ_λ such that $\bar{\delta}_2/\bar{\delta}_1 \leqq m(m + m(m + 2)^{-1}$ and $\bar{\delta}_1 + \bar{\delta}_2 = \epsilon_\lambda$ then there exists $(\eta^{(\lambda)}, \mathrm{I}_p)$ in the closure of Φ_λ for which $\eta^{(\lambda)}$ satisfies (27). Hence, when $p = r = 2$, the condition $\bar{\delta}_2/\bar{\delta}_1 \leqq m(m + 2)^{-1}$ and $\bar{\delta}_2 + \bar{\delta}_1 = \epsilon_\lambda$ for some (ξ, Σ) in the closure of Φ_λ is equivalent to the existence of $\eta^{(\lambda)}$ satisfying (26).

For a general $p \times r$ matrix η with $r \leqq p$, *if* $a_1 \geqq a_2 \geqq \cdots \geqq a_r$ are the latent roots of $\eta'\eta$ and if $\eta_i\eta_i' \leqq \eta_{i+1}\eta_{i+1}'$ for $1 \leqq i \leqq p - 1$, then $\sum_{i=k}^r a_i \leqq \sum_{i=1}^{p-k+1} \eta_i\eta_i'$ for $1 \leqq k \leqq r$, with equality for $k = 1$. It therefore seems reasonable to conjecture that, for any values of $a_1 \geqq a_2 \geqq \cdots \geqq a_r$ satisfying

$$\textstyle\sum_{i=k}^r a_i \leqq \epsilon_\lambda p^{-1}(m + r)(m + r - p)\ \sum_{i=1}^{p-k+1} (m + r - i)^{-1}(m + r - i + 1)^{-1}$$

for $1 \leqq k \leqq r$, with equality for $k = 1$, there exists a $p \times r$ matrix $\eta^{(\lambda)}$ satisfying (26) and such that $a_1 \geqq a_2 \geqq \cdots \geqq a_r$ are the latent roots of $\eta'\eta$. This would yield a condition on $\{\Phi_\lambda\}$ analogous to the one at the end of the previous paragraph in the case $p = r = 2$.

(b) Suppose $p = r$ and let

$$\Phi_\lambda = \{(\xi, \Sigma) \mid \bar{\delta}_1 = \bar{\delta}_2 = \cdots = \bar{\delta}_r = \lambda\}. \tag{28}$$

This choice of Φ_λ does not satisfy the conditions of Theorem 2. Although for each λ the set Φ_λ is full invariant, the group $G_T(p) \times O(r)$ acts transitively on Φ_λ. Hence for each $\lambda > 0$ there is a unique best $G_T(p) \times O(r)$-invariant test of $\xi = 0$ vs. $(\xi, \Sigma) \varepsilon \Phi_\lambda$. Since $G_T(p) \times O(r)$ satisfies the Hunt-Stein theorem this test is minimax on Φ_λ. This best $G_T(p) \times O(r)$-invariant test is given by letting $\xi^* = \lambda^{\frac{1}{2}} I_p H$ in (19) and averaging wrt the uniform distribution on $H \varepsilon O(r)$. This computation has not been done. However from (20) the local form of the best $G_T(p) \times O(r)$-invariant test is easily computed. (The computation is very similar to the proof of Lemma 2.) The resulting test is based on $\sum_{j=1}^{p} (p + m + r - 2j + 1) V_k' V_k$, where V_k is the kth row of V, which is not fully invariant.

This shows that some condition like (26) is necessary for the conclusion of Theorem 2. It also provides another example in which the best $GL(p)$ invariant test is not minimax.

(c) In addition to its local minimax properties, a fair amount is now known about the test with acceptance region tr $Y'(YY' + ZZ')^{-1}Y \leqq C_\alpha$. From Kiefer and Schwartz (1965), when $m \geqq p$, it is a proper Bayes test of $\xi = 0$ vs. H_1 : $\{(\xi, \Sigma) |$ tr $\xi'\Sigma^{-1}\xi \leqq \epsilon\}$ for any $\epsilon > 0$. Taking $p = r$ for simplicity, if Φ_λ is defined by (28) then, from Schwartz (1966a) the above test is admissible for testing $\xi = 0$ vs. $\xi \varepsilon \bigcup_{\lambda > K} \Phi_\lambda$ for any $K > 0$.

6. Local properties of invariant tests of the independence of sets of variates. The setup for testing independence of sets of variates is the following: For $m \geqq p$ the $p \times m$ matrix X is partitioned into $X' = (X_1' X_2' \cdots X_k')$ where X_i is $p_i \times m$ with $p_1 + \cdots + p_k = p$. The columns of X are independently and identically distributed normal vectors with zero mean and unknown non-singular covariance matrix Σ. Letting $\Sigma^{ij} = m^{-1} E X_i X_j$, the problem is to test

$$H_0 : \Sigma^{ij} = 0 \quad \text{for all} \quad i \neq j.$$

Again, we will be concerned with different families of invariant alternative hypotheses. (As far as minimax properties are concerned, the more usual set-up where each column of X has the same unknown expectation is reduced to the given set up, with m replaced by $m - 1$, by invoking the Hunt-Stein theorem applied to the real translation group of dimension p.)

Let

$$G = GL(p_1) \times GL(p_2) \times \cdots \times GL(p_k),$$

$$G_T = G_T(p_1) \times G_T(p_2) \times \cdots \times G_T(p_k),$$

$$G_T^+ = G_T^+(p_1) \times G_T^+(p_2) \times \cdots \times G_T^+(p_k),$$

$$F = O(p_1) \times O(p_2) \times \cdots \times O(p_k),$$

and let

$$\nu = \mu_{L(p_1)} \times \mu_{L(p_2)} \times \cdots \times \mu_{L(p_k)} ,$$

$$\nu_T = \mu_{T(p_1)} \times \mu_{T(p_2)} \times \cdots \times \mu_{T(p_k)} ,$$

$$\nu_F = \mu_{O(p_1)} \times \mu_{O(p_2)} \times \cdots \times \mu_{O(p_k)} ,$$

so that ν, ν_T and ν_F are (left) Haar measure on G, G_T, and F respectively. Also $\nu_F(F) = 1$.

Let A be any $p \times j$ matrix partitioned according to $A' = (A_1', \cdots, A_k')$ with A_i being $p_i \times j$. We shall think of G as a subgroup of $GL(p)$ with $g \varepsilon G$ acting on A by the usual matrix multiplication, that is,

$$g = \begin{Bmatrix} g_1 & & & 0 \\ & g_2 & & \\ & & \ddots & \\ 0 & & & g_k \end{Bmatrix}, \qquad gA = \begin{Bmatrix} g_1 A_1 \\ g_1 A_2 \\ \vdots \\ g_k A_k \end{Bmatrix}.$$

If the alternative hypothesis is

$$H_1 : \Sigma^{ij} \neq 0 \quad \text{for some} \quad i \neq j,$$

then the problem remains invariant under G acting on X according to the previous paragraph. The induced group on the parameter space is isomorphic to G and acts as

$$g \circ \Sigma = (g_1, \cdots, g_k) \circ \Sigma = g\Sigma g'.$$

We shall use the symbol "$\circ$" to distinguish the abstract group action on the parameter space from ordinary matrix multiplication.

Since G_T satisfies the Hunt-Stein theorem we proceed, as in the MANOVA example, to find the local form of the probability ratio of the maximal invariant under both G and G_T. Temporarily, let G_0 stand for either G or G_T and let ν_0 denote (left) Haar measure on G_0. We note that G_0 acts transitively on H_0 so that the maximal invariant has a single probability distribution under H_0.

Let $\Gamma = \Sigma^{-1}$ and (Γ^{ij}), $i, j = 1, \cdots, k$, be the partition of Γ corresponding to the partition (Σ^{ij}) of Σ. There exists $\bar{g} = (\bar{g}_1, \cdots, \bar{g}_k) \varepsilon G_T^+$ such that $\bar{g}_i \Gamma^{ii} \bar{g}_i' = I_{p_i}$. Let $\Sigma^* = (\bar{g}_1', \bar{g}_2', \cdots, \bar{g}_k')^{-1} \circ \Sigma$ and $\Gamma^* = (\Sigma^*)^{-1}$. Then the distribution of the maximal invariant under Σ is the same as under Σ^*, for which $(\Gamma^*)^{ii} = I_{p_i}$. We also let $\Delta = \sum_{i \neq j} \operatorname{tr} \Sigma^{ij}(\Sigma^{jj})^{-1}\Sigma^{ji}(\Sigma^{ii})^{-1}$, which is also equal to $\sum_{i \neq j} \operatorname{tr} (\Gamma^*)^{ij}(\Gamma^*)^{ij}$.

The density wrt Lebesgue measure Q, of X under Σ^* is

$$(2\pi)^{-\frac{1}{2}pm} |\Sigma^*|^{-m/2} \operatorname{etr} \{ -\tfrac{1}{2}(\Sigma^*)^{-1}xx' \}.$$

The measure with volume element $|xx'|^{-m/2} \, dQ$ is invariant under all linear transformations. With respect to this measure the density of X is

$$(29) \qquad f_{(\Sigma^*)}(x) = (2\pi)^{pm/2} |\Sigma^*|^{-m/2} (xx')^{m/2} \operatorname{etr} \{ -\tfrac{1}{2}(\Sigma^*)^{-1}xx' \}.$$

By Stein's representation the probability ratio of the maximal invariant under G_0

is given by

$$\int_{G_0} f_{(\Sigma^*)}(gx)\, d\nu_0(g) \left(\int_{G_0} f_{(I_p)}(gx)\, d\nu_0(g) \right)^{-1}. \tag{30}$$

There exists $\hat{g} = (\hat{g}_1, \cdots, \hat{g}_k) \varepsilon G_0$ such that $\hat{g}_i x_i x_i' \hat{g}_i' = I_{p_i}$ for $1 \leqq i \leqq k$.

Multiplying g on the right by $\hat{g}$ in the integrands leaves the value of (30) unchanged so that (30) can be rewritten as

$$\begin{aligned}(31)\quad \int_{G_0} |\Gamma^*|^{m/2} \textstyle\prod_{i=1}^k |g_i g_i'|^{m/2} \operatorname{etr} \{ -\tfrac{1}{2} \Gamma^*(gw)(gw)' \} \\ \cdot \left(\int_{G_0} \textstyle\prod_{i=1}^k |g_i g_i'|^{m/2} \operatorname{etr} \{ -\tfrac{1}{2} gg' \}\, d\nu_0(g) \right)^{-1}\end{aligned}$$

where $w = \hat{g}x$ and $w_i = \hat{g}_i x_i$ satisfies $w_i w_i' = I_{p_i}$. Let C denote the value of the denominator in (31), which by (33) below may be taken to have the same value whether $G_0 = G$ or $G_0 = G_T$. Then (31) becomes

$$\begin{aligned} & C^{-1} \int_{G_0} |\Gamma^*|^{m/2} \textstyle\prod_{i=1}^k |g_i g_i'|^{m/2} \exp \{ -\tfrac{1}{2} \sum_{i,j} \operatorname{tr} (\Gamma^*)^{ji} g_i w_i w_j' g_j' \}\, d\nu_0(g) \\ (32)\quad & = C^{-1} \int_{G_0} |\Gamma^*|^{m/2} \textstyle\prod_{i=1}^k |g_i g_i'|^{m/2} \exp \{ -\tfrac{1}{2} \sum_{i=1}^k g_i g_i' \} \\ & \cdot \textstyle\sum_{v=0}^{\infty} v!^{-1} \left(\sum_{i \neq j} -\tfrac{1}{2} \operatorname{tr} (\Gamma^*)^{ji} g_i w_i w_j' g_j' \right)^v d\nu_0(g). \end{aligned}$$

In analogy with (5), ν_T may be chosen so that for every function t integrable on G wrt μ

$$\int_G t(g)\, d\mu(g) = \int_{G_T} \int_F t(gh)\, d\nu_T(g)\, d\nu_F(h). \tag{33}$$

For the next part of the derivation let γ denote a probability measure on F which is either equal to ν_F or which assigns unit mass to the identity element of F. Then using (33) the right-hand side of (32) becomes

$$\begin{aligned}(34)\quad C^{-1} |\Gamma^*|^{m/2} \int_{G_T} \textstyle\prod_{i=1}^k |g_i g_i'|^{m/2} \exp \{ -\tfrac{1}{2} \sum_{i=1}^k \operatorname{tr} g_i g_i' \} \int_F \sum_{v=0}^{\infty} v!^{-1} \\ \cdot \left(\textstyle\sum_{i \neq j} -\tfrac{1}{2} \operatorname{tr} (\Gamma^*)^{ji} g_i h_i w_i w_j' h_j' g_j' \right)^v d\gamma(h)\, d\nu_T(g),\end{aligned}$$

where setting $\gamma = \nu_F$ gives equality with (32) when $G_0 = G$ and letting γ assign mass one to the identity corresponds to $G_0 = G_T$ in (32).

Letting g_{ijl} denote the (j, l) entry of g_i,

$$d\nu_T = \textstyle\prod_{i=1}^k \prod_{j \geqq l} dg_{ijl} \left(\prod_{i=1}^k \prod_{j=1}^{p_i} |g_{ijj}|^j \right)^{-1} \tag{35}$$

from which it is clear that $\prod_{i=1}^k | g_i g_i' |^{n/2} \operatorname{etr} \{ -\frac{1}{2} \Sigma\, g_i g_i' \} \prod_{i=1}^k \prod_{j=1}^{p_i} |g_{ijj}|^{-j}$ is separable and symmetric about zero in the individual variables g_{ijl}. Hence for odd indices v in the integrand of (34) the resulting integral is zero. Similarly when $v = 2$ the integral of all cross-product terms will be zero and only terms of the form $(\operatorname{tr} (\Gamma^*)^{ji} g_i h_i w_i w_j' h_j' g_j')^2$ will contribute. Hence (34) can be rewritten as

$$\begin{aligned} & C^{-1} |\Gamma^*|^{m/2} \int_{G_T} \textstyle\prod_{i=1}^k |g_i g_i'|^{m/2} \exp \{ -\tfrac{1}{2} \Sigma \operatorname{tr} g_i g_i' \} \\ (36)\quad & \cdot \left[\int_F \{ 1 + \tfrac{1}{8} \textstyle\sum_{i \neq j} (\operatorname{tr} (\Gamma^*)^{ji} g_i h_i w_i w_j' h_j' g_j')^2 \}\, d\gamma(h) \right. \\ & \left. + \int_F \textstyle\sum_{v=2}^{\infty} (2v)!^{-1} \left(\sum_{i \neq j} \tfrac{1}{2} \operatorname{tr} (\Gamma^*)^{ji} g_i h_i w_i w_j' h_j' g_j' \right)^{2v} d\gamma(h) \right] d\nu_T(g). \end{aligned}$$

The following simple results are needed in estimating the contribution of the sum over v in (36). First, from the definition of w_i following (31) we have that $\operatorname{tr} w_i w_j' w_j w_i'$ is precisely the sum of the squares of the sample canonical correlations between the ith and jth sets of variates. Hence $\operatorname{tr} w_i w_j' w_j w_i' \leqq p$ for all i and j. Next it is easily shown that for any real matrices a, b, c, d

$$|\operatorname{tr} abcd| \leqq (\operatorname{tr} aa' \operatorname{tr} bb' \operatorname{tr} cc' \operatorname{tr} dd')^{\frac{1}{2}}. \tag{37}$$

This follows from two applications of the inequality $|\operatorname{tr} AB| \leqq (\operatorname{tr} AA' \operatorname{tr} BB')^{\frac{1}{2}}$ together with the inequality $|\operatorname{tr} [(AA')^2]| \leqq (\operatorname{tr} AA')^2$, which are valid for real matrices A and B.

Applying the preceding paragraph to the sum in (36) and noting that h_i and h_j are orthogonal yields

$$\begin{aligned}\textstyle\sum_{v=2}^{\infty} (2v)!^{-1}\big(\sum_{i\neq j} \tfrac{1}{2} \operatorname{tr} (\Gamma^*)^{ji} g_i h_i w_i w_j' h_j' g_j'\big)^{2v} \\ &\leqq \textstyle\sum_{v=2}^{\infty} p^v (2v)!^{-1}\big(\sum_{i\neq j} (\operatorname{tr} (\Gamma^*)^{ij} (\Gamma^*)^{ji} \operatorname{tr} g_i g_i' \operatorname{tr} g_j g_j')^{\frac{1}{2}}\big)^{2v} 2^{-2v} \\ &\leqq \textstyle\sum_{v=2}^{\infty} p^v (2v)!^{-1} 2^{-2v} \Delta^v \big(\sum_{i\neq j} \operatorname{tr} g_i g_i' \operatorname{tr} g_j g_j'\big)^v\end{aligned}$$

by the Schwarz inequality. If $\Delta \leqq p^{-1}$, then, since $\sum_{i\neq j} \operatorname{tr} g_i g_i' \operatorname{tr} g_j g_j' \leqq (\sum_{i=1}^{k} \operatorname{tr} g_i g_i')^2$, we have

$$\begin{aligned}\textstyle\sum_{v=2}^{\infty} (2v)!^{-1}\big(\sum_{i\neq j} \tfrac{1}{2} \operatorname{tr} (\Gamma^*)^{ji} g_i h_i w_i w_j' h_j' g_j'\big)^{2v} \\ &\leqq \textstyle\sum_{v=2}^{\infty} (2v)!^{-1} 2^{-2v} p^2 \Delta^2 \big(\sum_{i=1}^{k} \operatorname{tr} g_i g_i'\big)^{2v} \\ &\leqq \textstyle\sum_{v=2}^{\infty} p^2 \Delta^2 v!^{-1} \big(\tfrac{1}{4} \sum_{i=1}^{k} \operatorname{tr} g_i g_i'\big)^v \leqq p^2 \Delta^2 \exp\{\tfrac{1}{4} \sum_{i=1}^{k} \operatorname{tr} g_i g_i'\}.\end{aligned} \tag{38}$$

Applying (38) to (36) it follows that (34) can be rewritten as

$$\begin{aligned}C^{-1} |\Gamma^*|^{m/2} \textstyle\int_{G_T} \prod_{i=1}^{k} |g_i g_i'|^{m/2} \exp\{-\tfrac{1}{2} \sum_{i=1}^{k} \operatorname{tr} g_i g_i'\} \\ \cdot \textstyle\int_F \big(1 + \tfrac{1}{4} \sum_{i\neq j} (\operatorname{tr} (\Gamma^*)^{ji} g_i h_i w_i w_j' g_j' h_j')^2\big)\, d\gamma(h)\, d\nu_T(g) + M,\end{aligned} \tag{39}$$

where $M = o(\Delta)$ uniformly in Γ^* and the w_i.

We proceed to evaluate (39) when γ assigns mass one to the identity element of F, and thereby obtain the local form of the probability ratio of the maximal invariant under G_T. From (35) for this γ, (39) becomes

$$\begin{aligned}C^{-1} |\Gamma^*|^{m/2} \textstyle\int_{G_T} \prod_{i=1}^{k} \prod_{j=1}^{p_i} |g_{ijj}|^{m-j} \exp\{-\tfrac{1}{2} \sum_{i=1}^{k} \sum_{j\geqq l} g_{ijl}^2\} \\ \cdot \big(1 + \tfrac{1}{4} \textstyle\sum_{i\neq j} (\operatorname{tr} (\Gamma^*)^{ji} g_i w_i w_j' g_j')^2\big) \prod dg_{ijl} + M.\end{aligned} \tag{40}$$

Let $w^{ij} = w_i w_j'$. Then using the fact that

$$\textstyle\prod_{i=1} \prod_{j=1} |g_{ijj}|^{m-j} \exp\{-\tfrac{1}{2} \sum_{i=1} \sum_{j\geqq l} g_{ijl}^2\}$$

is separable and symmetric about zero in all of the variables, a straightforward computation shows that (39) is equal to

$$\begin{aligned}|\Gamma^*|^{m/2}\big(1 + \tfrac{1}{2} \textstyle\sum_{i\neq j} \sum_{r=1}^{p_i} \sum_{s=1}^{p_j} (w_{rs}^{ij})^2 \big[\sum_{l>r, n>s} ((\overset{*}{}_{ln})\Gamma^{ij})^2 \\ + (m - r + 1) \textstyle\sum_{n>s} ((\Gamma_{rn}^*)^{ij})^2 + (m - s + 1) \sum_{l>r} ((\Gamma_{ls}^*)^{ij})^2 \\ + (m - r + 1)(m - s + 1)(\Gamma_{rs}^{ij})^2\big]\big) + M.\end{aligned} \tag{41}$$

From the obvious fact that $|\Gamma^*| = 1 - \Delta + o(\Delta)$ we obtain finally that the local form of the probability ratio of the maximal invariant under G_T is given by

$$\begin{aligned} &1 - \tfrac{1}{2}m\Delta + \tfrac{1}{2}\textstyle\sum_{i\neq j}\sum_{r=1}^{p_i}\sum_{s=1}^{p_j}(w_{rs}^{ij})^2[\sum_{l>r,n>s}((\Gamma_{ln}^*)^{ij})^2 \\ (42)\qquad &\quad + (m-r+1)\textstyle\sum_{m>s}(\Gamma_{rm}^{*ij})^2 + (n-s+1)\sum_{l>r}((\Gamma_{ls}^*)^{ij})^2 \\ &\quad + (m-r+1)(m-s+1)((\Gamma_{rs}^*)^{ij})^2] + o(\Delta) \end{aligned}$$

uniformly in the w_{rs}^{ij} and the $(\Gamma_{rs}^*)^{ij}$.

Next we evaluate (39) when $\gamma = \nu_F$ is normalized Haar measure on F. With this choice of γ, (39) becomes

$$\begin{aligned} &C^{-1}|\Gamma^*|^{m/2}\textstyle\int_{G_T}\prod_{i=1}^{k}\prod_{j=1}^{p_i}|g_{ijj}|^{m-j}\exp\{-\tfrac{1}{2}\sum_{i=1}^{k}\sum_{j\geq l}g_{ijl}^2\} \\ (43)\qquad &\quad \cdot[1 + \tfrac{1}{4}\textstyle\sum_{i\neq j}\int_{O(p_i)}\int_{O(p_j)}(\operatorname{tr}(\Gamma^*)^{ji}g_ih_iw^{ij}h_jg_j')^2 \\ &\qquad\qquad \cdot d\mu_{O(p_i)}(h_i)\,d\mu_{O(p_j)}(h_j)]\textstyle\prod dg_{ijl} + M. \end{aligned}$$

Hence the first step is to evaluate

$$(44)\qquad \int_{O(p_i)}\int_{O(p_j)}(\operatorname{tr}(\Gamma^*)^{ji}g_ih_iw^{ij}g_jh_jg_j')^2\,d\mu_O(h_i)\,d\mu_O(h_j).$$

Integrating over $O(p_j)$ according to relation (9) of James (1961), we obtain that (44) equals

$$(45)\qquad \int_{O(p_i)}p_j^{-1}\operatorname{tr}(g_j'(\Gamma^*)^{ji}g_ih_iw^{ij}w^{ji}h_i'g_i'(\Gamma^*)^{ij}g_j)\,d\mu_O(h_i).$$

Integrating over $O(p_i)$ according to relation (11) of James (1961), we find that (45) equals

$$(46)\qquad p_i^{-1}p_j^{-1}(\operatorname{tr}g_ig_i'(\Gamma^*)^{ij}g_jg_j'(\Gamma^*)^{ji})\operatorname{tr}(w^{ij}w^{ji}).$$

Applying (46) to (43) yields

$$\begin{aligned} &C^{-1}|\Gamma^*|^{m/2}\textstyle\int_{G_T}\prod_{i=1}^{k}\prod_{j=1}^{p_i}|g_{ijj}|^{m-j}\exp\{-\tfrac{1}{2}\sum_{i=1}^{k}\sum_{j\geq l}g_{ijl}^2\} \\ (47)\qquad &\quad [1 + \tfrac{1}{4}\textstyle\sum_{i\neq j}p_i^{-1}p_j^{-1}(\operatorname{tr}(g_ig_i'(\Gamma^*)^{ij}g_jg_j'(\Gamma^*)^{ji}))(\operatorname{tr}w^{ij}w^{ji})] \\ &\qquad\qquad \cdot\textstyle\prod_{i=1}^{k}\prod_{j\geq l}dg_{ijl} + M. \end{aligned}$$

The last expression, (47), will now be shown to have the following form:

$$(48)\qquad |\Gamma^*|^{m/2}[1 + K\textstyle\sum_{i\neq j}p_i^{-1}p_j^{-1}\operatorname{tr}(w^{ij}w^{ij})(\operatorname{tr}(\Gamma^*)^{ij}(\Gamma^*)^{ji})] + M,$$

where K is a positive constant.

Consider the lefthand side of (32) with $G_0 = G$ and ν_0 equal to Haar measure on G, so that ν_0 is transpose invariant. It is readily seen, using elementary properties of the trace, that (32) is symmetric in the pair $(w^{ij}, (\Gamma^*)^{ij})$. Hence (47) must have the form

$$(49)\qquad |\Gamma^*|^{m/2}[1 + \textstyle\sum_{i\neq j}p_i^{-1}p_j^{-1}a_{ij}\operatorname{tr}(w^{ij}w^{ji})\operatorname{tr}((\Gamma^*)^{ij}(\Gamma^*)^{ji})] + M,$$

and it only remains to show a_{ij} does not depend on i and j. It is possible but tedious to calculate the a_{ij} explicitly from (46). Instead, we set $(\Gamma^*)^{rs} = 0$ for

$(r, s) \neq (i, j)$ in (49) and compare the result with equation (59) of Constantine (1963); it follows that a_{ij} is positive and depends only on m and not on p_i or p_j. This proves (48). Finally, from (48) and $|\Gamma^*| = 1 - \Delta + o(\Delta)$, we obtain that the local form of the probability ratio of the maximal invariant under G is

$$(50) \quad 1 - \tfrac{1}{2}m\Delta + K \textstyle\sum_{i\neq j} p_i^{-1}p_j^{-1} \operatorname{tr}(w^{ij}w^{ji}) \operatorname{tr}((\Gamma^*)^{ij}(\Gamma^*)^{ji}) + o(\Delta)$$

uniformly in the w^{ij} and $(\Gamma^*)^{ij}$.

Let $V^{ij} = (X_iX_i')^{-1}X_iX_j'(X_jX_j')^{-1}X_jX_i'$, let $U_{ij} = \operatorname{tr} V^{ij}$, and let $U = \sum_{i\neq j} U_{ij}$.

The notions of local admissibility and locally complete classes of tests appearing in the following definitions are natural adaptations of the usual definitions of admissibility and complete class.

Definition 3. The G-invariant test $\psi \in Q_\alpha$ is locally admissible among G-invariant tests (hereafter called locally G-admissible) if, given any other G-invariant test $\psi_1 \in Q_\alpha$ and any $\epsilon > 0$, $E(\psi_1 \mid \Sigma) \geqq E(\psi \mid \Sigma)$ for all Σ such that $0 < \Delta < \epsilon$ implies $E(\psi_1 \mid \Sigma) = E(\psi \mid \Sigma)$ for all Σ.

Definition 4. Let Ψ be a class of G-invariant tests. Then Ψ is a locally minimal essentially complete class of G-invariant tests (hereafter abbreviated by lmlcc-G) if given any G-invariant test ψ_1 there exists $\epsilon > 0$ and $\psi \in \Psi$ such that

(i) $E(\psi \mid \Sigma) \leqq E(\psi_1 \mid \Sigma)$ for $\Sigma \in H_0$,

(ii) $E(\psi \mid \Sigma) \geqq E(\psi_1 \mid \Sigma)$ for $0 < \Delta < \epsilon$

and if any proper of subset of Ψ does not have this property.

Theorem 3. *If $\psi \in Q_\alpha$ is G-invariant, then the power function of ψ has the form*

$$(51) \qquad E(\psi \mid \Sigma) = \alpha + \textstyle\sum_{i<j} B_{ij}(\psi) \operatorname{tr}(\Sigma^{ii})^{-1}\Sigma^{ij}(\Sigma^{jj})^{-1}\Sigma^{ji} + o(\Delta)$$

uniformly in Σ.

If Ψ is the class of tests with acceptance regions of the form $\{\sum_{i<j} d_{ij}U_{ij} \leqq C; d_{ij} \geqq 0\}$, then Ψ is lmlcc-G and every $\psi \in \Psi$ is locally G-admissible.

If $k = 2$, the test ψ^* with acceptance region $\{U \leqq C_\alpha\}$ maximizes $B_{12}(\psi)$ among all G-invariant tests of size α.

Proof. The proof is based entirely on (50). Since $\operatorname{tr}(\Gamma^*)^{ij}(\Gamma^*)^{ji} = \operatorname{tr}(\Sigma^{ii})^{-1}\Sigma^{ij}(\Sigma^{jj})^{-1}\Sigma^{ji}$, we have from (50)

$$E(\psi \mid \Sigma) = E(\psi \mid I_p) + \textstyle\sum_{i\neq j} [(Kp_i^{-1}p_j^{-1}E(\psi U_{ij} \mid I_p) - m/2) \cdot (\operatorname{tr}(\Sigma^{ii})^{-1}\Sigma^{ij}(\Sigma^{jj})^{-1}\Sigma^{ji})] + o(\Delta)$$

which proves (51) with

$$B_{ij}(\psi) = Kp_i^{-1}p_j^{-1}E(\psi U_{ij} \mid I_p) - m/2.$$

Let $C > 0$ and $d_{ij} \geqq 0$ be given for all $i < j$ with $d_{ij} \neq 0$ for all (i, j), this latter case being trivial. Let $\psi_0(c)$ be the test with acceptance region $\{\sum_{i\neq j} d_{ij}U_{ij} \leqq c\}$. For $0 < \lambda < 1$ let q_λ be an *a priori* probability measure which assigns mass $d_{ij}p_ip_j(\sum_{i<j} d_{ij}p_ip_j)^{-1}$ to each of the sets $D_{ij} = \{\Sigma \mid (\Gamma^*)^{rs} = 0$ for $(r, s) = (i, j)$; $\operatorname{tr}(\Gamma^*)^{ij}(\Gamma^*)^{ji} = \lambda\}$. Let q_0 assign mass one to $\Sigma = I_p$.

For any G-invariant test ψ let c_ψ be such that $E(\psi_0(c_\psi) \mid I_p) = E(\psi \mid I_p)$. Then, the expectation wrt q_λ of the power function of ψ has the form $E_{q_\lambda}E(\psi \mid \Sigma) = E(\psi \mid I_p) + \hat{B}(\psi)\lambda + o(\lambda)$ and, from (50), $\psi_0(c_\psi)$ is the essentially unique test which maximizes $\hat{B}$ among all G-invariant tests of size $E(\psi \mid I_p)$. Let $r_\lambda = tq_\lambda + (1 - t)q_0$ with $t^{-1}(1 - t) = C$, and let ψ be any G-invariant test and denote the Bayes risk of ψ wrt r_λ by $R(\psi \mid r_\lambda)$. The differences in Bayes risks wrt r_λ of ψ and $\psi_0(C)$ is

$$R(\psi \mid r_\lambda) - R(\psi_0(C) \mid r_\lambda) \geqq R(\psi \mid r_\lambda) - R(\psi_0(c_\psi) \mid r_\lambda)$$
$$= t\lambda[\hat{B}(\psi) - \hat{B}(\psi_0(c_\psi))] + o(\lambda).$$

The expression in square brackets is non-negative and is positive if ψ is essentially different from $\psi(c_\psi)$. This shows that $\psi_0(C)$ is the unique local Bayes test wrt r_λ as $\lambda \to 0$.

Hence any $\psi \varepsilon \Psi$ is a unique local Bayes test and therefore locally G-admissible. Conversely a local Bayes test among G-invariant tests belongs to Ψ. The proof that Ψ is lmlcc-G parallels completely the proof that the class of Bayes procedures is essentially complete for finite decision problems.

Finally, the proof that for $k = 2$ the test ψ^* maximizes $B_{12}(\psi)$ is a special case of the above discussion or, alternatively, a local form of the proof of the Neyman-Pearson lemma.

In Theorem 3 the conclusion for $k = 2$ is a direct analogue of Theorem 1 for the MANOVA problem. The other parts of Theorem 3 do not have analogues in the MANOVA problem.

We turn now to the local minimax problem for which a satisfactory solution has been obtained only in two cases. First, if there are only two sets of variates, the test ψ^* of Theorem 3 will be shown to be locally minimax for certain contours. Also if all p_i are equal the test which accepts when $U \leqq C_\alpha$ will be shown to be locally minimax for certain contours. These results apply to various families of contours $\{\Phi_\lambda\}$ including in particular

$$\{\{\Sigma \mid \Delta = \lambda\}\} \quad \text{and} \quad \{\{\Sigma \mid \max_{i<j} \operatorname{tr} (\Gamma^*)^{ij}(\Gamma^*)^{ji} = \lambda\}\}.$$

It will also become clear that, even for general p_i the test based on U is locally minimax wrt the contours on which it has constant power. Other tests belonging to Ψ with unequal d_{ij} are locally minimax for different families of contours, but no attempt has been made to delineate these.

Again the first step in the proof is to reduce the original problem by invariance under G_T and again it is unnecessary to compute explicitly sample and parameter maximal invariants under G_T. We know that any G_T-invariant test is a function of W where $W = (\hat{g}_1, \hat{g}_2, \cdots, \hat{g}_k)X$ and $\hat{g}_iX_iX_i'\hat{g}_i' = I_{p_i}$ and that its distribution under Σ depends only $\Sigma^* = (\bar{g}_1, \bar{g}_2, \cdots, \bar{g}k) \circ \Sigma$ where $\bar{g}_i\Gamma^{ii}\bar{g}_i' = I_{p_i}$ and $\Gamma = \Sigma^{-1}$. Even though Σ^* (and W) are not G_T-invariant but only $G_T{}^+$-invariant, it will suffice and make the development simpler to construct *a priori* measures on $\{\Sigma^*\}$. For Φ a subset of the parameter space let $\Phi = \{\Sigma^* \mid \Sigma \varepsilon \Phi\}$.

(a) *The Case* $k = 2$. The following lemma is an adaptation to the present setting of Lemma 1 of G-K (1964).

LEMMA 3. *Let* $\{\Phi_\lambda ; \lambda \varepsilon \Omega\}$ *be a family of G-invariant subsets of* $\{\Sigma \mid \Delta \neq 0\}$. *Suppose that for each* λ *there exists* S_λ, *a G-invariant subset of the closure of* Φ_λ *with* Δ *constant on* S_λ, *and probability measures* γ_λ *on* S_λ^*, *such that*

(i) $\lim_{\lambda\to 0} \sup_{S_\lambda} \{\Delta\} = 0$,

(ii) $\inf_{S_\lambda} \{\Delta\} = \inf_{\Phi_\lambda} \{\Delta\} > 0$,

(iii)
$$E_{\gamma_\lambda}[\textstyle\sum_{l>r,n>s} ((\Gamma_{lm}^*)^{12})^2 + (m - r + 1) \sum_{n>s} ((\Gamma_{rn}^*)^{12})^2 + (m - s + 1) \sum_{l>r} ((\Gamma_{ls}^*)^{12})^2 + (m - r + 1)(m - s + 1)((\Gamma_{rs}^*)^{12})^2] = d_\lambda$$

where d_λ *is independent of r and s.*

Then $\{\Phi_\lambda\}$ *is a local family and for each* α, *the test* ψ^* *with acceptance region* $\{U \leqq C_\alpha\}$ *is locally minimax* wrt $\{\Phi_\lambda\}$ *as* $\lambda \to 0$.

The proof of Lemma 3 parallels almost exactly the proof of Lemma 1. The monotonicity of the power function of ψ^* follows from the results of Anderson and Das Gupta (1964).

THEOREM 4. *Let* $\{\Phi_\lambda ; \lambda \varepsilon \Omega\}$ *be a family of G-invariant subsets of* $\{\Sigma \mid \Delta \neq 0\}$. *Let* $\epsilon_\lambda = \inf_{\Phi_\lambda} \{\Delta\}$ *and suppose* $\epsilon_\lambda > 0$ *and* $\lim_{\lambda\to 0} \epsilon_\lambda = 0$. *Suppose there exists* $\lambda_0 > 0$ *such that* $\lambda < \lambda_0$ *implies the existence of* Σ *belonging to the closure of* Φ_λ *for which* $(\Gamma^*)^{12}$ *has rank one and* tr $(\Gamma^*)^{12}(\Gamma^*)^{21} = \epsilon_\lambda$. *Then, for each* α, *the test* ψ^* *which accepts when* $U = U_{12} \leqq C_\alpha$ *is locally minimax* wrt $\{\Phi_\lambda\}$ *as* $\lambda \to 0$.

PROOF. Let S_λ be the intersection of $\{\Sigma \mid \text{tr}\,(\Gamma^*)^{12}(\Gamma^*)^{21} = \epsilon_\lambda ; \text{rank}\,(\Gamma^*)^{12} = 1$ and the closure of Φ_λ. Conditions (i) and (ii) are clearly satisfied for $\{\Phi_\lambda ; \lambda < \lambda_0\}$. Let γ_λ be the measure (not normalized) defined as follows: Assign mass $(m - r + 1)^{-1}$ to each of the points $\Sigma_{(r)}$ defined by $(\Gamma_{ls}^*)^{12} = 0$ if $l \neq r$, and $(\Gamma_{rn}^*)^{12} = (\epsilon_\lambda(m - n)^{-1}(m - n + 1)^{-1}P_2^{-1}m(m - p_2))^{\frac{1}{2}}$. Then the left-hand side of (iii) of Lemma 3 becomes

$$\begin{aligned}
&\textstyle\sum_{n>s} \epsilon_\lambda(m - n)^{-1}(m - n + 1)^{-1}p_2^{-1}m(m - p_2) \\
(52)\qquad &\quad + (m - s + 1)\epsilon_\lambda(m - s)^{-1}(m - s + 1)^{-1}p_2^{-1}m(m - p_2) \\
&= \epsilon_\lambda p_2^{-1}m(m - p_2)(\textstyle\sum_{m>s} (m - n)^{-1}(m - n + 1)^{-1} + (m - s)^{-1}) \\
&= \epsilon_\lambda p_2^{-1}m.
\end{aligned}$$

Hence condition (iii) of Lemma 3 is satisfied since, for each $\Sigma_{(r)}$, tr $(\Gamma^*)^{12}(\Gamma^*)^{21} = \epsilon_\lambda$ and rank $(\Gamma^*)^{12} = 1$.

Theorem 4 is an incomplete analogue of Theorem 2 since rank $(\Gamma^*)^{12} = 1$ on the support of γ_λ is much more restrictive than (26). Further work might produce a more complete analogue of Theorem 2 and of the discussion in Section 5. In any event Theorem 4 applies to $\{\{\Sigma \mid \Delta = \lambda\}\}$ and for each α the test based on U_{12} is locally minimax wrt $\{\{\Sigma \mid \Delta = \lambda\}\}$ as $\lambda \to 0$.

(b) *The Case* $p_1 = p_2 = \cdots = p_k$. When $p_1 = p_2 = \cdots = p_k$ the original problem remains invariant under all permutations of the k sets of variates. The test ψ^* which accepts when $U \leqq C_\alpha$ is invariant under all such permutations and therefore in (51) $B_{ij}(\psi^*) = B(\psi^*)$, independent of (i, j) Hence

$$E(\psi^* \mid \Sigma) = \alpha + K\Delta + o(\Delta)$$

uniformly in Σ and with $K > 0$, so that (2.1) of G-K (1964) is satisfied for $\Phi_\lambda = \{\Sigma \mid \Delta = \lambda\}$.

Also, let γ_λ be the measure defined as follows: γ_λ assigns mass $(m - r + 1)^{-1}$ to each of the points $\Sigma_{(r)}$ defined by $(\Gamma^*_{ls})^{ij} = 0$ if $l \neq r$, and

$$(\Gamma^*_{rm})^{ij} = (2\lambda k^{-1}(k-1)^{-1}(m-n+1)^{-1}p_i^{-1}m^{-1}(m-p_i)^{-1}),$$

then γ_λ is supported on $\{\Sigma \mid \Delta = \lambda\}$ and, from (42) and (52), ψ^* satisfies (2.2) of G-K (1964) so that Lemma 1 of G-K (1964) is satisfied and yields the local minimaxity of ψ^* wrt $\{\{\Sigma \mid \Delta = \lambda\}\}$ as $\lambda \to 0$.

Alternatively, let γ_λ' be the measure supported on $\{\Sigma \mid \max_{i<j} \operatorname{tr}(\Gamma^*)^{ij}(\Gamma^*)^{ji} = \Delta = \lambda\}$ and which assigns mass $k^{-1}(k-1)^{-1}(m-r+1)^{-1}$ to each of the points $\Sigma_{(r,i,j)}$ defined by $(\Gamma^*)^{st} = 0$ if $(s, t) \neq (i, j)$ and $(\Gamma^*_{rm})^{ij} = (2\lambda(m-n)^{-1} \cdot (m-n+1)^{-1}p_i^{-1}m^{-1}(m-p_i)^{-1})^{-1}$. Again from (42) and (52) ψ^* satisfies (2.2) of G-K (1964) and we obtain the local minimaxity of ψ^* wrt $\{\{\Sigma \mid \max_{i<j} \operatorname{tr}(\Gamma^*)^{ij}(\Gamma^*)^{ji} = \Delta = \lambda\}\}$.

We summarize in

THEOREM 5. *If $p_1 = p_2 = \cdots = p_k$, then, for each α, the test ψ^* which accepts when $U \leqq C_\alpha$ is locally minimax as $\lambda \to 0$ wrt $\{\{\Sigma \mid \Delta = \lambda\}\}$ and* wrt $\{\{\Sigma \mid \max_{i<j} \operatorname{tr}(\Gamma^*)^{ij}(\Gamma^*)^{ji} = \Delta = \lambda\}\}$.

In the special case $p_1 = p_2 = \cdots = p_k = 1$, ψ^* is the test based on the sum of the squares of the sample correlation coefficients.

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