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A RANDOM TIME CHANGE RELATING SEMI-MARKOV AND MARKOV PROCESSES[1]

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Abstract. We investigate the question of when a semi-Markov process is transformed by a random time change into a Markov process.

1. Introduction. In [2] Lévy asserts that every semi-Markov process is related to a Markov process in such a way that the sample functions of the Markov process are a suitable modification of the sample functions of the semi-Markov process. We study this relationship by defining a random time change, $\tau(\cdot, \omega)$, for each ω a function from the parameter space, the reals, into the reals. $\tau(\cdot, \cdot)$ is defined in such a way that the sample functions of the semi-Markov process are transformed into sample functions of a Markov process. Our results are quite general if the process has no instantaneous states. Otherwise we require a restrictive measurability condition which will be explicitly stated in the next section.

Lévy refers to this relationship by saying the two processes have the same succession of states. We now define that concept.

Definition 1.1. Two processes $\{U_t, t \geqq 0\}$ and $\{V_t, t \geqq 0\}$ which are defined on the same probability space $(\Omega, \mathbf{B}, P)$ will be said to have the same succession of states if there is a monotone non-decreasing continuous function of t, $\tau(t, \omega)$, defined for all t and almost all ω, such that for all t, $U_{\tau(t,\omega)}(\omega) = V_t(\omega)$ for those ω for which $\tau(t, \omega)$ is defined and such that τ is unbounded as t becomes infinite.

As it is assumed that $(\Omega, \mathbf{B}, P)$ remains fixed, an immediate consequence, is that $U_{\tau(t,\omega)}(\omega)$ and $V_t(\omega)$ will have the same finite dimensional distributions. For example, even if $\{U_t, t \geqq 0\}$ and $\{V_t, t \geqq 0\}$ can both be realized on the space of right continuous functions from $[0, \infty)$ to the appropriate phase space the transformation $\tau(\cdot, \cdot)$ will be the identity transformation only if U and V have the same finite dimensional distributions.

2. Specifaction of the process and preliminaries. Let $\{X_t, t > 0\}$ be a separable, Borel measurable process with a denumerable phase space denoted by I, a set of positive integers. Define.

$$\begin{aligned} Y_t(\omega) &= t \text{ if } X_s(\omega) = X_t(\omega) \quad \text{for all} \quad 0 \leqq s \leqq t, \\ &= t - \sup\{s: 0 \leqq s \leqq t, X_s(\omega) \neq X_t(\omega)\} \qquad \text{otherwise.} \end{aligned}$$

Definition 2.1. If the two dimensional process $\{(X_t, Y_t), t \geqq 0\}$ is a Markov process having the strong Markov property and strong stationary transition

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probabilities then $\{X_t, t \geqq 0\}$ will be called a semi-Markov process. (See [3, Chapter 12] for a discussion of these concepts.)

We will assume the sample functions satisfy lim inf as $s \to t^+ X_s = X_t$.

In terms of the transition probability $p_t(i, y; S)$, which denotes a version of the conditional probability $P[(X_{t+u}, Y_{t+u}) \varepsilon S \mid (X_u, Y_u) = (i, y)]$, we define for each $i \varepsilon I$ a measure on the events in the σ-field generated by $\{(X_t, Y_t), t \geqq 0\}$. This measure determined by $p_{\cdot}(i, 0; \cdot)$ will be written $P_i[\cdot]$.

Consider next the random variable

$$W_t = \inf \{s : s \geqq t, X_s \neq X_t\} \quad \text{if} \quad X_u = X_t, \quad 0 \leqq u \leqq t,$$
$$= \inf \{s : s \geqq t, X_s \neq X_t\} - \sup \{s : s \leqq t, X_s \neq X_t\} \qquad \text{otherwise.}$$

Then let $F_i(t) = P_i[W_0 \leqq t]$ for each $i \varepsilon I$.

This allows us to classify the set I according to the limit from the right at zero of F_i for $i \varepsilon I$. Call this limit a_i. Then

(i) if $a_i = 0$ we say that i is a stable state,

(ii) if $a_i = 1$ we say that i is instantaneous.

In general, we may have $0 < a_i < 1$ or $F_i(+\infty) < 1$ for some states i but since these generalizations cause technical difficulties without adding clarity to the results we will not consider them here.

Let J denote the set of stable states. Then $I-J$ is the set of instantaneous states. We assume $J \neq \emptyset$ since a semi-Markov process with only instantaneous states is already a Markov process according to our definition.

We now define the entrance times for each $j \varepsilon J$. The first entrance to state j

$$\theta(j, 1) = \infty \qquad \text{if} \quad \{t : t \geqq 0, X_t = j\} = \emptyset.$$
$$= \inf \{t : t \geqq 0, (X_t, Y_t) = (j, 0)\} \qquad \text{otherwise.}$$

The kth entrance to state j is successively defined by

$$\theta(j, k) = \infty \qquad \text{if } \{t : \infty > t > \theta(j, k-1), (X_t, Y_t) = (j, 0)\} = \emptyset.$$
$$= \inf \{t : t > \theta(j, k-1), (X_t, Y_t) = (j, 0)\} \qquad \text{otherwise.}$$

For each finite valued entrance time $\theta(j, k)$ there corresponds a finite exit time since $F_i(+\infty) = 1$. Call this exit time

$$\Delta(j, k) = \inf \{t : t > \theta(j, k), X_t \neq j\}.$$

Then for $j \varepsilon J$ and $k = 1, 2, \cdots$

$$[\theta(j, k) \leqq t] \,\varepsilon\, \mathbf{B}\{(X_s, Y_s), 0 \leqq s \leqq t\},$$

($\mathbf{B}$ denotes the field of measurable sets generated by the random variables listed between the braces)

$$[\Delta(j, k) \leqq t] \,\varepsilon\, \mathbf{B}\{(X_s, Y_s), 0 \leqq s \leqq t\},$$

so that θ and Δ are Markov times for the strong Markov process (X_t, Y_t), c.f.

[3], page 582. Then, if $j \varepsilon J$, we denote

$$N(j, t) = \sup [0, \{k: \theta(j, k) \leqq t\}]$$

and observe that $P[N(j, t) < \infty] = 1$ for all $t \geqq 0$ by the strong Markov property, right lower semi-continuity, and the fact that $\{\theta(k, k): k = 1, 2, \cdots\}$ is a renewal process.

Finally, if the process X_t has instantaneous states we require an additional measurability condition.

CONDITION A. The two dimensional Markov process $\{(X_t, Y_t), t \geqq 0\}$ is a strong Markov process with respect to the family of Borel fields

$$\mathbf{F}_t = \mathbf{B}\{X_u, u \leqq t; Z_{j,k}, k \leqq N(j, t), j \varepsilon J\}.$$

The definition of the random variables $Z_{j,k}$ appears in the next section. This condition is discussed in Section 5 where it is used.

3. Definition of the random time change. In general the triple $(\Omega, \mathbf{B}, P)$ on which $\{X_t, t \geqq 0\}$ is defined need not support the following constructions. However, one can easily enlarge the space and define an equivalent process on this enlarged space so that the new space will support this construction. We thus assume that our process is already defined on such a sufficient space.

Define for each $j \varepsilon J$ a sequence of independent random variables $\{Z_{j,k}\}$, $k = 1, 2, \cdots$, so that they are all mutually independent and mutually independent of the semi-Markov process $\{X_t, t \geqq 0\}$. For each $j \varepsilon J$ let $Z_{j,k}$, $k = 1, 2, \cdots$, be identically distributed negative exponential

$$P[Z_{j,k} > t] = \exp(-\lambda_j t)$$

where $(\lambda_j)^{-1}$ is a median of the distribution F_j.

Next we define

$$\begin{aligned}\Psi(s, \omega) &= Z_{x(s,\omega), N(X(s,\omega),s)}/W_s(\omega) && \text{if } 0 < W_s(\infty) < \infty, \quad X(s, \omega) \varepsilon J,\\ &= 1 && \text{if } X(s, \omega) \varepsilon I - J,\\ &= 0 && \text{otherwise.}\end{aligned}$$

PROPOSITION 3.1. *$\Psi(s, \omega)$ is a measurable function of (s, ω) on the product space $T \times \Omega$ with respect to the product field generated by the Lebesgue sets on T and by $\mathbf{B}$ on Ω.*

The proof of this proposition is routine and will be omitted.

DEFINITION 3.1. For almost all ω write $\tau(t, \omega) = \inf \{s: \int_0^s \Psi(v, \omega)\, dv \geqq t\}$.

DEFINITION 3.2. We let $X_{\tau(t,\omega)}(\omega) = X_t'(\omega)$.

PROPOSITION 3.2. *τ is a measurable function of the pair (t, ω).*

The proof of this proposition is omitted. However several comments about τ will aid to its understanding:

(i) τ is a monotone non-decreasing function of t for a.a. ω. Note that $\Psi \geqq 0$ almost surely.

(ii) Ψ remains constant on intervals of constancy of X_t, thus τ varies linearly on such an interval and the interval of constancy of X_s' has length determined by the appropriate random variable Z.

4. X and X' have the same succession of states. Here and in what follows primed symbols such as θ', N', etc. will be defined in terms of the X' process exactly as the unprimed symbols were defined in terms of X.

THEOREM 4.1. *$\{X_t', t \geqq 0\}$ is a measurable process. $\{X_t, t \geqq 0\}$ and $\{X_t', t \geqq 0\}$ have the same succession of states.*

PROOF. The first assertion follows from Proposition 3.2 and our assumption that $\{X_t, t \geqq 0\}$ is Borel measurable.

In view of the remarks at the end of Section 3, concerning the properties of τ, we need only show that the total time spent by X' in a succession of states determined by $\{X_t, t \leqq M\}$ is a.s. finite. Since $\Psi(s, \omega) = 1$ for $X_s(\omega)$ in an instantaneous state we need only compare times spent by X' and X in the stable states.

This latter question can be answered by consideration of an increasing sequence $\{\alpha(n)\}$ of entrance times of X_t.

Let $U_n = \min(W_{\alpha(n)}, 1)$ and form $Z_n = \sum_{k=1}^{n} \{U_k - E\{U_k \mid W_{\alpha(1)}, \cdots, W_{\alpha(k-1)}\}\}$. $\{Z_n\}$ is a convergent martingale, see [1], page 323. Thus $\sum_{k=1}^{\infty} U_k$ converges or diverges with $\sum_{k=1}^{\infty} E\{U_k \mid W_{\alpha(1)}, \cdots, W_{\alpha(k-1)}\}$ but this is almost surely equal to $\sum_{k=1}^{\infty} E\{U_k \mid X_{\alpha(k)}\}$ ($\alpha(k)$ is a Markov time for the process) which converges or diverges according to $\sum_{k=1}^{\infty} \lambda_{X_{\alpha(k)}}^{-1}$ which in turn implies the convergence or divergence of the associated sequence of exponential random variables. This assures a finite duration of the X' process in any succession of finite duration for the X process.

The X' process may have a finite life, i.e., the sample functions may not be defined beyond some point, say $\zeta(\omega)$.

5. The main theorem. In view of existing literature on random time changes for Markov processes some comments are in order regarding our method of proof. We do not use existing theorems or techniques because they all require that for fixed t, $\tau(t, \cdot)$ be an optional random variable and our time change fails this condition in its dependence on W_t. Our attack will be to use the random times at which τ is an optional random variable to get a generous supply of times for which the Markov property holds for the X' process. Then we use the exponential distributions to complete the argument.

We say a random variable T is an optional random variable if for all $t \geqq 0$, $\{T \leqq t\} \in \mathbf{B}\{X_s, s \leqq t\}$. The field of events A such that $A \in \mathbf{B}$ and $A \cap [T \leqq t] \in \mathbf{B}\{X_s, s \leqq t\}$ is denoted by $\mathbf{B}\{X_s, s \leqq T\}$. With a strict inequality above we call this the pre-T field and with a reversed strict inequality it is called the post-T field.

PROPOSITION 5.1. *The optional random variables $\theta'(\Delta')$ have the property that the pre-θ' (pre-Δ') and post-θ' (post-Δ') fields are conditionally independent given $\mathbf{B}\{X'_{\theta'}\}$ ($\mathbf{B}\{X'_{\Delta'}\}$).*

PROOF. Let $i \varepsilon J$ and k be fixed in the following discussion: Denote by

$$\begin{aligned}
\mathbf{B}_1' &= \mathbf{B}\{X_u', u < \theta'(i, k)\} \quad \text{the pre-}\theta'(i, k) \text{ field},\\
\mathbf{B}_2' &= \mathbf{B}\{X_u', u > \theta'(i, k)\} \quad \text{the post-}\theta'(i, k) \text{ field},\\
\mathbf{Z}_1 &= \mathbf{B}\{Z_{j,n}, n \leqq N(j, \theta(i, k)) - \delta_{i,j} j \varepsilon J\},\\
\mathbf{Z}_2 &= \mathbf{B}\{Z_{j,n}, n > N(j, \theta(i, k)) - \delta_{i,j}, j \varepsilon J\},\\
\mathbf{B}_1 &= \mathbf{B}\{\mathbf{B}\{X_u, u < \theta(i, k)\}, \mathbf{Z}_1\},\\
\mathbf{B}_2 &= \mathbf{B}\{\mathbf{B}\{X_u, u > \theta(i, k)\}, \mathbf{Z}_2\},\\
\mathbf{F} &= \mathbf{B}\{X_{\theta(i,k)}\}.
\end{aligned}$$

By our assumption, $X_{\theta(i,k)} = [\lim/s \downarrow \theta(i, k)]X_s = i$ for a stable state. Hence $\mathbf{F}$ is degenerate. Note that $\tau(\theta'(i, k), \cdot)$ a.s.

The strong Markov property at time $\theta(i, k)$ implies $\mathbf{B}\{X_u, u > \theta(i, k)\}$ and $\mathbf{B}\{X_u, u < \theta(i, k)\}$ are independent, (conditionally independent given $\mathbf{F}$). $\mathbf{B}\{X_u, u > \theta(i, k)\}$ is conditionally independent of $\mathbf{Z}_1$ given $\mathbf{F}$ by virtue of the strong Markov property at $\theta(i, k)$ and the independence of the $Z_{j,k}$ and the X_t process.

$\mathbf{Z}_1$ and $\mathbf{Z}_2$ are conditionally independent owing to the independence of the $Z_{j,n}$ and the strong Markov property for $\theta(i, k)$. Similarly $\mathbf{Z}_2$ and $\mathbf{B}\{X_u, u < \theta(i, k)\}$ are conditionally independent.

Now $\mathbf{B}\{X_u, u > \theta(i, k)\}$ and $\mathbf{B}_1$ are conditionally independent given $\mathbf{F}$ since if $A \varepsilon \mathbf{B}\{X_u, u > \theta(i, k)\}$, $B \varepsilon \mathbf{B}\{X_u, u < \theta(i, k)$ and $C = [Z_{j,n} \varepsilon M] \cap [n \leqq N(j, \theta(i, k))]$ then

$$P[A \cap B \cap C \mid \mathbf{F}] = P[A \cap B \cap [n \leqq N(j, \theta(i, k))] \mid \mathbf{F}]P[Z_{j,n} \varepsilon M] \text{ a.s.}$$

and

$$B \cap [n \leqq N(j, \theta(i, k))] \varepsilon \mathbf{B}\{X_u, u < \theta(i, k)\}$$

so that this becomes

$$P[A \mid \mathbf{F}]P[B \cap \{m \leqq N(j, \theta(i, k))\} \mid \mathbf{F}]P[Z_{j,n} \varepsilon M] \text{ a.s.}$$

Finally, using the independence of the Z's we can write this $P[A \mid \mathbf{F}]P[B \cap C \mid \mathbf{F}]$ a.s.

Now by repeating the above argument with an additional factor for a generating set from $\mathbf{Z}_2$ the conditional independence follows since it is sufficient to show the independence for sets which generate the fields.

This is slightly stronger than the statement of the Proposition for θ' since $\mathbf{B}_1' \subseteq \mathbf{B}_1$ and $\mathbf{B}_2' \subseteq \mathbf{B}_2$. The assertion concerning Δ' is proved similarly, $\mathbf{B}\{X'_{\Delta'}\}$ is not trivial in this case but that does not complicate the proof. Note that $\tau(\Delta'(i, k), \cdot) = \Delta(i, k)$ a.s. Our proof uses now the strong Markov property at $\Delta(i, k)$. The details are omitted.

PROPOSITION 5.2. *If $i \varepsilon J$ then for $t > s$,*

$$P[X_t' = j \mid X_u', u \leqq s] = P[X_t' = j \mid X_s' = i]$$

for almost all $\omega \varepsilon [X_s = i]$.

PROOF. Assume WLOG we have a regular version of $P[\cdot \mid X_u', u \leqq s]$, i.e., we assume the conditional probability has the properties of a measure for almost all ω, c.f. [1], page 31.

We write

$$P[\cdot \mid X_u', u \leqq s; X_s' = i]$$

to denote the value of $P[\cdot \mid X_u', u \leqq s]$ for a.a. $\omega \varepsilon [X_s' = i]$. Consider then

$$P[X_t' = j \mid X_u', u \leqq s; X_s' = i] = \textstyle\sum_{k=0}^{\infty} P[X_t' = j, N'(i, s) = k \mid X_u', u \leqq s; X_s' = i]$$

which equals, by Proposition 5.1,

$$\begin{aligned} &\textstyle\sum_{k=0}^{\infty} \int_s^t P[X'_{t-v} = j \mid X'_{\Delta'(i,k)}]P[\Delta'(i, k) \varepsilon dv, N'(i, s) \\ (1) \qquad &= k \mid X_u', u \leqq s; X_s' = i] + \delta_{i,j} \textstyle\sum_{k=0}^{\infty} P[\Delta'(i, k) > t, N'(i, s) \\ &\qquad = k \mid X_u', u \leqq s; X_s' = i] \end{aligned}$$

where

$$\begin{aligned} \delta_{i,j} &= 0 \quad \text{if} \quad i \neq j, \\ &= 1 \quad \text{if} \quad i = j. \end{aligned}$$

Now by the exponential form of the distribution of $Z_{i,k}$ which determines the value of $\Delta'(i, k) - \theta'(i, k)$

$$\begin{aligned} P[\Delta'(i, k) \varepsilon dv, N'(i, s) &= k \mid X_u', u \leqq s; X_s' = i] \\ &= P[\Delta'(i, k) \varepsilon dv, N'(i, s) = k \mid X_u', \theta'(i, k) \leqq u \leqq s; X_s' = i] \\ &= \lambda_i e^{-\lambda_i(v-s)} dv = P[\Delta'(i, k) \varepsilon dv, N'(i, s) = k \mid X_s' = i] \end{aligned}$$

with all equalities holding almost surely.

Thus (1) becomes

$$\begin{aligned} &\textstyle\sum_{k=0}^{\infty} \int_t^s P[X'_{t-v} = j \mid X'_{\Delta'(i,k)}]P[\Delta'(i, k) \varepsilon dv, N'(i, s) = k \mid X_s' = i] \\ &\qquad + \delta_{i,j} \textstyle\sum_{k=0}^{\infty} P[\Delta'(i, k) > t, N'(i, s) = k \mid X_s' = i] \end{aligned}$$

which is equal to

$$\textstyle\sum_{k=0}^{\infty} P[X_t' = j, N'(i, s) = k \mid X_s' = i] = P[X_t' = j \mid X_s' = i] \text{ a.s.}$$

This proves Proposition 5.2.

THEOREM 5.1. *$\{X_t', t \geqq 0\}$ is a Markov process.*

In the case that $J = I$ the proof follows from Proposition 5.2. If the semi-

Markov process has instantaneous states then our methods thus far do not suffice. Note that for fixed t, on the set $[X_t = i]$, for an instantaneous state i, $Y_t = 0$ a.s. and the Markov property holds for the X process. This need not carry over to the X' process however since in general $[X_t' = i]$ cannot be described in terms of an optional random variable so that the strong Markov property cannot be applied here. Our Condition A (see Section 2) allows us to handle this case.

Clearly $\mathbf{F}_s \subset \mathbf{F}_t$ for $s \leqq t$ and $P[X_t = j \mid \mathbf{F}_s] = P[X_t = j \mid X_s, Y_s]$ a.s. Since X_t is independent of the Z's and the Markov property of $\{(X_t, Y_t), t \geqq 0\}$ applies, hence with no assumptions required (X_t, Y_t) has the Markov property with respect to the larger family of fields, $\{\mathbf{F}_t, t \geqq 0\}$.

We now have however that $[X_{\tau(s,\cdot)} = i, \tau(s, \cdot) \leqq u] \varepsilon \mathbf{F}_u$ for instantaneous states i and the Condition A asks for the strong Markov property to hold here so that this condition gives us a larger class of Markov times.

We now state the final result needed to complete Theorem 5.1.

PROPOSITION 5.3. *Assume the process satisfies Condition A and let i be an instantaneous state*

$$\Gamma \varepsilon \mathbf{B}\{X_u', u < s\},$$

then

$$P[X'_{t+s} = j, \Gamma \mid X_s' = i] = P[X_t' = j \mid X_0' = i]P[\Gamma \mid X_s' = i].$$

PROOF. Define

$$\begin{aligned} \eta(\omega) &= \tau(s, \omega) \quad \text{if} \quad X_{\tau(s,\omega)}(\omega) = i, \\ &= \inf \{\theta(j, k) > \tau(s, \omega) \quad \text{for} \quad j \varepsilon J \text{ fixed}\} \qquad \text{otherwise.} \end{aligned}$$

$\eta(\omega)$ is an optional random variable with respect to the fields $\mathbf{F}_t$ defined in Condition A since

$$[\eta \leqq t] = [X_{\tau(s,\omega)} = i, \tau(s, \cdot) \leqq t] \cup [X_{\theta(j,k)} = j, \tau(s, \cdot) < \theta(j, k) < t] \varepsilon \mathbf{F}_t.$$

By Condition A the strong Markov property applies and hence

$$P[X'_{t+s} = j, \Gamma \mid X_s' = i] = P[X'_{t+s} = j \mid X_\eta]P[\Gamma \mid X_\eta]$$

and on the set where $X_\eta = i$ this reduces to the statement of the proposition by stationarity.

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