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ON THE BAYES CHARACTER OF A STANDARD MODEL II ANALYSIS OF VARIANCE TEST[1]

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1. Introduction and notations. We consider here the model II analysis of variance problem in which there are two effects and an interaction term. The standard test of whether the first effect is zero uses the ratio of two chi-square statistics, as described below and as may be found in Scheffé [2]. It is the purpose of this paper to show the standard test described here is in fact a Bayes test. In as much as the sufficient statistics of the problem are complete, Bayes tests are admissible.

Below, in this section, we set up notation adequate for the problem. In the next section we explicitly describe *a priori* probability measures on the hypothesis and alternative sets. An easy calculation then produces the desired Bayes test.

Let $X_{ij} = \mu + A_i + B_j + E_{ij}$, $1 \leqq i \leqq I$, $1 \leqq j \leqq J$. μ is the unknown general mean (a constant), $A_1, \cdots, A_I$ are the (random) first effects, $B_1, \cdots, B_J$ the (random) second effects, and $E_{11}, \cdots, E_{IJ}$ the (random) interaction terms. We make the usual assumption that $A_1, \cdots, A_I, B_1, \cdots, B_J, E_{11}, \cdots, E_{IJ}$ are mutually independent normally distributed random variables having zero means. In addition, $\operatorname{var} A_1 = \cdots = \operatorname{var} A_I = \sigma_A^2$, $\operatorname{var} B_1 = \cdots = \operatorname{var} B_J = \sigma_B^2$, and $\operatorname{var} E_{11} = \cdots = \operatorname{var} E_{IJ} = \sigma^2$. We let $X^T = (X_{11}, \cdots, X_{IJ})$ be the transposed random vector of observations.

We will use $IJ \times IJ$ matrices in the sequel. The row indices (i, j) and column indices (i', j') will be integer pairs, $1 \leqq i, i' \leqq I$, $1 \leqq j, j' \leqq J$. We let P_1 be the $IJ \times IJ$ matrix such that if $1 \leqq i, i' \leqq I$ and $1 \leqq j, j' \leqq J$ then the (i, j), (i', j') entry of P_1 is zero if and only if $j \neq j'$; the (i, j), (i', j') entry of P_1 is $1/I$ if and only if $j = j'$. Similarly P_2 is the $IJ \times IJ$ matrix such that if $1 \leqq i, i' \leqq I$, $1 \leqq j, j' \leqq J$, then the (i, j), (i', j') entry of P_2 is zero if and only if $i \neq i'$, the (i, j), (i', j') entry of P_2 is $1/J$ if and only if $i = i'$. It is easily verified that P_1 and P_2 are symmetric matrices, that $P_1^2 = P_1$, $P_2^2 = P_2$, and that $P_1P_2 = P_2P_1$ is the $IJ \times IJ$ matrix having every entry equal to $1/(IJ)$. Write Id for the $IJ \times IJ$ identity matrix.

A routine calculation shows that

$$
\begin{aligned}
\Sigma = E(X - \mu)(X - \mu)^T &= (J\sigma_A^2 + I\sigma_B^2 + \sigma^2)P_1P_2 \\
&+ (I\sigma_B^2 + \sigma^2)P_1(Id - P_2) + (J\sigma_A^2 + \sigma^2)(Id - P_1)P_2 \\
&+ \sigma^2(Id - P_1)(Id - P_2).
\end{aligned} \tag{1.1}
$$

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From this it follows that the exponent of the joint normal density function of X is

$$\begin{aligned}(-\tfrac{1}{2})\quad \text{trace}\quad &[(x-\mu)(x-\mu)^T((J\sigma_A^2+I\sigma_B^2+\sigma^2)^{-1}P_1P_2 \\ &+(I\sigma_B^2+\sigma^2)^{-1}P_1(Id-P_2)+(J\sigma_A^2+\sigma^2)^{-1}(Id-P_1)P_2 \\ &+\sigma^{-2}(Id-P_1)(Id-P_2))] \\ =\ &(-\tfrac{1}{2})[(\bar{x}-\mu)^2(J\sigma_A^2+I\sigma_B^2+\sigma^2)^{-1}+y_1(I\sigma_B^2+\sigma^2)^{-1} \\ &+y_2(J\sigma_A^2+\sigma^2)^{-1}+y_3\sigma^{-2}],\end{aligned} \tag{1.2}$$

where the corresponding sufficient statistics are

$$\begin{aligned}\bar{X} &= \textstyle\sum_{i=1}^{I}\sum_{j=1}^{J}X_{ij}/(IJ); \qquad & Y_2 &= X^T(Id-P_1)P_2X; \\ Y_1 &= X^TP_1(Id-P_2)X; & Y_3 &= X^T(Id-P_1)(Id-P_2)X.\end{aligned} \tag{1.3}$$

In the sequel we need to know that Y_2 and Y_3 are independent random variables. This may be seen as follows.

$$E(Id-P_1)(Id-P_2)X=0 \quad\text{and}\quad EP_2(Id-P_1)X=0.$$

Also the covariance matrix $E(Id-P_1)P_2X\ X^T(Id-P_1)(Id-P_2) = (Id-P_1)P_2\Sigma(Id-P_1)(Id-P_2)=0$ since all matrices involved commute.

The classical testing problem is to test the hypothesis $\sigma_A=0$ against the alternative $\sigma_A>0$. The standard test of this problem uses the statistic

$$Y=Y_2/Y_3\,. \tag{1.4}$$

It is clear that $\sigma^2(J-1)Y_2/(J\sigma_A^2+\sigma^2)Y_3$ has a central F distribution and the family of F density functions with scale parameters has monotone likelihood ratios, so that within the class of tests based on Y there is a UMP size α test. We wish to show these UMP tests to be Bayes tests within the class of all tests based on the sufficient statistic $(\bar{X}, Y_1, Y_2, Y_3)$.

2. Description of the *a priori* measures. In order to write down the *a priori* measures we need the constant that multiplies the exponential part of the density of X. This constant is:

$$\begin{aligned}\{(2\pi)^{-IJ}(J\sigma_A^2+I\sigma_B^2+\sigma^2)^{-1}(J\sigma_A^2+\sigma^2)^{-(I-1)} \\ \cdot(I\sigma_B^2+\sigma^2)^{-(J-1)}\sigma^{-2(I-1)(J-1)}\}^{\frac{1}{2}}.\end{aligned} \tag{2.1}$$

We consider first the alternative set. We restrict the parameters σ_A, σ_B and σ to be in that subset of the alternative such that $1=(I\sigma_B^2+\sigma^2)^{-1}$ and such that the following equations are solvable for real numbers z_1 and z_2.

$$(1+z_1^2)=(J\sigma_A^2+\sigma^2)^{-1}; \qquad 1+z_1^2+z_2^2=\sigma^{-2}. \tag{2.2}$$

These equations clearly imply that

$$\sigma^2\leqq 1; \qquad J\sigma_A^2\leqq 1; \qquad \text{and} \quad I\sigma_B^2\leqq 1. \tag{2.3}$$

We follow the method developed by Kiefer and Schwartz [1]. Conditional on the values of z_1 and z_2 we let μ be a normal $(0, 3 - J\sigma_A^2 - I\sigma_B^2 - \sigma^2)$ random variable. We let z_1, z_2 have the density function

$$\text{constant} \quad (1 + z_1^2)^{-(I-1)/2}(1 + z_1^2 + z_2^2)^{-(I-1)(J-1)/2}. \tag{2.4}$$

Thus the joint density of μ, z_1, z_2 is integrable if and only if

$$(I - 1)(J - 1) \geqq 2. \tag{2.5}$$

Multiplication of the density functions of X and of (μ, z_1, z_2) together and integration gives the value of the integral to be

$$\text{constant} \quad [\exp((-\tfrac{1}{2})(\bar{x}^2/3 + y_1 + y_2 + y_3))][(y_2 + y_3)y_3]^{-\frac{1}{2}}. \tag{2.6}$$

On the hypothesis set we restrict the parameters σ_A, σ_B and σ to satisfy $\sigma_A = 0$, $(I\sigma_B^2 + \sigma^2) = 1$, and $\sigma^2 < 1$. We consider all possible real number pairs (z_1, z_2) such that $1 + z_1^2 + z_2^2 = \sigma^{-2}$. Conditional on z_1 and z_2 we let μ be a normal $(0, 3 - J\sigma_A^2 - I\sigma_B^2 - \sigma^2)$ random variable, and assign to z_1, z_2 the density function

$$\text{constant} \quad (1 + z_1^2 + z_2^2)^{-(I-1)J/2}. \tag{2.7}$$

This is integrable if and only if

$$(I - 1)J \geqq 3. \tag{2.8}$$

Multiplying the joint density functions of X and (μ_1, z_1, z_2) together and integrating gives the answer

$$\text{constant}\,[\exp((-\tfrac{1}{2})(\bar{x}^2/3 + y_1 + y_2 + y_3))][y_2 + y_3]^{-1}. \tag{2.9}$$

Taking the ratio of the quantities in (2.6) and (2.9) gives for the Bayes test statistic

$$(Y_2 + Y_3)/Y_3 = 1 + (Y_2/Y_3). \tag{2.10}$$

Therefore, by considering all possible relative weighings of the *a priori* probability measures on the hypothesis and alternative sets, we see that the following theorem has been proven.

THEOREM. *Suppose* $(I - 1)(J - 1) \geqq 2$. *Within the class of all tests based on the sufficient statistic the class of tests having acceptance regions*

$$\{x \mid x^T(Id - P_1)P_2\, x < c\, x^T(Id - P_1)(Id - P_2)x\}, \qquad 0 \leqq c < \infty,$$

are Bayes tests and hence are admissible tests.

NOTE. $(I - 1)(J - 1) \geqq 2$ implies $I - 1 \geqq 1$. Therefore $(I - 1)J \geqq 2 + (I - 1) \geqq 3$, which is (2.8).

3. More complex designs. The next level of generality discussed by Scheffé [2], Section 7.4, allows several observations on each level. The random variable formulation becomes

$$\text{if} \quad 1 \leqq i \leqq I, \; 1 \leqq j \leqq J, \; 1 \leqq k \leqq K \quad \text{then} \tag{3.1}$$

$$X_{ijk} = \mu + A_i + B_j + C_{ij} + E_{ijk}.$$

Notationally we now need to consider $IJK \times IJK$ matrices. The calculations are aided if we introduce $IJK \times IJK$ orthogonal projections P_1, P_2 and P_3 defined in a manner similar to those of Section 1 such that P_1 averages over the first index position, P_2 averages over the second index position, and P_3 averages over the third index position.

Let $A_1, \cdots, A_J, B_1, \cdots, B_J, C_{11}, \cdots, C_{IJ}, E_{111}, \cdots, E_{IJK}$ be mutually independent normally distributed random variables having zero means. The assumption on variances is now

$$\sigma_A^2 = \operatorname{var} A_1 = \cdots = \operatorname{var} A_I, \qquad \sigma_B^2 = \operatorname{var} B_1 = \cdots = \operatorname{var} B_J,$$
$$\sigma_C^2 = \operatorname{var} C_{11} = \cdots = \operatorname{var} C_{IJ}, \qquad \sigma^2 = \operatorname{var} E_{111} = \cdots = \operatorname{var} E_{IJK}.$$

With these notations we may write the covariance matrix as

$$\begin{aligned}\Sigma = {} & (JK\sigma_A^2 + IK\sigma_B^2 + K\sigma_C^2 + \sigma^2)P_1P_2P_3 \\ & + (JK\sigma_A^2 + K\sigma_C^2 + \sigma^2)(Id - P_1)P_2P_3 \\ & + (IK\sigma_B^2 + K\sigma_C^2 + \sigma^2)P_1(Id - P_2)P_3 \\ & + (K\sigma_C^2 + \sigma^2)(Id - P_1)(Id - P_2)P_3 \\ & + \sigma^2(Id - P_3).\end{aligned} \tag{3.2}$$

The stochastic independence of

$$Y_2 = X^T(Id - P_1)P_2P_3X \quad \text{and} \quad Y_3 = X^T(Id - P_1)(Id - P_2)P_3X$$

follows from the fact that P_1, P_2 and P_3 commute so that

$$\begin{aligned}&E(Id - P_1)P_2P_3X\, X^T(Id - P_1)(Id - P_2)P_3 \\ &\qquad = (Id - P_1)P_2P_3\Sigma(Id - P_1)(Id - P_2)P_3 = 0.\end{aligned}$$

Therefore if σ_A is zero then $(I - 1)(J - 1)Y_1/(I - 1)Y_3$ has a central F distribution. By reconsidering the calculations of Section 2 it is easily seen that UMP tests based on Y_2/Y_3 are Bayes tests in which the prior measures are supported on the subsets of the hypothesis and alternative sets such that $\sigma = 0$.

The tests which have been discussed here are similar tests. In the complete n-way random effects layout, $n \geqq 3$, we do not know of good similar tests based on the sufficient statistic of the hypothesis that the variance of a specific effect is zero.

REFERENCES

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[2] Scheffé, H. (1959). *The Analysis of Variance.* Wiley, New York.