The Annals of Mathematical Statistics
1969, Vol. 40, No. 6, 2178–2179

NOTES

ON EXTENDED RATE OF CONVERGENCE RESULTS FOR THE INVARIANCE PRINCIPLE

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Let $\{X_i, i = 1, 2, 3, \cdots\}$ be a sequence of independent and identically distributed random variables for which $EX_i = 0$, $\text{var } X_i = \sigma^2$, $E|X_i|^{2+a} = b < \infty$, $a > 0$. $C[0, 1]$ is the space of continuous real-valued functions on the interval $[0, 1]$ with the sup norm topology. We define a "random broken line" as follows:

$$\xi_n(t) = \sum_{i=1}^{k} X_i/(\sigma n^{\frac{1}{2}}) \qquad \text{for } t = n^{-1}k$$

$$\xi_n(t) = \xi_n(n^{-1}k) + n[\xi_n(n^{-1}(k+1)) - \xi_n(n^{-1}k)](t - n^{-1}k)$$
$$\text{for } n^{-1}k \leqq t \leqq n^{-1}(k+1),\ 1 \leqq k \leqq n.$$

THEOREM. *Let g be any uniformly continuous functional, $g: C[0, 1] \to$ real line, such that there exists a constant $L > 0$ with the property that*

$$|\Pr\{g(\xi(t)) \leqq x + h\} - \Pr\{g(\xi(t)) \leqq x\}| \leqq L|h|$$

where $\xi(t)$ is the standard Wiener process. Then, there exists a constant $A > 0$ such that for all $n > 1$,

$$(1) \qquad |\Pr\{g(\xi_n(t)) \leqq x\} - \Pr\{g(\xi(t)) \leqq x\}| \leqq A(\log n)^{\lambda} n^{-\mu},$$

where $\lambda = (1 + \frac{1}{2}a)/(a+3)(<\frac{1}{2})$, $\mu = \min(a, 1 + \frac{1}{2}a)/2(a+3)$. This result extends that of Theorem 5 of Rosenkrantz [3] where the bound (1) is, in essence, given in the form

$$|\Pr\{g(\xi_n(t)) \leqq x\} - \Pr\{g(\xi(t)) \leqq x\}| \leqq A(\log n)^{\frac{1}{2}} n^{-\mu}$$

subject to the restriction that $a \leqq 2$. Only minor modifications of the work of Rosenkrantz and necessary to furnish this extension; we shall just indicate these and refer the reader to [3] for full details of the proof.

The condition $a \leqq 2$ was imposed by Rosenkrantz as a consequence of employing a moment inequality of von Bahr and Esseen [1] which is valid for exponent r, $1 \leqq r \leqq 2$ (note that $2r = 2 + a$). This is used to obtain the vital inequality (25) of [3]. For the case $a > 2$ ($r > 2$), we make use of a result of Dharmadhikari, Fabian and Jogdeo [2] which gives in place of (25),

$$(2) \qquad E|z_{nn}|^r \leqq C_r n^{\frac{1}{2}} \nu_n,$$

where $C_r = [8(r-1) \max(1, 2^{r-3})]^r$.
Thus, combining (2) with (25) of [3], we can replace the estimate (27) of [3]

Received 14 February 1969.

by the general estimate

$$\Pr\left(\max_{1 \leqq k \leqq n} |z_{nk}| > \delta\right) \leqq \max(2, C_r) n^{-a(r)} \delta^{-r} \tag{3}$$

where $a(r) = \min(r - 1, \frac{1}{2}r)$, $r > 1$.
The details of proof of [3] are then followed exactly up to the stage of choosing ϵ_n and δ_n tending to zero at appropriate rates. Instead of the versions chosen in (37), we employ

$$\delta_n = n^{-2a(r)(2r+1)^{-1}} (\beta \log n)^{-(2r+1)^{-1}}, \qquad \epsilon_n^2 = 2\beta\delta_n \log n,$$

where $\beta > 1 + a(r)/(2r + 1)$. The result of the theorem then follows as with [3].

REFERENCES

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