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SOME CONDITIONS UNDER WHICH TWO RANDOM VARIABLES ARE EQUAL ALMOST SURELY AND A SIMPLE PROOF OF A THEOREM OF CHUNG AND FUCHS[1]

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Let (X, Y) be an ordered pair of real-valued random variables. Say that (X, Y) is *fair* if $E(Y \mid X) = X$ a.s.

It is shown, for example, that if X has a finite mean and the pair (X, Y) is fair, then X and Y cannot be stochastically ordered unless $X = Y$ a.s. The conclusion is in general false, if X does not have a mean. On the other hand, if X is independent of the increment $Y-X$, the preceding statement remains in force without any moment restrictions on X.

The last assertion, combined with a gambling idea of Dubins and Savage, yields a simple proof of a theorem of Chung and Fuchs on the upper limit of a random walk with mean zero.

0. Introduction and summary. Let (X, Y) be an ordered pair of real-valued random variables. We define (X, Y) to be *subfair* if $E(Y \mid X) \leqq X$ almost surely, *superfair* if $E(Y \mid X) \geqq X$ almost surely, and *fair* if it is both subfair and superfair. Here and in the sequel, $E(Y \mid X)$, denotes the mean of (a regular version of) the conditional distribution of Y given X which, as well known, may exist even if Y has no mean.

The following result is proved on page 314 of Doob's book (1953): If both (X, Y) and (Y, X) are fair then $X = Y$ almost surely, provided $E|X| < \infty$. Theorem 1 shows that the same conclusion holds under the weaker hypothesis that the pairs (X, Y) and (Y, X) are both superfair with $EX^+ < \infty$ or subfair with $EX^- < \infty$.

If $P[X > u] \geqq P[Y > u]$ for all real numbers u, we say that X is *stochastically larger* than Y and denote this relation by $X \succ Y$. If either $X \succ Y$ or $Y \succ X$, the pair (X, Y) is said to be *stochastically ordered.* Theorem 1 also provides that $X = Y$ almost surely, if the pair (X, Y) is superfair (subfair) with $EX^+ < \infty$ and $X \succ Y (EX^- < \infty$ and $Y \succ X)$. When $EX^+ = \infty = EX^-$, neither of the assertions of Theorem 1 is valid, even if X and Y have the same distribution and (Y, X) as well as (X, Y) is fair (Example 3). Thus, in general, the conclusion of Theorem 1 is false when X does not have a mean (at least an infinite one). On the other hand, as shown in Section 2, the integrability condition in Theorem 1 can be replaced by the independence of X and the increment $Y-X$. Thus, for example, if X and

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$Y-X$ are independent, the fair pair (X, Y) cannot be stochastically ordered unless $X = Y$ almost surely (Theorem 2). The treatment of this case is partly based on a theorem of P. Lévy about concentration functions (Théorème 29 on page 91 of Lévy (1954)). There seems to be a gap in Lévy's proof and Lemma 2 was designed to make up for it. Corollary 3 is a slightly stronger version of Lévy's theorem.

Lemma 6, which is another tool in the proof of Theorem 2, is due to L. J. Savage (1970) and is reproduced here with his kind permission. Theorem 2, when coupled with a gambling idea I learned from L. E. Dubins and L. J. Savage, yields a simple proof of a well-known theorem of Chung and Fuchs (1951) concerning the upper limit of a random walk with mean zero. This is presented in Section 3.

Finally, in Section 4, we study the situation when conditional means are replaced by conditional percentiles. This was motivated by Lemma 1.4 of Bickel and Blackwell (1967) which, when vaguely stated, asserts that $X = Y$, provided the conditional median of Y given X equals X, and the conditional median of X given Y equals Y. Theorem 4 makes a similar assertion for all percentiles.

Throughout this paper, with the possible exception of Section 3, P is a probability measure on the Borel sets of the plane; X and Y are the projection maps defined by $X(x, y) = x$ and $Y(x, y) = y$ for every point (x, y) in the plane. P is then the joint distribution of X and Y, while PX^{-1} and PY^{-1} are their respective marginal distributions. E stands for expectation with respect to P, and $E(Y \mid X)$ as well as similar conditional expressions are as explained in the beginning of this introduction. As usual, for any real number x, $x^+ = \max(0, x)$ and $x^- = (-x)^+$.

Equalities and inequalities between random variables are meant in the almost sure sense; thus, for example, we write $X \geqq Y$ for $P[X \geqq Y] = 1$.

1. The semi-martingale case.

LEMMA 1. *Let X be a random variable with $EX^+ < \infty$. Then the distribution of X is completely determined by the values of $E(X-u)^+$, u ranging over all real numbers.*

PROOF. Let F be the distribution function of X. Using integration by parts it is not hard to show that

$$E(X-u)^+ = \int_u^\infty (1-F(x))dx.$$

Differentiation with respect to u recovers F.

THEOREM 1. *Suppose $EX^+ < \infty$ $(EX^- < \infty)$. Then the following conditions are equivalent:*

(a) *Both (X, Y) and (Y, X) are superfair (subfair).*

(b) *(X, Y) is superfair (subfair) and $X \succ Y$ $(Y \succ X)$.*

(c) *$X = Y$.*

In particular, a fair pair (X, Y) with $E|X| < \infty$ cannot be stochastically ordered unless $X = Y$.

PROOF. For each real number u the function $x \to (x-u)^+$ is convex and non-decreasing. Hence Jensen's inequality (Doob (1953)) applies to conclude from (a) that $EY^+ < \infty$ (since $EX^+ < \infty$), and that for each real u the pairs $((X-u)^+, (Y-u)^+)$ and $((Y-u)^+, (X-u)^+)$ are both superfair. Consequently,

$$E(X-u)^+ = E(Y-u)^+ < \infty \qquad \text{for all } u, \tag{1}$$

and by Lemma 1, X and Y have the same distribution, so (b) follows from (a). On the other hand, assuming (b), one has for each u, $E(X-u)^+ \leqq E(Y-u)^+$ by superfairness of (X, Y), and the reverse inequality by $X \succ Y$. Thus (b) implies (1). The pairs $((X-u)^+, (Y-u)^+)$ being all superfair, (1) is possible only if they are in fact all fair. Using the defining property of conditional expectations one gets:

$$\int_{[X \leqq u]}(Y-u)^+ dP = \int_{[X \leqq u]}(X-u)^+ dP = 0$$

which implies

$$P[X \leqq u, (Y-u)^+ > 0] = P[X \leqq u,\ Y > u] = 0. \tag{2}$$

Since (2) holds for all u, one concludes that $Y \leqq X$, which implies

$$(Y-u)^+ \leqq (X-u)^+ \qquad \text{for all } u. \tag{3}$$

In view of (1) (since $EX^+ < \infty$), (3) is possible only as an equality. Finally, equality in (3) for every u yields $Y = X$. So (b) implies (c). That (c) implies (a) is self-evident. To prove the dual assertion where $EX^- < \infty$, simply apply the proven part of the theorem to the pair $(-X, -Y)$.

Recall that the definition of semi-martingale entails the appropriate one-sided integrability. The following corollaries of Theorem 1 are thus straightforward.

COROLLARY 1. *If $X_1, X_2, \cdots$ is both a sub (super)-martingale and a reverse sub (super)-martingale, then $X_1 = X_2 = \cdots$ with probability one.*

COROLLARY 2. *If $X_1, X_2, \cdots$ is a sub (super)-martingale with $X_1 \succ X_2 \succ \cdots$ $(X_1 \prec X_2 \prec \cdots)$, then $X_1 = X_2 = \cdots$ with probability one. In particular, if $X_1, X_2, \cdots$ is either a stationary semi-martingale, or a martingale such that for each n, the pair (X_n, X_{n+1}) is stochastically ordered, then $X_1 = X_2 = \cdots$ with probability one.*

The examples below are designed to show that the existence of the mean of X is essential for the validity of Theorem 1. In fact if $EX^+ = +\infty = EX^-$, we can have (b) but not (a) (Example 1); (a) but not (b) (Example 2); and both (a) and (b) but not (c) (Example 3).

EXAMPLE 1. Let α be the discrete probability measure on the real line R defined by:

$$\alpha\{0\} = \tfrac{1}{3};\ \alpha\{2^n\} = 2^{-n}/3 = \alpha\{-2^n\} \qquad \text{for } n = 1, 2, \cdots.$$

Notice that α does not have a mean. For each $x \neq 0$ let Q_x be the two-point measure $Q_x\{0\} = \tfrac{1}{2} = Q_x\{2x\}$. Define Q_0 by $Q_0\{2\} = \tfrac{1}{2} = Q_0\{-2\}$. Put α on the X-axis in the plane and let $Q = \{Q_x : x \in R\}$ be the conditional distribution of Y given X. This determines a (discrete) probability measure P on the plane. Observe that x is the mean of Q_x for every $x \in R$, and therefore the projections

X, Y (in this order) form a fair pair. Moreover, as is easy to check, this P has equal marginals, i.e. $PY^{-1} = PX^{-1} = \alpha$. Nevertheless $P[Y = X] = 0$. Notice that for this P the pair (Y, X) is neither sufair nor superfair.

EXAMPLE 2. For each $\rho > 0$ let φ_ρ be the map

$$(x, y) \to (x, x+\rho(x-y)) \qquad \text{if } 0 \leqq y \leqq x$$

$$(x, y) \to (y+\rho(y-x), y) \qquad \text{if } 0 \leqq x \leqq y.$$

Let $p_0, p_1, p_2, \cdots$ be any sequence of strictly positive numbers, such that $\Sigma p_n = \frac{1}{2}$. Put $\rho_n = p_{n-1}/p_n$. Let $\omega_0 = (1, 0)$ and $\omega_n = \omega_{\rho_n}(\omega_{n-1})$ for $n \geqq 1$. If $\omega = (x, y)$ let $-\omega = (-x, -y)$. Define a (discrete) probability measure P on the plane by: $P\{\omega_n\} = p_n = P\{-\omega_n\}$ for $n \geqq 0$. It is now easy to verify that P enjoys the following property:

(*) For every vertical or horizontal line L in the plane: If $P(L) > 0$ then L contains exactly two points of positive P-mass—one on each side of the diagonal $D = \{(x, y): x = y\}$—whose distances from D are inversely proportional to their probabilities.

Conclude from (*) that the pairs (X, Y) and (Y, X) are both fair. Nevertheless $P(D) = 0$, so that $X \neq Y$ with probability one.

EXAMPLE 3. Note that the P of the previous example does not have equal marginals (e.g. $P\{Y = 0\} = 2p_0 > 0$ while $P\{X = 0\} = 0$). We could of course reflect P with respect to the diagonal to obtain $\hat{P} = P\pi^{-1}$ where π is the flip $(x, y) \to (y, x)$, and then consider $(\frac{1}{2})(P+\hat{P})$ which does have equal marginals. However, in general we may lose the fairness property in the process. We now show that fairness is preserved provided the sequence $\{p_n\}$, used in the above construction of P, is properly selected. To this end, take $p_n = p_0\rho^{-n}$ for $n \geqq 0 (\rho > 0,\ p_0\Sigma\rho^{-n} = \frac{1}{2})$, so that $\rho_n = \rho$ for $n \geqq 1$. In this case

$$\begin{aligned} \omega_n &= (\sigma_n, \sigma_{n-1}) && \text{if } n \text{ is even} \\ &= (\sigma_{n-1}, \sigma_n) && \text{if } n \text{ is odd} \end{aligned}$$

for $n \geqq 0$, where $\sigma_{-1} = 0$ and $\sigma_n = 1+\rho+ \cdots +\rho^n$, $n \geqq 1$. So, the sets

$$\{X(\omega_n) : n \geqq 0\} = \{\sigma_n : n \geqq 0 \text{ even}\}$$

$$\{Y(\omega_n) : n \geqq 0\} = \{\sigma_n : n \geqq -1 \text{ odd}\}$$

are disjoint. Thus if P and $\hat{P}$ are as above, then $P(L) > 0 \Rightarrow \hat{P}(L) = 0$ and $\hat{P}(L) > 0 \Rightarrow P(L) = 0$ for all vertical and horizontal lines L. The measure $(\frac{1}{2})(P+\hat{P})$ thus satisfies (*). Consequently it makes both (X, Y) and (Y, X) fair, in addition to having equal marginals.

2. The independent case. If, for $u \geqq 0$, $C_X(u) = \sup_x P[x \leqq X \leqq x+u]$, C_X is the *concentration function* of the random variable X introduced by Lévy (1954). Here are some well-known and easily verified properties of concentration functions:

(4i) if X and Y–X are independent, then $C_Y \leqq C_X$;

(4ii) C_X is nondecreasing and right-continuous;

(4iii) For each $u \geqq 0$ $C_X(u)$ is attained.

Say that C_X is *uniquely attained* at u, or simply that $C_X(u)$ is uniquely attained, if $C_X(u) = P[x \leqq X \leqq x+u] = P[y \leqq X \leqq y+u]$ implies $x = y$.

LEMMA 2. *For any random variable X there is at least one $u \geqq 0$ at which C_X is uniquely attained.* (There may be only one such u, as for example when X has a symmetric U-shaped density on a bounded interval.)

PROOF. Let $u \geqq 0$. If $C_X(u)$ is not uniquely attained, then there are a and b with $b-a = d > 0$, such that

$$C_X(u) = P[a \leqq X \leqq a+u] = P[b \leqq X \leqq b+u].$$

Under these circumstances P. Lévy (1954) shows (page 91) that

$$C_X(u+d) \geqq 2C_X(u) - C_X(u-d)$$

(Note: If $C_X(u) \geqq \frac{1}{2}$ then $u-d \geqq 0$; when $v < 0$ put $C_X(v) = 0$.)
Thus every u for which $C_X(u)$ is not uniquely attained satisfies

$$2C_X(u) \leqq C_X(u-d)+C_X(u+d). \tag{5}$$

The geometric meaning of (5) (when $u-d \geqq 0$) is that the point $(u, C_X(u))$ on the graph of C_X lies on or below the midpoint of some chord. We now construct a u with $C_X(u) \geqq \frac{1}{2}$ (which implies $u-d \geqq 0$) for which (5) is violated. If $C_X(0) = 1$ then $C_X \equiv 1$ (i.e. X is degenerate) and $u = 0$ violates (5). If $C_X(0) < 1$ then, since C_X is right-continuous (4ii), it is bounded away from one in some neighborhood of zero. Since in addition C_X is bounded, there is a line $L(u) = \alpha u+\beta$, with $\alpha > 0$ and $\frac{1}{2} \leqq \beta < 1$, such that $C_X(u) \leqq L(u)$ for all $u \geqq 0$. Consider such a line L and rotate it clockwise until it hits the graph of C_X for the first time. Denote by L_0 the final position of L. Then $L_0 \geqq C_X$. Furthermore, by boundedness and right-continuity of C_X, there exists a u^* such that: $C_X(u) = L_0(u^*)$ and $C_X(u) < L_0(u)$ for all $u > u^*$. It is now easy to argue that u^* cannot satisfy (5). Thus C_X is uniquely attained at u^*.

COROLLARY 3. *Suppose X and $Y-X$ are independent. If $C_Y(u) = C_X(u)$ for some $u \geqq 0$ at which C_X is uniquely attained, then $Y = X+k$ for some constant k.* In particular (Théorème 29 of P. Lévy (1954): *If $C_Y(u) = C_X(u)$ for all $u \geqq 0$, then $Y-X$ is a constant.*

PROOF. Let $u \geqq 0$ be such that $C_X(u)$ is uniquely attained and equals $C_Y(u)$. By (4iii) pick $y = y(u)$ so that

$$C_Y(u) = P[y \leqq Y \leqq y+u] \tag{6}$$

which, by independence of X and $Y-X$, $= \int P[y-t \leqq X \leqq y-t+u]\theta(dt) = C_X(u)$, where θ is the distribution of $Y-X$.

Since the integrand in (6) is at most $C_X(u)$, one concludes that the set $T = \{t : P[y-t \leqq X \leqq y-t+u] = C_X(u)\}$ has θ-measure one. But since C_X is uniquely attained at u, T consists of a single point and hence θ must be degenerate.

These facts, implicit in Lévy's book (1954), are now straightforward from (4i) and Corollary 3.

LEMMA 3. *If X and Y are each independent of their difference $Y-X$, then $Y-X$ is a constant.*

LEMMA 4. *If X and the increment $Y-X$ are independent, then X and Y do not have the same distribution unless they are equal.*
Lemmas 3 and 4 have very short proofs using characteristic functions. It seems, however, that the proof given here is more revealing.

LEMMA 5. *Suppose X and $Y-X$ are independent. If $Y-X$ is symmetric (around zero), then the pair (X, Y) is not stochastically ordered, unless $X = Y$.*

PROOF. Let F and H be the distribution functions of X and $Y-X$. The convolution F^*H is then the distribution function of Y and the lemma amounts to saying that if H is symmetric but not concentrated at zero, then the difference $\Delta = F-F^*H$ assumes both positive and negative values. If $\Delta \geqq 0$ or $\Delta \leqq 0$ then so is $\Delta' = \Delta^*F'$, where F' is the distribution function of $-X$. So, we may assume without loss that not only H but also F is symmetric, in which case Δ is the difference of two symmetric distribution functions. Δ is then easily verified to satisfy: $\Delta(-x) = -\Delta(x)$ for all x at which F is continuous. Thus, unless Δ is identically zero it must change sign. Since, however, $\Delta \equiv 0$ is ruled out by Lemma 4, the proof is complete.

LEMMA 6. (L. J. Savage (1970). *If G and H are distribution functions with finite means g and h, then*

$$\int [(G^*F)(x)-(H^*F)(x)]dx = h-g \tag{7}$$

whatever be the distribution function F.

PROOF. In the case in which G and F are concentrated at zero, (7) reduces to the well-known formula

$$-\int_{-\infty}^{0} H(x)dx+\int_{0}^{\infty} (1-H(x))dx = h \tag{8}$$

(see for example page 149 of Feller (1966)).
Replacing H by G in (8) and differencing, yields

$$\int [G(x)-H(x)]dx = h-g \tag{9}$$

which proves the case in which F, but not necessarily G, is concentrated at 0. The general case now follows by changing the order of integration in (7) and then applying (9) to the inner integral.

REMARK. The preceding Lemma was discovered by Savage in connection with the proof of the next theorem which was independently conjectured by the author.

THEOREM 2. *If (X, Y) is a fair pair with X and $Y-X$ independent, then (X, Y) is not stochastically ordered unless $X = Y$.* (When $E|X| < \infty$, Theorem 2 becomes a special case of Theorem 1.)

PROOF. Let F, H and $\Delta = F-F*H$ be as in the proof of Lemma 5. Theorem 2 amounts to saying that if H has mean zero (fairness) but is not concentrated at zero, then Δ assumes both positive and negative values. Indeed, by (7) (with G concentrated at zero) $\int \Delta(x)dx = 0$, and since $\Delta \equiv 0$ is ruled out by Lemma 4, Δ must change sign.

3. A simple proof of a theorem of Chung and Fuchs. Let $X_1, X_2, \cdots$ be a sequence of independent random variables with the common distribution F, and let $S_n = X_1 + \cdots + X_n$ for $n \geqq 1$. For each $s < 0$, define $U(s)$ as the probability that $s+S_n \geqq 0$ for some $n \geqq 1$. Extend the definition of U by $U(s) = 1$ for all $s \geqq 0$. Then:

$$\int U(s+x)dF(x) \leqq U(s) \qquad \text{for all } s. \tag{10}$$

In fact, (10) can be regarded as a very special case of Theorem 2.14.1 (page 32) in Dubins and Savage (1965). Being, however, such a narrow case of a very general theorem, it merits a direct proof. For $s \geqq 0$ there is nothing to prove, whereas for $s < 0$, using Fubini's theorem, the left-hand side of (10) can be interpreted as the probability of the event $\{s+S_n \geqq 0$ for some $n \geqq 2\}$, which is obviously a sub-event of $\{s+S_n \geqq 0$ for some $n \geqq 1\}$ whose probability is $U(s)$.

THEOREM 3 (Chung and Fuchs (1951)). *If F has mean zero but is not concentrated at zero, then* $\limsup_{n\to\infty} S_n = +\infty$ *with probability one.* (When F has a positive mean the same conclusion follows trivially from the law of large numbers.)

PROOF. It is not hard to argue, as for example on page 188 of Dubins and Savage (1965), that the conclusion of Theorem 3 is equivalent to U being identically one. In any event, since U is nondecreasing, $\lambda = \lim_{s\to-\infty} U(s)$ exists. If $\lambda > 0$, then $\limsup_{n\to\infty} S_n = +\infty$ with positive probability, and in view of the Hewitt–Savage (1955) zero–one law, this probability must be one, so that $U(s) = 1$ for all s. Thus, if for some s, $U(s) < 1$, then $\lambda = 0$, in which case U is a distribution function (except perhaps for necessary modifications at discontinuity points). But then to say that U satisfies (10) contradicts Theorem 2 (with U as the distribution of X and F as that of $X-Y$).

THEOREM 3. *The same as Theorem* 3 *except that F is assumed to be symmetric instead of having mean zero.*

PROOF. In the proof of Theorem 3, replace "Theorem 2" by "Lemma 5" to get the final contradiction.

4. Conditional percentiles. Let R be the real line and $\mathscr{B}$ the sigma algebra of linear Borel sets. Let Q be a regular conditional distribution of Y given X, i.e. Q maps

$R \times \mathscr{B}$ into [0, 1] in such a way that for each $x \in R$, Q_x is a probability measure on $\mathscr{B}$; for each $B \in \mathscr{B}$, $Q_X(B)$ is a Borel function of x, and Q is related to P by the requirement:

$$Ef(X, Y) = \int [\int f(x, y) Q_x(dy)] PX^{-1}(dx)$$

for every bounded Borel function f on the plane. Let π be the map $(x, y) \to (y, x)$ and let $\hat{P} = P\pi^{-1}$. Let $\hat{Q}$ be related to $\hat{P}$ as Q is to P, i.e. $\hat{Q}$ is a regular conditional distribution of X given Y.

THEOREM 4. *If for some ρ, $0 \leqq \rho \leqq 1$,*

(i) $Q_x(-\infty, x) \leqq \rho Q_x(x, \infty)$ *and*

(ii) $\hat{Q}_y(-\infty, y) \leqq \rho Q_y(y, \infty)$

hold for P-almost all (x, y); or if the reverse inequalities are almost surely satisfied for some $\rho \geqq 1$, then $X = Y$ almost surely.

PROOF. For each real number u consider the following four, pairwise disjoint, subsets of the plane:

$$C_1(u) = \{(x, y) : x > u, y < u\}$$
$$W_1(u) = \{(x, y) : x > y > u\}$$
$$W_2(u) = \{(x, y) : y > x > u\}$$
$$C_2(u) = \{(x, y) : x < u, y > u\}.$$

Condition (i) implies

$$\begin{aligned} P\{C_1(u)\} + P\{W_1(u)\} &= P\{C_1(u) \text{ or } W_1(u)\} \\ &= \int_{(u,\infty)} Q_X(-\infty, x) PX^{-1}(dx) \\ &\leqq \rho \int_{(u,\infty)} Q_x(x, \infty) PX^{-1}(dx) = \rho P\{W_2(u)\} \end{aligned} \tag{11}$$

for all u.

Similarly, from Condition (ii) deduce

$$P\{C_2(u)\} + P\{W_2(u)\} \leqq \rho P\{W_1(u)\} \qquad \text{for all } u. \tag{12}$$

Adding (12) to (11) yields

$$P\{C_1(u)\} + P\{C_2(u)\} + P\{W_1(u)\} + P\{W_2(u)\} \leqq \rho(P\{W_1(u)\} + P\{W_2(u)\})$$

which, since $\rho \leqq 1$, is impossible, unless

$$P\{C_1(u)\} = 0 = P\{C_2(u)\} \qquad \text{for all } u.$$

So, $P[X > Y] = P[(X, Y) \in C_1(u)$ for some rational $u] = 0$, and likewise $P[Y > X] = 0$. The proof of the assertion for $\rho \geqq 1$ is similar and is hence omitted.

REMARKS. (a) The p-percentile $(0 \leqq p \leqq 1)$ of a probability measure θ on the real line is customarily defined as any number x for which $\theta(-\infty, x) \leqq p$ and $\theta(x, \infty) \leqq 1-p$. Thus for example when $1-\theta\{0\} = \frac{1}{3} = \theta\{1\}$, the point $x = 0$

qualifies as a median ($p = \frac{1}{2}$) of θ. If we exclude such irregularities (which can occur only if θ has atoms) and admit as a p-percentile of θ only those x for which $(1-p)\theta(-\infty, x) = p\theta(x. \infty)$, then Theorem 3 can be stated in terms of the $p = \rho/(1+\rho)$—conditional percentiles. It is easy to give examples where Theorem 3 would fail, were it stated in terms of the customary wide sense definition of percentiles.

(b) Example 1 in Section 1 shows that Condition (ii) (or (i)) cannot in general be replaced by $PX^{-1} = PY^{-1}$. However, in view of Theorem 1, we suspect but cannot prove that when X is integrable, equality of marginals together with (i) (or(ii)) alone are enough to guarantee $X = Y$.

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