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AN ASYMPTOTIC EXPANSION FOR THE DISTRIBUTION OF THE EIGENVALUES OF A 3 BY 3 WISHART MATRIX[1]

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A parametrization of the rotation group $O^+(p)$ of p by p orthogonal matrices with determinant $+1$ in terms of their skew symmetric parts is used to derive, for $p = 3$, an explicit expansion for ${}_0F_0^{(p)}(Z, \Omega)$, a hypergeometric function of two matrix arguments appearing in the distribution of the eigenvalues of a p by p Wishart matrix. On the basis of a numerically derived simplification of the low order terms of this series, an asymptotic expansion of ${}_0F_0^{(3)}$ in terms of products of ordinary confluent hypergeometric series is conjectured. Limited numerical exploration indicates the new series to be several orders of magnitude more accurate than the series from which it was derived.

0. Summary. Anderson (1965) derived an expression to order n^{-2} for a hypergeometric function of two matrix arguments appearing in the distribution of the eigenvalues of a p by p Wishart matrix [James (1964)] when $p = 3$ and 4, and the complete series for $p = 2$. A novel parametrization of the group of rotation matrices in terms of their skew symmetric parts is used to derive an explicit expression for the complete series when $p = 3$. Numerical methods are used to simplify the series to usable form through terms in n^{-8}. Examination of these terms leads to a conjectured representation of the function as a sum of products of confluent hypergeometric functions. Limited numerical experimentation suggests that this latter representation provides a much better approximation than any truncation of the asymptotic series. The first term, expressible as a product of Bessel functions, seems to be sufficiently accurate for many applications.

1. Introduction. Let W be a p by p random symmetric matrix with the central Wishart distribution $W(n, \Sigma)$, where Σ is positive definite (see Rao 1965). We may assume without loss of generality that $\Sigma = \operatorname{diag}[\lambda_1, \lambda_2, \cdots, \lambda_p]$, $\lambda_{i+1} \leqq \lambda_i$, since we are concerned only with the eigenvalues of W. As is well known, W can be expressed as

$$W = HSH^T \tag{1.1}$$

where $H \in O^+(p)$, the group of p by p rotation matrices (orthogonal with determinant $= +1$), and $S = \operatorname{diag}[s_1, s_2, \cdots, s_p]$, $s_{i+1} \leqq s_i$, is the diagonal matrix of the eigenvalues of W. The joint distribution of $s_1, \cdots, s_p$ is [James (1964), Anderson (1965)]

$$dF(s_1, \cdots, s_p) = \text{const}(\det S)^{(n-p-1)/2}(\det \Sigma)^{-n/2} \prod_{i>j} (s_j - s_i) \times {}_0F_0^{(p)}(\Sigma^{-1}, -(n/2)S)\, ds_1 \cdots ds_p \tag{1.2}$$

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where

(1.3) $${}_0F_0^{(p)}(Z, \Omega) = (2/V_p) \int_{O^+(p)} \exp(\operatorname{tr} ZH\Omega H^T)(dH)$$

is a generalized hypergeometric function of two p by p matrix arguments [James (1964)]. The differential form (dH) represents invariant Haar measure on $O^+(p)$ having total content $\frac{1}{2}V_p$. James (1964) has given a power series expansion for ${}_0F_0^{(p)}$:

(1.4) $${}_0F_0^{(p)}(Z, \Omega) = \sum_{k=0}^{\infty} [(1/k!) \sum_\kappa C_\kappa(Z)C_\kappa(\Omega)/C_\kappa(I_p)]$$

where the inner summation is over all partitions κ of k into p or fewer parts, C_κ are zonal polynomials, and I_p is the p by p identity matrix. ${}_0F_0^{(p)}$ depends only on the eigenvalues of Z and Ω and hence these matrices may be assumed to have the form $Z = \operatorname{diag}[\zeta_1, \cdots, \zeta_p]$, $\Omega = \operatorname{diag}[\omega_1, \cdots, \omega_p]$, with $\zeta_i \leqq \zeta_{i+1}$ and $\omega_i \leqq \omega_{i+1}$.

Anderson's result is essentially an asymptotic representation of ${}_0F_0^{(p)}$ valid as all

(1.5) $$\Delta_{ij} = (\zeta_i - \zeta_j)(\omega_i - \omega_j)\,, \qquad i \neq j$$

increase without limit. Thus it is applicable to the current problem for large n, provided the eigenvalues of Σ are distinct. Anderson showed that

(1.6) $${}_0F_0^{(p)}(Z, \Omega) = \frac{\prod_{j=1}^{p} \Gamma(j/2)}{\pi^{p/2}} \frac{\exp(\operatorname{tr} Z\Omega)}{(\prod_{i=1}^{N_p} \delta_i)^{\frac{1}{2}}} \cdot F_p\,,$$

where $N_p = p(p-1)/2$, $\delta_1, \delta_2, \cdots, \delta_{N_p}$ is some ordering of Δ_{ij} and $F_p = 1 + o(1)$ as the δ's simultaneously approach $+\infty$. When $p = 2$, it is known [Anderson (1965)] that

(1.7) $${}_0F_0^{(2)}(Z, \Omega) = \exp(\operatorname{tr} Z\Omega)\,{}_1F_1(\tfrac{1}{2}; 1; -\delta_1)$$

where ${}_1F_1$ is a confluent hypergeometric function [Erdelyi (1953), page 248]. We use Kummer's notation

(1.8) $$\begin{aligned} &{}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; z) \\ &\qquad = \sum_{k=0}^{\infty} [\prod_{i=1}^{p} (\alpha_i)_k / \prod_{j=1}^{q} (\beta_j)_k] z^k/k! \end{aligned}$$

where $(\gamma)_0 = 1$, $(\gamma)_k = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+k-1)$. A standard result [Erdelyi (1953), page 278] yields the asymptotic series, equivalent to that given by Anderson,

(1.9) $$F_2 = 1 + \sum_{j=1}^{\infty} [(\tfrac{1}{2})_j^2/j!]\delta_1^{-j} = {}_2F_0(\tfrac{1}{2}, \tfrac{1}{2}; \delta_1^{-1})\,, \qquad \delta_1 \to +\infty\,.$$

For $p = 3$ and 4, Anderson derives for F_p the series through terms in δ^{-2}

(1.10) $$F_p = 1 + (\tfrac{1}{4}) \sum_i (1/\delta_i) + (\tfrac{9}{32}) \sum_i (1/\delta_i^2) + (\tfrac{1}{16}) \sum_{i<j} 1/(\delta_i\delta_j) + \cdots\,.$$

For $p > 4$, Anderson confirms the correctness of the term in δ^{-1} and conjectures that the term in δ^{-2} is also correct. He also conjectures that the remainder in Equation (1.10) is $O(\delta^{-3})$.

A somewhat different approach from that used by Anderson simplifies the algebra for finding successive terms in F_p. When $p = 3$, it allows the derivation of an explicit expression for the term in F_3 of order r in δ^{-1}.

2. Parametrization of a rotation matrix in terms of its skew symmetric part. Let $H = U + V \in O^+(p)$, where $U = \frac{1}{2}(H + H^T)$ and $V = \frac{1}{2}(H - H^T)$ are the symmetric and skew symmetric parts of H, respectively. Then it can be shown [Bingham (1972)] that H can be represented as

$$(2.1) \qquad H = V + \sum_{j=0}^{m'} h_j^{(p)}(\rho) V^{2j}, \qquad m' \equiv [(p-1)/2],$$

where the h's satisfy

$$(2.2) \qquad \begin{aligned} \sum_{j=0}^{m'} (-1)^j h_j^{(p)}(\rho)\rho_k^{2j} &= \varepsilon_k(1-\rho_k^2)^{\frac{1}{2}}, \qquad & k = 1, 2, \cdots, \quad m \equiv [p/2], \\ h_0^{(p)}(\rho) &= 1, & \text{when} \quad p = 2m+1; \end{aligned}$$

$\rho_j = \sin r_j$, $j = 1, \cdots, m$, are the eigenvalues of $i \cdot V$ and the $r_j \in (-\pi, +\pi]$ are the angles of rotation induced by H in its eigen planes. ε_j is defined by $\varepsilon_j(1 - \rho_j^2) = \cos r_j$. The sum in (2.1) is an expression for the appropriate symmetric matrix symbolized by $U = (I_p + V^2)^{\frac{1}{2}}$. In the neighborhood of $I_p \in O^+(p)$ defined by $\{H \mid |r_j| < \pi/2, j = 1, \cdots, m\}$, $\varepsilon_j = +1$, $j = 1, \cdots, m$, and H can be expressed as a convergent power series in V:

$$(2.3) \qquad H = V + \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} V^{2k}.$$

For $p = 2$, $m' = 0$, $m = 1$, and

$$V = \begin{bmatrix} 0 & -\rho_1 \\ \rho_1 & 0 \end{bmatrix}.$$

By (2.2), $h_0^{(2)}(\rho) = \varepsilon_1(1 - \rho_1^2)^{\frac{1}{2}}$ and

$$H = V + \varepsilon_1(1 - \rho_1^2)^{\frac{1}{2}} I_2 = \begin{bmatrix} \cos r_1 & -\sin r_1 \\ \sin r_1 & \cos r_1 \end{bmatrix}.$$

For $p = 3$, $m = 1$,

$$(2.4) \qquad V = \begin{bmatrix} 0 & v_{12} & v_{13} \\ -v_{12} & 0 & v_{23} \\ -v_{13} & -v_{23} & 0 \end{bmatrix} \equiv \begin{bmatrix} 0 & z_3 & -z_2 \\ -z_3 & 0 & z_1 \\ z_2 & -z_1 & 0 \end{bmatrix},$$

and $\rho_1^2 = \sum_{i=1}^{3} z_i^2$. By (2.1) and (2.2),

$$(2.5) \qquad H = V + I_3 + h_1^{(3)}(\rho)V^2$$

where

$$(2.6) \qquad h_1^{(3)}(\rho) = [1 - \varepsilon_1(1 - \rho_1^2)^{\frac{1}{2}}]/\rho_1^2 = [1 + \varepsilon_1(1 - \rho_1^2)^{\frac{1}{2}}]^{-1}.$$

Here $\varepsilon_1 = +1$ is equivalent to $\operatorname{tr} H = 3 - 2\rho_1^2 h_1^{(3)}(\rho) > 1$.

It is shown in Bingham (1972) that invariant (Haar) measure (dH) on $O^+(p)$ can be expressed as a differential form in the elements of V as follows:

$$(2.7) \qquad (dH) = [g_0(\rho)g_1(\rho)/g_2(\rho)] \prod_{i<j}^{p} dv_{ij}$$

where

$$\begin{aligned} g_0(\rho) &= \prod_{i=1}^m (1-\rho_i^2)^{-\frac{1}{2}}, & p &= 2m \\ &= 2^m \prod_{i=1}^m [(1-\varepsilon_i(1-\rho_i^2)^{\frac{1}{2}})(1-\rho_i^2)^{-\frac{1}{2}}], & p &= 2m+1 \\ g_1(\rho) &= 2^{m^2-m} \prod_{i<j}^m [\varepsilon_i(1-\rho_i^2)^{\frac{1}{2}} - \varepsilon_j(1-\rho_j^2)^{\frac{1}{2}}]^2 & & \\ g_2(\rho) &= \prod_{j<k}^m (\rho_j^2-\rho_k^2)^2, & p &= 2m \\ &= \prod_{i=1}^m \rho_i^2 \prod_{j<k}^m (\rho_j^2-\rho_k^2)^2, & p &= 2m+1. \end{aligned}$$

In that neighborhood of I_p in which all $\varepsilon_j = +1$, one can readily express (2.7) as

$$(dH) = \prod_{i=d}^m \prod_{j=1}^m T(\rho_i, \rho_j) \prod_{l<k}^p dv_{lk}, \tag{2.8}$$

where $d = 2m + 1 - p$, $\rho_0 = 0$, and $T(\rho_i, \rho_j) = 2[(1-\rho_i^2)^{\frac{1}{2}} + (1-\rho_j^2)^{\frac{1}{2}}]^{-1}$.

For $p = 2$,

$$V = \begin{bmatrix} 0 & -\rho_1 \\ \rho_1 & 0 \end{bmatrix},$$

$(dH) = T(\rho_1, \rho_1)\, d\rho_1 = (1-\rho_1^2)^{-\frac{1}{2}}\, d\rho_1$. For $p = 3$ and V as in (2.4),

$$(dH) = T(0, \rho_1)T(\rho_1, \rho_1)\, dz_1\, dz_2\, dz_3 = 2h_1^{(3)}(\rho)(1-\rho_1^2)^{-\frac{1}{2}}\, dz_1\, dz_2\, dz_3. \tag{2.9}$$

3. Asymptotic expansion when $p = 3$. The results summarized in Section 2 can be exploited to develop a formal asymptotic series to represent F_p in a manner analogous to that used by Anderson (1965). With about the same degree of algebraic effort the $O(\delta^{-1})$ term in (1.10) can be verified. For arbitrary p computation of higher order terms is still formidable and has not been attempted. However, separating out the skew symmetric part of H does seem to simplify things and in fact permits the development of an explicit form for the term of $O(\delta^{-r})$ when $p = 3$.

Consider the exponent $\operatorname{tr} ZH\Omega H^T$ in the integrand of (1.3). Letting $H = U + V$ as in Section 2, $\operatorname{tr} ZH\Omega H^T = \operatorname{tr} Z(V+U)\Omega(-V+U) = -\operatorname{tr} ZV\Omega V - 2\operatorname{tr} ZU\Omega V + \operatorname{tr} ZU\Omega U$. This can be simplified by the following.

LEMMA. *Let $B = [b_{ij}]$ be a p by p matrix such that $b_{ij}^2 = b_{ji}^2$, all i and j. Then* $\operatorname{tr} ZB\Omega B^T = -\sum_{i<j}^p \Delta_{ij} b_{ij}^2 + \operatorname{tr} Z\Omega BB^T$, *where* $Z = \operatorname{diag}[\zeta_i]$, $\Omega = \operatorname{diag}[\omega_j]$ *and* Δ_{ij} *is as in* (1.5).

PROOF. $\operatorname{tr} ZB\Omega B^T = \sum_{i,j=1}^p \zeta_i \omega_j b_{ij}^2 = \sum_{i<j}^p (\zeta_i\omega_j + \zeta_j\omega_i) b_{ij}^2 + \sum_{i=1}^p \zeta_i\omega_i b_{ii}^2 = -\sum_{i<j}^p \Delta_{ij} b_{ij}^2 + \sum_{i<j}^p (\zeta_i\omega_i + \zeta_j\omega_j) b_{ij}^2 + \sum_{i=1}^p \zeta_i\omega_i b_{ii}^2 = -\sum_{i<j}^p \Delta_{ij} b_{ij}^2 + \sum_{i,j=1}^p \zeta_i\omega_i b_{ij}^2$. □

Since both $U = U^T$ and $V = -V^T$ satisfy the conditions of the lemma $\operatorname{tr} ZH\Omega H^T = -\sum_{i<j}^p \Delta_{ij} v_{ij}^2 - 2\operatorname{tr} ZU\Omega V - \sum_{i<j}^p \Delta_{ij} u_{ij}^2 + \operatorname{tr} Z\Omega(-V^2 + U^2)$. Now $-V^2 + U^2 = HH^T = I_p$ and $\operatorname{tr} ZU\Omega V = \sum_{i,j=1}^p \zeta_i\omega_j u_{ij} v_{ji} = -\sum_{i<j}^p (\zeta_i\omega_j - \zeta_j\omega_i) u_{ij} v_{ij}$. Thus

$$\operatorname{tr} ZH\Omega H^T = \operatorname{tr} Z\Omega - \sum_{i<j}^p \Delta_{ij} v_{ij}^2 + 2\sum_{i<j}^p \Gamma_{ij} u_{ij} v_{ij} - \sum_{i<j}^p \Delta_{ij} u_{ij}^2 \tag{3.1}$$

where

$$\Gamma_{ij} = \zeta_i\omega_j - \zeta_j\omega_i. \tag{3.2}$$

Defining $t_{ij}^{(k)}$ by $V^{2k} = [t_{ij}^{(k)}]$, (2.1) and (2.3) imply

$$u_{ij} = \sum_{k=0}^{m'} h_p^{(p)}(\rho) t_{ij}^{(k)} \tag{3.3}$$

$$= \sum_{k=0}^{\infty} ((-1)^k (-\tfrac{1}{2})_k / k!) t_{ij}^{(k)} . \tag{3.4}$$

Either (3.3) or (3.4) may be used to develop tr $ZH\Omega H^T -$ tr $Z\Omega$ as a power series in the v_{ij}.

When $p = 3$, using z_i, $i = 1, 2, 3$, defined by (2.4) and $h(\rho) \equiv h_1^{(3)}(\rho)$ given in (2.6), (3.3) yields $u_{ij} = h(\rho) z_i z_j$ and hence

$$\begin{aligned} \operatorname{tr} ZH\Omega H^T = \operatorname{tr} Z\Omega - \textstyle\sum_{i=1}^3 \delta_i z_i^2 + 2G(\delta) h(\rho) z_1 z_2 z_3 \\ - h^2(\rho) z_1^2 z_2^2 z_3^2 \textstyle\sum_{i=1}^3 \delta_i z_i^{-2} \end{aligned} \tag{3.5}$$

where

$$G(\delta) = \Gamma_{12} - \Gamma_{13} + \Gamma_{23} \tag{3.6}$$

and

$$\delta_1 = \Delta_{23} , \qquad \delta_2 = \Delta_{13} , \qquad \delta_3 = \Delta_{23} . \tag{3.7}$$

Anderson (1965) gives the identity

$$G^2(\delta) = \sum_{j=1}^3 \delta_j^2 - 2 \sum_{i<j}^3 \delta_i \delta_j . \tag{3.8}$$

Following Anderson (1965), we note that tr $ZH\Omega H^T$ is unchanged by multiplication of H by any of the orthogonal matrices of the form diag$[\pm 1, \pm 1, \pm 1]$ of which the four with an even number of -1's are in $O^+(3)$. Thus we can replace the domain of integration in (1.3) by a region D containing the identity matrix, provided we multiply by a factor of 4. Since $V_3 = 16\pi^2$ [James (1954)], using (2.9) and (3.5),

$$\begin{aligned} {}_0F_0^{(3)}(Z, \Omega) = \tfrac{1}{2}\pi^{-2} \exp(\operatorname{tr} Z\Omega) \textstyle\iiint_D \{\exp(-\sum_{j=1}^3 \delta_j z_j^2) \\ \times \exp[2G(\delta) h(\rho) z_1 z_2 z_3 - h^2(\rho) z_1^2 z_2^2 z_3^2 \textstyle\sum_{j=1}^3 \delta_j z_j^{-2}] \\ \times 2h(\rho)(1 - \rho^2)^{-\frac{1}{2}}\} \, dz_1 \, dz_2 \, dz_3 . \end{aligned} \tag{3.9}$$

The exact boundaries of D need not be specified since the integrand is highly concentrated around the origin ($z_i = 0$, $i = 1, 2, 3$) as $\delta_j \to \infty$. In fact, this implies that asymptotically we can replace D by the smaller neighborhood of I_3, D^*, defined by $H > 1$ (equivalent to specifying $\varepsilon_1 = +1$ and $\rho^2 < 1$ in (2.6)). This is permissible since D^* is disjoint from its images under multiplication by diag$[1, -1, -1]$, diag$[-1, 1, -1]$ and diag$[-1, -1, 1]$. We obtain the required expansion by expanding $\exp[2G(\delta) h(\rho) z_1 z_2 z_3 - h^2(\rho) z_1^2 z_2^2 z_3^2 \sum_{j=1}^3 \delta_j z_j^{-2}] \times [2h(\rho)(1 - \rho^2)^{-\frac{1}{2}}]$ in a power series in the z's, replacing the finite region D^* by all of 3-space (E_3), and integrating term by term. Terms of odd order in the z's make no contribution. Thus

$$\begin{aligned} {}_0F_0^{(3)}(Z, \Omega) = \tfrac{1}{2}\pi^{-2} \exp(\operatorname{tr} Z\Omega) \textstyle\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \dfrac{2^{2j}(G^2(\delta))^j(-1)^k}{(2j)!\, k!} \\ \times \textstyle\iiint_{D^*} \exp(-\sum_{i=1}^3 \delta_i z_i^2)\{P_1^j(z) P_2^k(z) 2h(\rho)^{2j+2k+1} \\ \times (1 - \rho^2)^{-\frac{1}{2}}\} \, dz_1 \, dz_2 \, dz_3 \end{aligned} \tag{3.10}$$

where $P_1(z) = z_1^2z_2^2z_3^2$, $P_2(z) = P_1(z)\sum_{i=1}^3 \delta_i z_i^{-2}$. Define coefficients $\alpha_l^{(r)}$ by

$$2h^{2r+1}(\rho)/(1 - \rho^2)^{\frac{1}{2}} = \sum_{l=0}^\infty \alpha_l^{(r)}\rho^{2l} . \tag{3.11}$$

By (2.6), $2h(\rho)(1-\rho^2)^{-\frac{1}{2}} = \rho^{-2}((1-\rho^2)^{-\frac{1}{2}} - 1) = 2\sum_{k=0}^\infty[(\frac{1}{2})_{k+1}/(k+1)!]\rho^{2k}$ and $(h(\rho))^2 = \rho^{-4}(2 - \rho^2 - 2(1-\rho^2)^{\frac{1}{2}}) = \sum_{k=0}^\infty[(\frac{1}{2})_{k+1}/(k+2)!]\rho^{2k}$. Successive power series multiplications yield the recurrence relations for $\alpha_l^{(r)}$

$$\begin{aligned}\alpha_l^{(0)} &= 2(\tfrac{1}{2})_{l+1}/(l+1)! \\ \alpha_l^{(r)} &= \textstyle\sum_{i=0}^l ((\tfrac{1}{2})_{i+1}/(i+2)!)\alpha_{l-i}^{(r-1)} .\end{aligned} \tag{3.12}$$

Also define

$$\begin{aligned} K(j,k,l;\delta) &\\ = G^{2j}(\delta)(\delta_1\delta_2\delta_3)^{\frac{1}{2}}\pi^{-\frac{3}{2}} &\textstyle\iiint_{E_3} \exp(-\sum_{i=1}^3 \delta_i z_i^2)P_1^j(z)P_2^k(z)\rho^{2l}\, dz_1\, dz_2\, dz_3 .\end{aligned} \tag{3.13}$$

Then, formally, as $\delta_i \to +\infty$, $i = 1, 2, 3$, we have

$${}_0F_0^{(3)}(Z,\Omega) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\exp(\operatorname{tr} Z\Omega)\textstyle\prod_{i=1}^3 \delta_i^{-\frac{1}{2}} \sum_j\sum_k\sum_l \alpha_l^{(j+k)} \dfrac{(-1)^k}{(\frac{1}{2})_j j!\, k!} K(j,k,l;\delta) .$$

We can evaluate $K(j, k, l; \delta)$ as follows:

$$\begin{aligned} &K(j,k,l;\delta) \\ &\quad = G^{2j}(\delta)\pi^{-\frac{3}{2}}(\delta_1\delta_2\delta_3)^{\frac{1}{2}} \textstyle\iiint_{E_3}\exp(-\sum_{i=1}^3\delta_i z_i^2)\{(z_1^2z_2^2z_3^2)^{j+k} \\ &\qquad \times (\textstyle\sum_{i=1}^3 \delta_i z_i^{-2})^k(\sum_{i=1}^3 z_i^2)^l\}\, dz_1\, dz_2\, dz_3 \\ &\quad = G^{2j}(\delta)\textstyle\sum_{k_1+k_2+k_3=k}\sum_{l_1+l_2+l_3=l}\Bigg[\dfrac{k!\, l!}{k_1!\,k_2!\,k_3!\,l_1!\,l_2!\,l_3!}\prod_{i=1}^3 \delta_i^{k_i}(\delta_1\delta_2\delta_3)^{\frac{1}{2}}(\pi^3)^{-\frac{1}{2}} \\ &\qquad \times \textstyle\iiint_{E_3}\{\exp(-\sum_{i=1}^3\delta_i z_i^2)\prod_{i=1}^3 (z_i^2)^{j+k-k_i+l_i}\}\, dz_1\, dz_2\, dz_3\Bigg] .\end{aligned}$$

We evaluate the integral using the standard identity $(\delta/\pi)^{\frac{1}{2}}\int_{-\infty}^{+\infty}\exp(-\delta z^2)z^{2r}\,dz = (\frac{1}{2})_r\delta^{-r}$ obtaining

$$\begin{aligned} K(j,k,l;\delta) = G^{2j}(\delta)(\delta_1\delta_2\delta_3)^{-j-k} &\textstyle\sum_{k_1+k_2+k_3=k}\sum_{l_1+l_2+l_3=l} \\ &\times \{k!\, l! \textstyle\prod_{i=1}^3 [(\frac{1}{2})_{j+k-k_i+l_i}\delta_i^{2k_i-l_i}/(k_i!\, l_i!)]\} .\end{aligned} \tag{3.14}$$

$K(j, k, l; \delta)$ is a symmetric function of δ^{-1} of total degree $-2j + 3(j+k) - 2k + l = j + k + l$ in δ^{-1} since $G^2(\delta)$ is of degree -2 in δ^{-1}. We can thus collect all terms of order r as

$$T_r(\delta) = \textstyle\sum_{j+k+l=r}\alpha_l^{(j+k)}(-1)^k K(j,k,l;\delta)/[(\frac{1}{2})_j j!\, k!] . \tag{3.15}$$

Since $T_0(\delta) = 1$, the desired asymptotic series for ${}_0F_0^{(3)}$ is

$${}_0F_0^{(3)}(Z,\Omega) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\exp(\operatorname{tr} Z\Omega)(\delta_1\delta_2\delta_3)^{-\frac{1}{2}}[1 + \textstyle\sum_{r=1}^\infty T_r(\delta)] \tag{3.16}$$

where T_r is given by (3.15), (3.14) and (3.11).

4. Numerical simplification of low order terms. It is desirable to simplify (3.16). $T_r(\delta)$ is symmetric in the δ's and Anderson's result, Equation (1.10), suggests the conjecture that $T_r(\delta)$ is also a polynomial in δ_i^{-1}, $i = 1, 2, 3$. I have been unable to prove this conjecture. However, assuming its truth, the following numerical method of simplification can be used. For a particular value of r, a sufficiently large sample of triples $(\delta_1, \delta_2, \delta_3)$ is chosen randomly and $Y = T_r(\delta)$ computed using (3.12), (3.14) and (3.15). For the same values of δ a complete set $X_{\kappa_1}, \cdots, X_{\kappa_s}$ of symmetric polynomials of degree r in δ^{-1} is computed, indexed by the partitions κ_j of r into 3 or fewer parts. Such a set is

$$X_{(r_1 r_2 r_3)}(\delta) = \sum_{j_1 \neq j_2 \neq j_3 \neq j_1} \prod_{i=1}^{3} \delta_{j_i}^{-r_i} \tag{4.1}$$

where the sum is over all *distinct* terms. The least squares regression of Y on the X's (omitting, of course, a constant term) can be then computed. Ideally, if the conjecture is true, the fit should be perfect and the "regression" coefficients provide the desired simplification. In practice, rounding error prevents an exact fit and the coefficients are approximate.

This procedure has been carried out for $r \leqq 8$ using double precision arithmetic. The fit was almost perfect $(1 - R^2 \sim 10^{-17})$ confirming the conjecture. In Table 1 are hypothesized rational forms for the desired coefficients. A conjecture yielding these coefficients is given in Section 5. For $r = 1$ and 2 these are known to be correct (Equation (1.10)). For $3 \leqq r \leqq 7$, the calculated coefficients multiplied by the hypothesized denominators yield the hypothesized integer numerators to within 2×10^{-6} absolute error. For $r = 8$, the absolute error in the computed numerators was less than 2.7 (6 out of 10 were accurate to the nearest integer), with maximum relative error of 6.5×10^{-7}. I believe that it can be taken as established that the rational coefficients are those in Table 1. Note that for $1 \leqq r \leqq 8$ the coefficients of $\sum_{i=1}^{3} \delta_i^{-r}$ have the form $(\frac{1}{2})_r{}^2/r!$ as conjectured by Anderson (1965).

5. Approximations and conjectures. It can be verified that when the asymptotic series formally expressed as

$$\begin{aligned} \tilde{F}_3 = {} & \prod_{i=1}^{3} [{}_2F_0(\tfrac{1}{2}, \tfrac{1}{2}; \delta_i^{-1})] - (\tfrac{1}{2}) \prod_{i=1}^{3} [{}_2F_0(\tfrac{3}{2}, \tfrac{3}{2}; \delta_i^{-1})/(2\delta_i)] \\ & + (\tfrac{3}{8}) \prod_{i=1}^{3} [3\,{}_2F_0(\tfrac{5}{2}, \tfrac{5}{2}; \delta_i^{-1})/(4\delta_i^2)] \end{aligned} \tag{5.1}$$

is expanded in terms of the symmetric polynomials X_κ (Equation (4.1)), it has coefficients that are identical through order 8 in δ^{-1} to the hypothesized coefficients for F_3 given in Table 1. By standard formulae [Erdelyi (1953), page 278], asymptotically as $z \to \infty$

$$\pi^{\frac{1}{2}}\,{}_1F_1(j + \tfrac{1}{2}; 1; -z) \cong (-1)^j z^{-\frac{1}{2}}[(\tfrac{1}{2})_j z^{-j}\,{}_2F_0(\tfrac{1}{2} + j, \tfrac{1}{2} + j; z^{-1})]\,. \tag{5.2}$$

Substitution of (5.2) in (5.1), together with bold extrapolation from the form of the constants, yields the conjecture

$${}_0F_0^{(3)}(Z, \Omega) \cong (\tfrac{1}{2}\pi) \exp(\operatorname{tr} Z\Omega) \sum_{j=0}^{\infty} \{[(\tfrac{1}{2})_j/j!] \prod_{i=1}^{3} [{}_1F_1(j + \tfrac{1}{2}; 1; -\delta_i)]\}\,. \tag{5.3}$$

(5.3) yields the correct asymptotic series up to order 8 in δ^{-1}. However, for $Z = \Omega = 0$, ${}_0F_0^{(3)} = 1$, while the right-hand side of (5.3) diverges to infinity. Thus, at best, (5.3) holds in an asymptotic sense.

Limited numerical experimentation indicates that (5.3) is a vast improvement over the direct use of (3.16). Values of ${}_0F_0^{(3)}(Z, \Omega)$ were computed using the direct power series expansion (Equation (1.4)) for $Z = \mathrm{diag}[0, 1, 2]$ and $\Omega = \mathrm{diag}[0, \omega, 2\omega]$ for $\omega = 2.5, 3.0, 3.5$, and 4.0. These were compared with values computed from (3.16) and (5.3), truncated at various points. Table 2 gives the relative error ([approx. − exact]/exact) for various truncations of (3.16), and Table 3 gives relative errors for (5.3). Clearly for these values of Z and Ω, (3.16) can provide only two to three significant figures, while (5.3) yields five or more. The $j = 0$ term alone is probably accurate enough for many purposes. This corresponds to the approximation

$$(5.4) \qquad {}_0F_0^{(3)}(Z, \Omega) = (\tfrac{1}{2}\pi)\exp(\mathrm{tr}\, Z\Omega) \textstyle\prod_{i=1}^{3} [\exp(-\tfrac{1}{2}\delta_i) I_0(\tfrac{1}{2}\delta_i)] ,$$

where we have used ${}_1F_1(\frac{1}{2}; 1; -\delta) = \exp(-\frac{1}{2}\delta)I_0(\frac{1}{2}\delta)$ [Erdelyi (1953), Equation (10), page 265]. Apparently, the confluent hypergeometric functions in (5.3) are summing substantial parts of the asymptotic series in (3.16).

TABLE 1

Coefficients β_κ of the elementary symmetric functions in the expansion $T_r = \sum_\kappa \beta_\kappa X_\kappa(\delta)$ (see Equation (3.16))

r	κ	β_κ
1	(1)	$1/2^2$
2	(2)	$3^2/2^5$
	(1^2)	$1/2^4$
3	(3)	$3\cdot5^2/2^7$
	(21)	$3^2/2^7$
	(1^3)	$-3/2^6$
4	(4)	$3\cdot5^2\cdot7^2/2^{11}$
	(31)	$3\cdot5^2/2^9$
	(2^2)	$3^4/2^{10}$
	(21^2)	$-3^2\cdot7/2^9$
5	(5)	$3^5\cdot5\cdot7^2/2^{13}$
	(41)	$3\cdot5^2\cdot7^2/2^{13}$
	(32)	$3^3\cdot5^2/2^{12}$
5	(31^2)	$-3\cdot5^2\cdot11/2^{11}$
	(2^21)	$-3^5\cdot5/2^{12}$
6	(6)	$3^4\cdot5\cdot7^2\cdot11^2/2^{16}$
	(51)	$3^5\cdot5\cdot7^2/2^{15}$
	(42)	$3^3\cdot5^2\cdot7^2/2^{16}$
	(3^2)	$3^2\cdot5^4/2^{14}$
	(41^2)	$-3^2\cdot5^3\cdot7^2/2^{15}$
	(321)	$-3^3\cdot5^2\cdot23/2^{14}$
	(2^3)	$-3^4\cdot5\cdot43/2^{15}$
7	(7)	$3^4\cdot5\cdot7\cdot11^2\cdot13^2/2^{18}$
	(61)	$3^4\cdot5\cdot7^2\cdot11^2/2^{18}$
	(52)	$3^7\cdot5\cdot7^2/2^{18}$
	(43)	$3^2\cdot5^4\cdot7^2/2^{18}$
7	(51^2)	$-3^5\cdot5\cdot7^2\cdot19/2^{17}$
	(421)	$-3^3\cdot5^2\cdot7^2\cdot31/2^{18}$
	(3^21)	$-3^2\cdot5^5\cdot7/2^{16}$
	(32^2)	$-3^4\cdot5^2\cdot7\cdot11/2^{17}$
8	(8)	$3^6\cdot5^3\cdot7\cdot11^2\cdot13^2/2^{23}$
	(71)	$3^4\cdot5\cdot7\cdot11^2\cdot13^2/2^{20}$
	(62)	$3^6\cdot5\cdot7^2\cdot11^2/2^{21}$
	(53)	$3^6\cdot5^3\cdot7^2/2^{20}$
	(4^2)	$3^2\cdot5^4\cdot7^4/2^{22}$
	(61^2)	$-3^4\cdot5\cdot7^2\cdot11^2\cdot23/2^{20}$
	(521)	$-3^8\cdot5\cdot7^2\cdot13/2^{20}$
	(431)	$-3^2\cdot5^4\cdot7^2\cdot47/2^{20}$
	(42^2)	$-3^4\cdot5^2\cdot7^2\cdot61/2^{21}$
	(3^22)	$-3^4\cdot5^4\cdot7/2^{19}$

TABLE 2

Relative error of Equation (3.16) truncated after the $O(\delta^{-r})$ term, evaluated at $Z = \mathrm{diag}[0, 1, 2]$, $\Omega = \mathrm{diag}[0, \omega, 2\omega]$

	ω			
r	2.5	3.0	3.5	4.0
1	-9.7×10^{-2}	-8.7×10^{-2}	-7.4×10^{-2}	-6.1×10^{-2}
2	-1.7×10^{-2}	-9.0×10^{-3}	-3.2×10^{-2}	-2.7×10^{-2}
3	4.6×10^{-2}	4.7×10^{-2}	-5.0×10^{-3}	-9.4×10^{-3}
4	1.2×10^{-1}	1.8×10^{-1}	1.6×10^{-2}	3.4×10^{-3}
5				1.6×10^{-2}

TABLE 3

Relative error of Equation (5.3) *truncated after the* jth *term, evaluated at* $Z = \mathrm{diag}[0, 1, 2]$, $\Omega = \mathrm{diag}[0, \omega, 2\omega]$

	ω			
j	2.5	3.0	3.5	4.0
0	5.4×10^{-4}	9.9×10^{-4}	9.7×10^{-4}	7.9×10^{-4}
1	-5.3×10^{-4}	-1.7×10^{-4}	-3.7×10^{-5}	-1.7×10^{-6}
2	1.8×10^{-5}	-1.7×10^{-5}	-1.6×10^{-6}	3.8×10^{-6}
3	1.1×10^{-5}	-2.1×10^{-5}	-1.1×10^{-5}	-3.4×10^{-6}
4	-2.0×10^{-5}	-1.6×10^{-5}	-3.4×10^{-6}	-8.0×10^{-7}
5	7.2×10^{-5}	6.1×10^{-6}	-2.6×10^{-6}	-8.0×10^{-7}

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