The Annals of Mathematical Statistics
1972, Vol. 43, No. 6, 2053-2057

A CONVERSE TO A COMBINATORIAL LIMIT THEOREM

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Let $a_n(i)$, $b_n(i)$, $i=1, \cdots, n$, be $2n$ numbers defined for every n and let $\bar{A}(k) = \sum_{i=1}^n |a_n(i)|^k$ and $\bar{B}(k) = \sum_{i=1}^n |b_n(i)|^k$. Let $(I_{n1}, \cdots, I_{nn})$ be a random permutation of $(1, \cdots, n)$ and let $S_n = \sum_{i=1}^n b_n(i)a_n(I_{ni})$. If

$$\bar{A}(k)/[\bar{A}(2)]^{\frac{1}{2}k} \to 0 \quad \text{and} \quad \bar{B}(k)/[\bar{B}(2)]^{\frac{1}{2}k} \to 0 ,$$

then it is known that the condition of Hoeffding,

$$n^{\frac{1}{2}k-1}\bar{A}(k)\bar{B}(k)/[\bar{A}(2)\bar{B}(2)]^{\frac{1}{2}k} \to 0 , \qquad k = 3, 4, \cdots ,$$

is sufficient for the standardized moments of S_n to tend to the moments of a standard normal variate. It is shown here that these conditions are also necessary. The relationship of these conditions to the Liapounov conditions is pointed out.

1. Introduction. Let $(I_{n1}, \cdots, I_{nn})$ be a random vector which takes on the $n!$ permutations of $(1, \cdots, n)$ with equal probabilities. Sufficient conditions for the asymptotic normality of

$$S_n = \sum_{i=1}^n b_n(i)a_n(I_{ni}) \tag{1}$$

were given by Wald and Wolfowitz (1944) and weakened conditions were given by Noether (1949). Hoeffding (1951) obtained more general sufficient conditions. These authors obtained sufficient conditions for the stronger result that the standardized moments of S_n tend to the moments of the standardized normal distribution. Motoo (1957) obtained weaker conditions for sufficiency which Hájek (1961) subsequently proved were also necessary for asymptotic normality of S_n.

It is the purpose of this note to prove that the conditions of Hoeffding (1951) are necessary and sufficient for S_n to have standardized moments tending to the moments of the normal distribution. This result is similar to the results of Bernstein (1939) and Brown and Eagleson (1970) who prove that the Liapounov conditions are necessary and sufficient for the convergence of the moments of a sum of independent random variables to the moments of the normal distribution.

2. A combinatorial limit theorem. We assume that for every integer n there are $2n$ numbers $a_n(i)$, $b_n(i)$, $i = 1, \cdots, n$, which are not all equal. For notational convenience we will assume that $a_n(\cdot) = b_n(\cdot) = 0$, where the dot denotes the arithmetic mean. Further, we will write

$$A(e_1, \cdots, e_m) = \sum_{i_1 \cdots i_m} a_n^{e_1}(i_1) \cdots a_n^{e_m}(i_m) ,$$
$$B(e_1, \cdots, e_m) = \sum_{i_1 \cdots i_m} b_n^{e_1}(i_1) \cdots b_n^{e_m}(i_m) ,$$
$$\bar{A}(e) = \sum_{i=1}^n |a_n(i)|^e \qquad \text{and} \qquad \bar{B}(e) = \sum_{i=1}^n |b_n(i)|^e .$$

Received October 6, 1971; revised April 27, 1972.

We assume that the $a_n(i)$ and $b_n(i)$ satisfy the conditions

$$\lim_{n\to\infty} \frac{\bar{A}(k)}{[A(2)]^{\frac{1}{2}k}} = 0\,, \qquad k = 3, 4, \cdots \tag{2}$$

and

$$\lim_{n\to\infty} \frac{\bar{B}(k)}{[B(2)]^{\frac{1}{2}k}} = 0\,, \qquad k = 3, 4, \cdots . \tag{3}$$

THEOREM. *If $a_n(i)$ and $b_n(i)$ satisfy conditions* (2) *and* (3), *then*

$$\begin{aligned} \lim_{n\to\infty} \frac{ES_n^{\,k}}{[VS_n]^{\frac{1}{2}k}} &= 0\,, && k \text{ odd}, \\ &= (k-1)\cdots 3\,, && k \text{ even}\,, \end{aligned} \tag{4}$$

if, and only if,

$$\lim_{n\to\infty} n^{\frac{1}{2}k-1} \frac{A(k)B(k)}{[A(2)B(2)]^{\frac{1}{2}k}} = 0\,, \qquad k = 3, 4, \cdots . \tag{5}$$

PROOF. Sufficiency has been proved by Hoeffding (1951).

We note that $VS_n = A(2)B(2)/(n-1)$. In the following we will assume $n^{-1}A(2)B(2) = 1$. Then it is sufficient to prove that $ES_n^{\,2k} \to (2k-1)\cdots 3$ implies that $n^{-1}A(2k)B(2k) \to 0$ as $n \to \infty$, for $k = 2, 3, \cdots$.

The $2k$th moment of S_n

$$ES_n^{\,2k} = E \sum_{i_1=1}^{n} \cdots \sum_{i_{2k}=1}^{n} b_n(i_1) \cdots b_n(i_{2k}) a_n(I_{ni_1}) \cdots a_n(I_{ni_{2k}}) \tag{6}$$

can be written as a sum of terms of the form $[n(n-1)\cdots(n-m+1)]^{-1}$ $A(e_1, \cdots, e_m)B(e_1, \cdots, e_m)$, where $e_i \geqq 1$, $e_1 + \cdots + e_m = 2k$. The number of each of these terms occurring in the sum is independent of n. We may write $A(e_1, \cdots, e_m)B(e_1, \cdots, e_m)$ as a linear combination of terms of the form $A(\alpha_1)\cdots$ $A(\alpha_g)B(\beta_1)\cdots B(\beta_h)$ where $\alpha_1 + \cdots + \alpha_g = 2k$, $\beta_1 + \cdots + \beta_h = 2k$ and the $\alpha_1, \cdots, \alpha_g$ and $\beta_1, \cdots, \beta_h$ correspond to sums of the $e_1, \cdots, e_m$. We may assume that $\alpha_i \geqq 2$, $\beta_j \geqq 2$ $(i = 1, \cdots, g;\ j = 1, \cdots, h)$, since $A(1) = B(1) = 0$.

First, consider the case $2k = 4$. The only terms appearing in the sum (6) are $A(4)B(4)$, $[A(2)]^2B(4)$, $A(4)[B(2)]^2$ and $[A(2)]^2[B(2)]^2$. The first of these appears in the terms of the sum for $m = 1, 2, 3, 4$ with coefficients of order n^{-1}, n^{-2}, n^{-3}, n^{-4}, respectively. The others appear in the terms of the sum with $m = 2, 3, 4$ with coefficients of order n^{-2}, n^{-3}, n^{-4}. Using (2) and (3) we see that the terms involving $[A(2)]^2B(4)$ and $A(4)[B(2)]^2$ tend to zero. The terms in $[A(2)B(2)]^2$ with coefficients of order n^{-3} and n^{-4} tend to zero and the term with coefficient of order n^{-2} appears in the limit with coefficient 3, since this is the number of times that $A(2, 2)B(2, 2)/n(n-1)$ appears in the sum (6). Thus

$$\lim_{n\to\infty} ES_n^{\,4} = \lim_{n\to\infty} n^{-1}A(4)B(4) + 3\,,$$

but $ES_n^{\,4} \to 3$, so $n^{-1}A(4)B(4) \to 0$, as $n \to \infty$. The proof will be completed by induction. We assume that

$$\lim_{n\to\infty} n^{-1}A(2l)B(2l) = 0\,, \qquad l = 2, 4, \cdots, k-1\,.$$

From this assumption and the inequality

$$A(2k)A(2k-2) \geqq [\bar{A}(2k-1)]^2 ,$$

it is clear that

$$\lim_{n\to\infty} n^{-1}\bar{A}(r)\bar{B}(r) = 0 , \qquad r = 3, 4, \cdots, 2k-2 . \tag{7}$$

Let s be the greatest possible number such that there exist numbers $\gamma_1, \cdots, \gamma_s$ which are sums of disjoint subsets of both $\alpha_1, \cdots, \alpha_g$ and $\beta_1, \cdots, \beta_h$. For $m > 1$, consider the following cases for $A(\alpha_1) \cdots A(\alpha_g)B(\beta_1) \cdots B(\beta_h)$:

(i) $s > 1$. Then Hoeffding (1951) has shown that $h + g \leqq m + s$ and that

$$n^{-m}|A(\alpha_1) \cdots B(\beta_h)| \leqq n^{h+g-m-s}|n^{-1}A(\gamma_1)B(\gamma_1)| \cdots |n^{-1}A(\gamma_s)B(\gamma_s)| \tag{8}$$

(ii) $s = 1$. Then $h + g \leqq m + 1$.

(iia) $h \geqq 2$, $g \geqq 2$. Using the following inequality due to Brown and Eagleson ((1970) Lemma 4),

$$\prod_{r=1}^{l} |EX^{w_r}| \leqq EX^2E|X|^{k-2} , \tag{9}$$

where $w_r \geqq 2$, $l \geqq 2$, $w_1 + \cdots + w_l = k$, we have

$$n^{-m}|A(\alpha_1) \cdots B(\beta_h)| \leqq n^{h+g-4-m}A(2)A(2k-2)B(2)B(2k-2) . \tag{10}$$

(iib) $g \geqq 2$, $h = 1$. Then using (9), we have

$$n^{-m}|A(\alpha_1) \cdots A(\alpha_g)B(2k)| \leqq n^{g-m-2}A(2)A(2k-2)B(2k) . \tag{11}$$

(iic) $h \geqq 2$, $g = 1$. As in (iib), we have

$$n^{-m}|A(2k)B(\beta_1) \cdots B(\beta_h)| \leqq n^{h-m-2}A(2k)B(2)B(2k-2) . \tag{12}$$

(iid) $h = 1$, $g = 1$.

For $m > 1$, the terms appearing in (6) fall into one of these cases. For case (i), (7) implies that these terms tend to zero, except for $s = k$ and $\gamma_1 = \cdots = \gamma_k = 2$. The case $\gamma_1 = \cdots = \gamma_k = 2$ appears with coefficients of order n^{-k}, $n^{-k-1}, \cdots, n^{-2k}$ and these terms tend to zero except for the term with coefficient of order n^{-k}, occurring when $m = k$, $e_1 = \cdots = e_m = 2$. This term appears in the limit with coefficient $(2k-1) \cdots 3$, since this is the number of times that $A(2, \cdots, 2)B(2, \cdots, 2)/n(n-1) \cdots (n-k+1)$ appears in the sum (6). For case (iia), (7) implies that these terms tend to zero. For case (iib) we note that

$$\frac{n^{-1}A(2)A(2k-2)}{A(2k)} = \frac{\sum_{i=1}^{n} g_{ni}^{2k-2}}{n\sum_{i=1}^{n} g_{ni}^{2k}}$$

where $g_{ni} = |a_n(i)|/[A(2)]^{\frac{1}{2}}$. Now if

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} g_{ni}^{2k-2}}{n\sum_{i=1}^{n} g_{ni}^{2k}} = 0 , \tag{13}$$

then case (iib) gives terms of smaller order than $n^{-1}A(2k)B(2k)$. Otherwise, we must have a number K, independent of n, such that

$$n\sum_{i=1}^{n} g_{ni}^{2k}/\sum_{i=1}^{n} g_{ni}^{2k-2} < K .$$

Then

$$n \sum_{i=1}^{n} g_{ni}^{2k} < K \sum_{i=1}^{n} g_{ni}^{2k-2} .$$

Now $\sum_{i=1}^{n} g_{ni}^{2m} / \sum_{i=1}^{n} g_{ni}^{2m-2}$ is strictly monotone increasing in $m = 2, 3, \cdots, k$, so

$$n^{k-1} \sum_{i=1}^{n} g_{ni}^{2k} < n^{k-2} K \sum_{i=1}^{n} g_{ni}^{2k-2} < \cdots < K^{k-1} \sum_{i=1}^{n} g_{ni}^{2} = K^{k-1} .$$

That is, $n^{k-1} A(2k)/[A(2)]^k$ is bounded. Using this result and (3), we have

$$\lim_{n\to\infty} n^{k-1} A(2k)B(2k)/[A(2)B(2)]^k = 0 ,$$

which is the result we require. In the following we assume that (13) holds. Case (iic) is similar to case (iib). For $m = 1$ and case (iid), we have terms in $A(2k)B(2k)$ with coefficients of order n^{-1} for $m = 1$ and of order $n^{-2}, n^{-3}, \cdots, n^{-2k}$ for case (iid). Thus

$$\lim_{n\to\infty} ES^{2k} = \lim_{n\to\infty} n^{-1} A(2k)B(2k) + (2k-1) \cdots 3 ,$$

but $ES_n^{2k} \to (2k-1) \cdots 3$, so $n^{-1}A(2k)B(2k) \to 0$, as $n \to \infty$. So the theorem is proved by induction.

3. Relationship with sums of independent variables. It is interesting to notice a relationship with the theorem of Bernstein (1939). Let the sequence $a_n(i)$ be renumbered such that $a_n(1) \leqq a_n(2) \leqq \cdots \leqq a_n(n)$. Let

$$a'(\lambda) = a_n(i) , \qquad (i-1)/n < \lambda \leqq i/n , \qquad 1 \leqq i \leqq n .$$

Let $U_1, \cdots, U_n$ be a sequence of independent uniform variates and let I_{ni} be the rank of U_i, $1 \leqq i \leqq n$, and let

$$T_n = \sum_{i=1}^{n} b_n(i) a_n'(U_i) .$$

This T_n was considered by Hájek (1961) and it was shown that

$$\lim_{n\to\infty} E(S_n - T_n)^2/(VT_n) = 0 .$$

We can prove a theorem on the limits of the moments of T_n.

THEOREM. *In order that for all* $k = 3, 4, \cdots$

$$\begin{aligned} (14) \qquad \lim_{n\to\infty} ET_n^{\,k}/[VT_n]^{\frac{1}{2}k} &= 0 , && k \ \textit{odd}, \\ &= (k-1)\cdots 3 , && k \ \textit{even}. \end{aligned}$$

it is necessary and sufficient that

$$(15) \qquad \lim_{n\to\infty} n^{-1} \bar{A}(k)\bar{B}(k) = 0 , \qquad k = 3, 4, \cdots .$$

PROOF. The theorem will be proved if we can show that (15) is equivalent to the Liapounov condition of order k. That is

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} E|b_n(i)a_n'(U_i)|^k}{[VT_n]^{\frac{1}{2}k}} = 0 .$$

Now the left-hand side is

$$\lim_{n\to\infty} [VT_n]^{-\frac{1}{2}k} \sum_{i=1}^{n} \int |x|^k \, dP[b_n(i)a_n'(U_i) < x] = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} |b_n(i)|^k \sum_{i=1}^{n} |a_n(i)|^k .$$

Then the result follows from the theorem of Bernstein (1939) and Brown and Eagleson ((1970) Corollary 2).

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