*The Annals of Mathematical Statistics*
1972, Vol. 43, No. 6, 1961-1967

# MARKOVIAN INTERACTION PROCESSES WITH FINITE RANGE INTERACTIONS[1]

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An elementary proof of an existence theorem for infinite particle systems interacting through a finite range interaction is given. The results are then used to prove that the system preserves ergodicity of the initial measure.

**1. Introduction.** Existence theorems for systems of infinitely many interacting particles have been proved by Dobrushin (1971), Harris (1972), Liggett (1972), and Holley (1970). All of these proofs have either been quite long or have used strong theorems concerning semigroups of operators. In this note we present a short elementary proof of an existence theorem by combining the ideas in the papers of Harris and Liggett with techniques that were first used by Robinson (1968). The price of the simplicity of our proof is that the resulting theorem is not as powerful as the theorems of either Liggett or Harris. Liggett treats interactions which may have an infinite range, and Harris' method can be applied to the individual trajectories. Our theorem covers neither of these situations.

In the third section we use the estimates obtained in the existence proof to prove that under suitable conditions the adjoint semigroup preserves ergodicity of the initial measure. When this result is applied to examples such as the speed change with exclusion model of Spitzer (1970), the physical interpretation is that if the system is initially in a pure phase, then it remains in a pure phase at all future times.

**2. The existence theorem.** Let $Z$ be the integers and $Z^\nu$ the $\nu$ dimensional cubic lattice. For each $a \in Z^\nu$ let $E_a$ be a compact Hausdorff space. We will take

$$E = \prod_{a \in Z^\nu} E_a$$

with the product topology for our state space.

Let $\mathscr{C}$ be the Banach space of continuous functions on $E$ with the uniform norm.

If $\Lambda$ is a finite set, $|\Lambda|$ will denote the cardinality of $\Lambda$.

For each finite $\Lambda \subset Z^\nu$ let $A_\Lambda$ be those elements of $\mathscr{C}$ which only depend on the coordinates in $\Lambda$. Then

$$\mathscr{D} = \bigcup_\Lambda A_\Lambda$$

is dense in $\mathscr{C}$.

Let $\mathscr{S}$ be the set of $\nu$ dimensional cubes in $Z^\nu$ whose sides are of length $L$ and parallel to the axes. For each $X \in \mathscr{S}$ we have a linear operator, $\Omega_X$, on $\mathscr{C}$, and we make the following assumptions about the $\Omega_X$.

(2.1) There is a finite $K$ such that $\|\Omega_X\| \leqq K$ for all $X \in \mathscr{S}$,

Received December 22, 1971; revised March 1972.

[1] Research partially supported by N.S.F. grant GP-33314X.

(2.2) If $f \in A_\Lambda$ and $X \cap \Lambda = \emptyset$, then

$$\Omega_X f = 0 .$$

(2.3) If $f \in A_\Lambda$ and $X \cap \Lambda \neq \emptyset$, then $\Omega_X f \in A_{\Lambda \cup X}$.

(2.4) There is an increasing sequence $\{M_n\}$ of finite subsets of $Z^\nu$ with $\bigcup_n M_n = Z^\nu$ such that for all $n$

$$\mathscr{A}_n = \sum_{X \subset M_n} \Omega_X$$

is the infinitesimal generator of a continuous positive contraction semigroup, $T_t^{(n)}$, of operators on $\mathscr{C}$. (Clearly $\mathscr{A}_n$ is bounded and therefore

$$T_t^{(n)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathscr{A}_n^{\,k} .$$

The assumption is that $T_t^{(n)}$ is positive and has norm less than or equal to one.)

Finally, for $f \in \mathscr{D}$, let

$$\mathscr{A} f = \sum_{X \in \mathscr{S}} \Omega_X f . \tag{2.5}$$

Note that because of (2.2) and (2.3) the summation in the definition of $\mathscr{A} f$ is really only a finite summation, and $\mathscr{A} f$ is again in $\mathscr{D}$.

Before stating the theorem we identify each of the objects mentioned so far in the special case of speed change with exclusion. We first take each $E_a$ equal to $\{0, 1\}$. Let $L = 2L_0$. Then each $X \in \mathscr{S}$ is a cube centered at some point $a \in Z^\nu$ and having sides of length $2L_0$. The operators $\Omega_X$ have the following form. For each fixed $X \in \mathscr{S}$ we are given a function $c(X, \eta)$, in $A_X$ which is positive and bounded by $K$. We are also given a transition function $p(a, b)$ with the property that $p(a, b) = 0$ if $|a - b| > L_0$. If $a$ is the element in the center of $X$, $f \in \mathscr{C}$, and $\eta \in E$, then

$$\Omega_X f(\eta) = \sum_{b \in X} c(X, \eta) p(a, b) \eta(a)[1 - \eta(b)][f(\eta_{a,b}) - f(\eta)] ,$$

where

$$\begin{aligned} \eta_{a,b}(c) &= \eta(c) && \text{if} \quad c \neq a \quad \text{and} \quad c \neq b \\ &= \eta(a) && \text{if} \quad c = b \\ &= \eta(b) && \text{if} \quad c = a . \end{aligned}$$

If $X_a$ denotes the element of $\mathscr{S}$ with center $a$, then

$$\mathscr{A} f(\eta) = \sum_{a,b \in Z^\nu} c(X_a, \eta) p(a, b) \eta(a)[1 - \eta(b)][f(\eta_{a,b}) - f(\eta)] .$$

It is easily checked that (2.1)—(2.4) hold in this example (see Spitzer (1970) where a description of the corresponding Markov process is also given).

(2.6) THEOREM. *There is a continuous positive contraction semigroup $T_t$, such that for all $t < \infty$ and all $f \in \mathscr{C}$*

$$\sup_{0 \le s \le t} \|T_s f - T_s^{(n)} f\| \to 0 \qquad \text{as } n \to \infty . \tag{2.7}$$

*Moreover for $f \in \mathscr{D}$ and $t < [K(L + 1)^\nu e^{(L+1)^\nu}]^{-1}$ we have*

$$T_t f = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathscr{A}^k f . \tag{2.8}$$

*$T_t$ is the unique semigroup whose infinitesimal generator, when restricted to $\mathscr{D}$, is given by* (2.5).

In the future we will denote $[K(L+1)^\nu e^{(L+1)^\nu}]^{-1}$ by $K_0$.

As in [3] we only do the proof for $t < K_0$. The extension to arbitrary $t$ follows easily from the semigroup properties of the $T_t^{(n)}$.

The following lemma is crucial. The proof is very similar to the proof of Lemma 1 in [8].

(2.9) LEMMA. *Let $f \in A_\Lambda$. Then for all $N$ and $n$*

$$\|\mathscr{A}_N{}^n f\| \leqq \|f\| n!\, e^{|\Lambda|} K_0^{-n}\,. \tag{2.10}$$

*The same bound holds for $\|\mathscr{A}^n f\|$.*

PROOF. We do the proof for $\|\mathscr{A}^n f\|$. The proof for $\|\mathscr{A}_N{}^n f\|$ is similar.

For any finite set, $\Lambda$, contained in $Z^\nu$ we let

$$S(\Lambda) = \{X \in \mathscr{S}\colon X \cap \Lambda \neq \emptyset\}\,.$$

Now for $f \in A_\Lambda$

$$\mathscr{A}^n f = \textstyle\sum_{X_n} \sum_{X_{n-1}} \cdots \sum_{X_1} \Omega_{X_n} \Omega_{X_{n-1}} \cdots \Omega_{X_1} f\,.$$

As noted after (2.5) this summation has only finitely many nonzero terms, and there is no difficulty in rearranging the terms to get

$$\textstyle\sum_{X_1} \sum_{X_2} \cdots \sum_{X_n} \Omega_{X_n} \Omega_{X_{n-1}} \cdots \Omega_{X_1} f\,.$$

For $X_1, \cdots, X_{n-1}$ fixed we see by (2.2) and (2.3) that the summation over $X_n$ can be restricted to $S(\Lambda \cup X_1 \cup X_2 \cup \cdots \cup X_{n-1})$. By repeated application of this argument we see that

$$\mathscr{A}^n f = \textstyle\sum_{X_1 \in S(\Lambda)} \sum_{X_2 \in S(\Lambda \cup X_1)} \cdots \sum_{X_n \in S(\Lambda \cup X_1 \cup \cdots \cup X_{n-1})} \Omega_{X_n} \cdots \Omega_{X_1} f\,.$$

Therefore

$$\begin{aligned} \|\mathscr{A}^n f\| &\leqq \textstyle\sum_{X_1 \in S(\Lambda)} \cdots \sum_{X_n \in S(\Lambda \cup X_1 \cup \cdots \cup X_{n-1})} \|\Omega_{X_n}\| \cdots \|\Omega_{X_1}\|\, \|f\| \\ &\leqq K^n \|f\| \textstyle\sum_{X_1 \in S(\Lambda)} \cdots \sum_{X_n \in S(\Lambda \cup X_1 \cup \cdots \cup X_{n-1})} 1\,. \end{aligned} \tag{2.11}$$

We next note that $|S(\{y\})| = (L+1)^\nu$. Therefore the multiple summation on the right side of (2.11) can be bounded by

$$\begin{aligned} &\textstyle\sum_{y_1 \in \Lambda} \sum_{X_1 \in S(\{y_1\})} \cdots \sum_{y_n \in \Lambda \cup X_1 \cup \cdots \cup X_{n-1}} \sum_{X_n \in S(\{y_n\})} 1 \\ &\quad \leqq (L+1)^\nu (|\Lambda| + (n-1)(L+1)^\nu) \\ &\qquad \times \textstyle\sum_{y_1 \in \Lambda} \sum_{X_1 \in S(\{y_1\})} \cdots \sum_{y_{n-1} \in \Lambda \cup X_1 \cup \cdots \cup X_{n-2}} \sum_{X_{n-1} \in S(\{y_{n-1}\})} \\ &\quad \leqq (L+1)^{\nu n} \textstyle\prod_{j=1}^n [|\Lambda| + (j-1)(L+1)^\nu] \\ &\quad \leqq (L+1)^{\nu n} [|\Lambda| + n(L+1)^\nu]^n \\ &\quad = (L+1)^{\nu n} n!\, [|\Lambda| + n(L+1)^\nu]^n / n! \\ &\quad \leqq (L+1)^{\nu n} n! \exp\{|\Lambda| + n(L+1)^\nu\}\,. \end{aligned}$$

The proof is completed by substituting this into (2.11).

Returning to the proof of the theorem, we next notice that if $\Lambda$, $M_j$, and $M_k$ are such that $\Lambda \subset M_j \subset M_k$ and the distance from $\Lambda$ to the complement of $M_j$ is larger than $\nu^{\frac{1}{2}}pL$, then

$$\mathscr{A}_j^m f = \mathscr{A}_k^m f \qquad \text{for all } f \in A_\Lambda \text{ and all } m \leqq p . \tag{2.12}$$

This follows from (2.2), (2.3), and the definitions of $\mathscr{A}_j$ and $\mathscr{A}_k$.

Let $\Lambda$ be any fixed finite subset of $Z^\nu$ and let $\alpha(p)$ be the largest integer such that $\nu^{\frac{1}{2}}L\alpha(p)$ is less than the distance from $\Lambda$ to the complement of $M_p$. From (2.4) it follows that $\alpha(p)$ goes to infinity as $p$ goes to infinity. Using (2.12) and Lemma (2.9) we see that for $f \in A_\Lambda$, $t < K_0$, and $j < k$,

$$\sup_{0 \leqq s \leqq t} ||T_s^{(j)}f - T_s^{(k)}f|| \leqq 2||f||e^{|\Lambda|} \sum_{n=\alpha(j)}^{\infty} (tK_0^{-1})^n ,$$

which converges to zero as $\alpha(j)$, and hence as $j$, goes to infinity.

Thus for all $f \in \mathscr{D}$ and all $s < K_0$, $T_s^{(j)}f$ converges to a limit, which we denote by $T_s f$. Furthermore the convergence is uniform for $0 \leqq s \leqq t < K_0$ (i.e., (2.7) holds for $f \in \mathscr{D}$).

An argument similar to the above shows that for $t < K_0$ and $f \in \mathscr{D}$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathscr{A}^n f = T_t f . \tag{2.13}$$

Since by (2.4) each $T_t^{(n)}$ is a contraction semigroup we must have $||T_t f|| \leqq ||f||$ for all $f \in \mathscr{D}$. Thus $T_t$ extends to a semigroup on all of $\mathscr{C}$ and $||T_t|| \leqq 1$. For the same reason (2.7) extends from $\mathscr{D}$ to all of $\mathscr{C}$. It follows immediately from (2.4) and (2.7) that $T_t$ is positive and continuous. Also (2.13) makes it clear that the infinitesimal generator of $T_t$, when restricted to $\mathscr{D}$, is given by (2.5).

Only the uniqueness remains to be proved. Suppose $T_t$ and $T_t'$ are two semigroups whose infinitesimal generators are given by (2.5) for $f \in \mathscr{D}$. Since $\mathscr{D}$ is dense it suffices to show that $T_t$ and $T_t'$ agree on $\mathscr{D}$ for $t < K_0$. Let $f \in A_\Lambda$. Then

$$T_t f = \sum_{n=0}^{N} \frac{t^n}{n!} \mathscr{A}^n f + \frac{1}{N!} \int_0^t (t-s)^N T_s \mathscr{A}^{N+1} f \, ds$$

and

$$T_t' f = \sum_{n=0}^{N} \frac{t^n}{n!} \mathscr{A}^n f + \frac{1}{N!} \int_0^t (t-s)^N T_s' \mathscr{A}^{N+1} f \, ds .$$

Therefore

$$\begin{aligned} ||T_t f - T_t' f|| &\leqq \frac{2}{N!} t^{N+1} ||\mathscr{A}^{N+1} f|| \\ &\leqq 2e^{|\Lambda|}||f||(N+1)(tK_0^{-1})^{N+1} . \end{aligned}$$

Since this is true for $N$ arbitrarily large, we see that if $t < K_0$ and $f \in \mathscr{D}$ then $T_t f = T_t' f$.

This completes the proof of the theorem.

$E$ is a compact metric space and $T_t$ is a continuous semigroup on $\mathscr{C}$. Therefore there is a standard Markov process, $\xi_t$, whose transition function is determined by $T_t$ (see Blumenthal and Getoor (1968) Theorem (9.4) page 46). For $\xi$ a configuration in $E$ and $A$ a Borel subset of $E$ we will let $P^\xi(\xi_s \in A)$ denote the probability that starting from $\xi$, the Markov process $\xi_t$ is at a configuration in $A$ at time $s$.

Let $\mu$ be a probability measure on the Borel sets of $E$. We define the measure $T_t{}^*\mu$ by the formula

$$T_t{}^*\mu(A) = \int_E P^\xi(\xi_t \in A)\mu(d\xi) .$$

Note that for $f \in \mathscr{C}$

$$\int f(s) T_t{}^*\mu(ds) = \int T_t f(\xi)\mu(d\xi) .$$

Therefore, since $T_t$ is positive and $T_t 1 = 1$ ($\mathscr{A}1 = 0$), it follows that $T_t{}^*\mu$ is again a probability measure. One is interested in the behavior of $T_t{}^*\mu$ for large $t$. The results in the next section are useful in this connection (see Holley (1972)).

**3. Preservation of ergodicity.** In this section we want to consider shifts in the lattice. Therefore we assume that

$$E_x = E_0 \qquad \text{for all } x .$$

For each $\xi \in E$ and $a \in Z^\nu$ define $\tau_a(\xi)$ by the formula

$$\tau_a(\xi)(x) = \xi(x - a) .$$

For each $f \in \mathscr{C}$ define

$$\tau_a(f) = f \cdot \tau_{-a} .$$

A probability measure, $\mu$, is shift invariant if for all $f \in \mathscr{C}$ and all $a \in Z^\nu$

$$\int \tau_a(f)(\xi)\mu(d\xi) = \int f(\xi)\mu(d\xi) .$$

Let $\mathscr{M}$ be the set of shift invariant probability measures. In addition to (2.2)—(2.4) we need two further assumptions about the operators $\Omega_X$. They are:

(3.1) $$\Omega_{X+a}\tau_a f = \tau_a \Omega_X f \qquad \text{for all } a \in Z^\nu \text{ and } f \in \mathscr{C}$$

and,

(3.2) if $f \in \mathscr{C}$ and $g \in A_\Lambda$ with $X \cap \Lambda = \varnothing$, then

$$\Omega_X(fg) = g\Omega_X f .$$

These two properties are easily seen to be true in all of the applications which we have in mind (see [5] or [6]).

(3.3) LEMMA. *Let $f \in \mathscr{C}$. Then for all $a \in Z^\nu$ and $0 \leqq t$, $\tau_a T_t f = T_t \tau_a f$.*

PROOF. By the semigroup properties of $T_t$ it clearly suffices to prove the lemma for $t < K_0$. Since both $\tau_a$ and $T_t$ are bounded operators it suffices to show that $\tau_a T_t f = T_t \tau_a f$ for all $f \in \mathscr{D}$. Note that $\tau_a$ maps $\mathscr{D}$ into itself. Therefore by Theorem (2.6)

$$\tau_a T_t f = \tau_a \sum_{n=0}^\infty \frac{t^n}{n!} \mathscr{A}^n f = \sum_{n=0}^\infty \frac{t^n}{n!} \tau_a \mathscr{A}^n f .$$

The definition of $\mathscr{A}$ together with (3.1) imply that $\tau_a \mathscr{A} = \mathscr{A}\tau_a$. Therefore

$$\tau_a T_t f = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathscr{A}^n \tau_a f = T_t \tau_a f .$$

(3.4) COROLLARY. *$T_t^*$ maps $\mathscr{M}$ into itself.*

PROOF. Let $\mu \in \mathscr{M}$

$$\int \tau_a f(\xi) T_t^* \mu(d\xi) = \int T_t \tau_a f(\xi)\mu(d\xi) = \int \tau_a T_t f(\xi)\mu(d\xi)$$
$$= \int T_t f(\xi)\mu(d\xi) = \int f(\xi) T_t^* \mu(d\xi) .$$

REMARK. This proves (3.2) in [5] which was left unproven there.

(3.5) LEMMA. *Let $f, g \in A_\Lambda$ and $0 \leqq t < K_0$. Then*

$$\lim_{|a|\to\infty} ||T_t(f\tau_a g) - (T_t f)(T_t \tau_a g)|| = 0 .$$

PROOF. From Theorem (2.6) and Lemma (2.9) it is seen that

$$(T_t f)(T_t \tau_a g) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{j=0}^{n} \binom{n}{j} (\mathscr{A}^j f)(\mathscr{A}^{n-j} \tau_a g) ,$$

and

$$T_t(f\tau_a g) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathscr{A}^n(f\tau_a g) .$$

Let $\varepsilon > 0$ be given and $N$ be so large that

$$\left|\left| \sum_{n=N+1}^{\infty} \frac{t^n}{n!} \sum_{j=0}^{n} \binom{n}{j} (\mathscr{A}^j f)(\mathscr{A}^{n-j} \tau_a g) \right|\right| < \varepsilon$$

and

$$\left|\left| \sum_{n=N+1}^{\infty} \frac{t^n}{n!} \mathscr{A}^n(f\tau_a g) \right|\right| < \varepsilon .$$

Then

$$||T_t(f\tau_a g) - (T_t f)(T_t \tau_a g)||$$
$$\leqq \left|\left| \sum_{n=0}^{N} \frac{t^n}{n!} [\mathscr{A}^n(f\tau_a g) - \sum_{j=0}^{n} \binom{n}{j} (\mathscr{A}^j f)(\mathscr{A}^{n-j} \tau_a g)] \right|\right| + 2\varepsilon .$$

Thus it suffices to show that for $|a|$ sufficiently large

$$\mathscr{A}^n(f\tau_a g) = \sum_{j=0}^{n} \binom{n}{j} (\mathscr{A}^j f)(\mathscr{A}^{n-j} \tau_a g) \tag{3.6}$$

for all $n \leqq N$.

If $|a|$ is large enough so that the distance from $\Lambda$ to $\Lambda + a$ is greater than $N\nu^{\frac{1}{2}}L$, then (3.6) follows by induction from the definition of $\mathscr{A}$, (2.3), (3.2), and the observation that if $g \in A_\Lambda$ then $\tau_a g \in A_{\Lambda+a}$.

We will say a measure $\mu$ in $\mathscr{M}$ is ergodic for the shifts $\tau_a$ if whenever $\tau_a(A) = A$ for all $a \in Z^\nu$, $\mu(A)$ is either zero or one. Let $f, g \in \mathscr{C}$ and consider the statement

$$\lim_{N\to\infty} (2N+1)^{-\nu} \sum_{a \in B_N} \int f(\xi)\tau_a g(\xi)\mu(d\xi) = \int f(\xi)\mu(d\xi) \int g(\xi)\mu(d\xi) , \tag{3.7}$$

where $B_N = \{a \in Z^\nu : a = (a_1, \cdots, a_\nu) \text{ and } \max|a_i| \leqq N\}$.

By using the ergodic theorem for groups of transformations with more than one generator (see Wiener (1939)), it is not difficult to see that a necessary condition for $\mu$ to be ergodic is that (3.7) holds for all $f, g \in \mathscr{C}$. It is also easily seen that a sufficient condition for $\mu$ to be ergodic is that $\mu \in \mathscr{M}$ and $\mu$ satisfies (3.7) for all $f, g \in \mathscr{D}$.

(3.8) THEOREM. *Let $\mu \in \mathscr{M}$ be ergodic for the shifts $\tau_a$, and assume that* (3.1) *and* (3.2) *hold. Then $T_t{}^*\mu$ is ergodic for all $t$.*

PROOF. As remarked above, it will suffice to show that if $\mu \in \mathscr{M}$ satisfies (3.7) for all $f, g \in \mathscr{C}$, then $T_t{}^*\mu \in \mathscr{M}$ and satisfies (3.7) for all $f, g \in \mathscr{D}$.

We do this for all $t < K_0$. The general case then follows because $T_t{}^*$ is a semigroup.

From Corollary (3.4) we know that $T_t{}^*\mu$ is in $\mathscr{M}$. Hence it suffices to choose $f, g \in \mathscr{D}$ and show that (3.7) holds for $T_t{}^*\mu$.

If $f, g \in \mathscr{D}$ then there is some finite $\Lambda$ such that $f, g \in A_\Lambda$. By Lemmas (3.3) and (3.5)

$$\lim_{|a|\to\infty} |\textstyle\int T_t(f\tau_a g)(\xi)\mu(d\xi) - \int (T_t f(\xi))(\tau_a T_t g(\xi))\mu(d\xi)| = 0 .$$

Thus

$$\lim_{N\to\infty}(2N+1)^{-\nu}|\textstyle\sum_{a\in B_N}\int T_t(f\tau_a g)(\xi)\mu(d\xi) - \sum_{a\in B_N}\int T_t f(\xi)(\tau_a T_t g(\xi))\mu(d\xi)| = 0.$$

But by the assumption on $\mu$

$$\lim_{N\to\infty}(2N+1)^{-\nu}\textstyle\sum_{a\in B_N}\int (T_t f(\xi))(\tau_a T_t g(\xi))\mu(d\xi) = \int T_t f(\xi)\mu(d\xi) \int T_t g(\xi)\mu(d\xi) .$$

Therefore (3.7) holds with $\mu$ replaced by $T_t{}^*\mu$.

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