The Annals of Applied Probability
1994, Vol. 4, No. 1, 241–253

ON THE RELATIONSHIP BETWEEN FRACTAL DIMENSION AND FRACTAL INDEX FOR STATIONARY STOCHASTIC PROCESSES

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For Gaussian processes there is a simple and well-known relationship between the fractal dimension of sample paths and the fractal index of the covariance function. This property is of considerable practical interest, since it forms the basis of several estimators of fractal dimension. Motivated by statistical applications involving non-Gaussian processes, we discuss the relationship in a wider context. We show that the relationship fails in some circumstances, but nevertheless does hold in a variety of cases.

1. Introduction and summary. The mathematical notion of fractal dimension provides a scale-free measure of roughness, with a rich variety of practical applications. For example, it may be used to rank surfaces in terms of increasing roughness, for purposes of quality management or wear monitoring. These applications often require estimates of the fractal dimension, D, of linear "sections" of the surface by vertical planes. (We shall call these sections *line transects*.) Such estimates can be difficult to produce directly, by appealing to the definition of D. An alternative approach is to calculate an estimate of a quantity called fractal index, α, which is determined by the behaviour of a covariance function at the origin and is generally more accessible than D itself, and then compute an estimate of D by using a formula that expresses D as a function of α. Relatively simple estimates of α may be based on the variogram or the periodogram of a line transect trace, or on the length or level crossings of a smoothed version of that trace. If the trace can be modelled by a stationary Gaussian process, then D and α are related by the formula

$$D = 2 - \tfrac{1}{2}\alpha \tag{1.1}$$

[see, e.g., Adler (1981), Chapter 8], and so an estimate of α leads immediately to an estimate of D.

Equation (1.1) is crucial to much of the applied work that has been done in the context of estimation of fractal properties. That work revolves around the issue of so-called scaling laws, which describe the way in which rather

Received August 1992; revised June 1993.
AMS 1991 *subject classifications*. Primary 60G10, 60G15; secondary 62G05.
Key words and phrases. Covariance, fractal dimension, fractal index, fractional index, Gaussian process, Hausdorff dimension, level crossing, variogram.

elementary physical measurements vary with the size of the measuring unit. We shall have more to say about this in Section 3, where we shall discuss the role of (1.1) in explicit mathematical detail. However, let us note here two of the simplest scaling laws, those based on the length of an approximating polygonal path and on the variogram. The first of these is perhaps the "classical" scaling law, very commonly associated with physical fractal properties. It declares that if (1.1) holds, then the length $l(s)$ of a polygonal approximation to a fractal curve, constructed on a grid of edge width s, should vary in such a way that $\log l(s)$ increases like $(D-1)|\log s|$ as s decreases:

$$\log l(s) = (D-1)|\log s| + \text{constant} + o(1) \tag{1.2}$$

as $s \to 0$. In practice, l may be estimated from data, and D estimated by fitting a simple linear regression model to a sequence of observed values of the pair $(\log l, \log s)$. For a rich and varied discussion of the applications of such length-based scaling laws to real data, ranging from measuring the roughness of polished metal, oxidized metal and brick surfaces to measurements of the roughness of coastlines, the reader is referred to Ling (1987, 1989, 1990) and Brown, Charles, Johnsen and Chester (1993). Related work of Majumdar and Bhushan (1991) discusses two-dimensional scaling laws.

The variogram method of estimating fractal dimension is founded on the observation that if $v(s)$ equals the mean square of the difference between two values of a fractal process at points distance s apart, then in the presence of (1.1), $|\log v(s)|$ should increase like $2(2-D)|\log s|$ as s decreases:

$$|\log v(s)| = 2(2-D)|\log s| + \text{constant} + o(1) \tag{1.3}$$

as $s \to 0$. Once again this equation forms the basis for many practical estimates of fractal dimension. For example, its applications to data on mineralogy, rainfall, geography and the measurement of surface roughness have been discussed by Serra (1968), Delfiner and Delhomme (1975), Journel and Huijbregts (1978), Burrough (1981), Constantine and Hall (1993) and authors cited in these articles.

The importance of (1.1) to the validity of results such as (1.2) and (1.3), and hence to the physical application of such scaling laws, cannot be understated. The latter two formulae hold if and only if (1.1) is valid. In particular, they are valid for many processes that tend to have heavier tails than Gaussian, but not for processes that tend to have shorter tails, such as powers of Gaussian processes when the exponent is less than $\frac{1}{2}$, as we shall prove.

The purpose of this note is to determine the validity of (1.1) for certain processes that are not Gaussian but may be expressed as functions of Gaussian processes. The study was motivated by a problem concerning wear of rollers used to produce aluminum sheet. Figure 1 depicts a line transect trace from the surface of an unused roller, and above it, a nonparametric estimate of the distribution of roller height above its mean. A Gaussian density with the same mean and variance as the data is also plotted, for the sake of comparison; it is strikingly close to the nonparametric estimate. The data illustrated in Figure 2 are from the same roller, but are recorded after

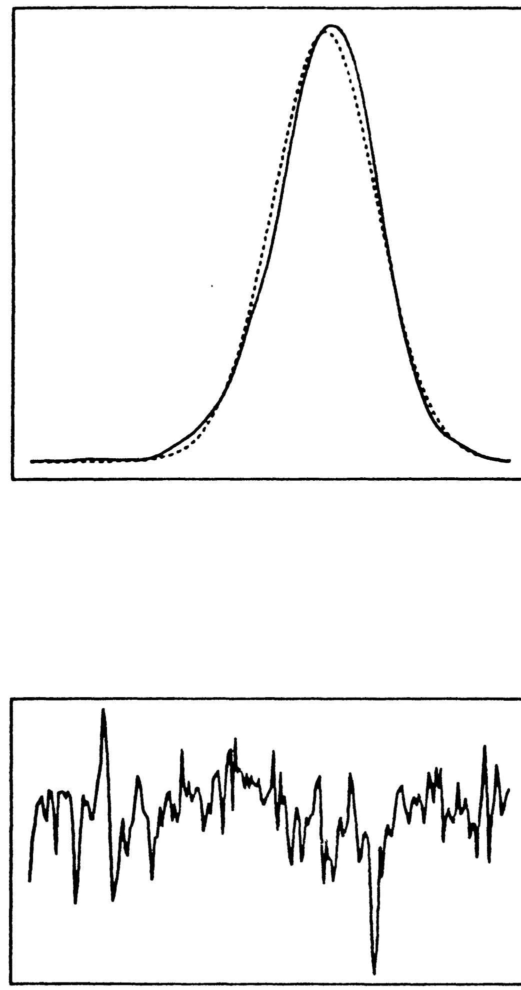

FIG. 1. *Surface trace and height density for unworn roller. The trace at the bottom represents a line transect sample obtained by drawing a stylus across a very small part of the surface of a new, unused roller. The trace is* 4 *mm wide and the standard deviation of its fluctuations equals* 0.5 *mm. The unbroken bell-shaped curve above the trace is a kernel estimate of the marginal density of height of the surface above its mean, computed using bandwidth equal to the fraction* 0.03 *of the length of the trace and employing the standard normal kernel. (The density estimate was obtained by integrating the continuous trace, analogous to summing for discrete data.) The broken bell-shaped curve is the Gaussian density with the same mean and variance as the data. The fractal index was estimated from the periodogram to be* $\hat{\alpha} = 0.3$, *leading to an estimate of fractal dimension equal to* $\hat{D} = 2 - \frac{1}{2}\hat{\alpha} = 1.85$.

a period of wear. Note the asymmetry of both the line transect and the nonparametric estimate of the density of height for this new data set. This time the fitted density is chi-squared rather than Gaussian.

Fitting distributions to data such as those in Figures 1 and 2 is an essential part of studying the properties of roller surfaces. Among other

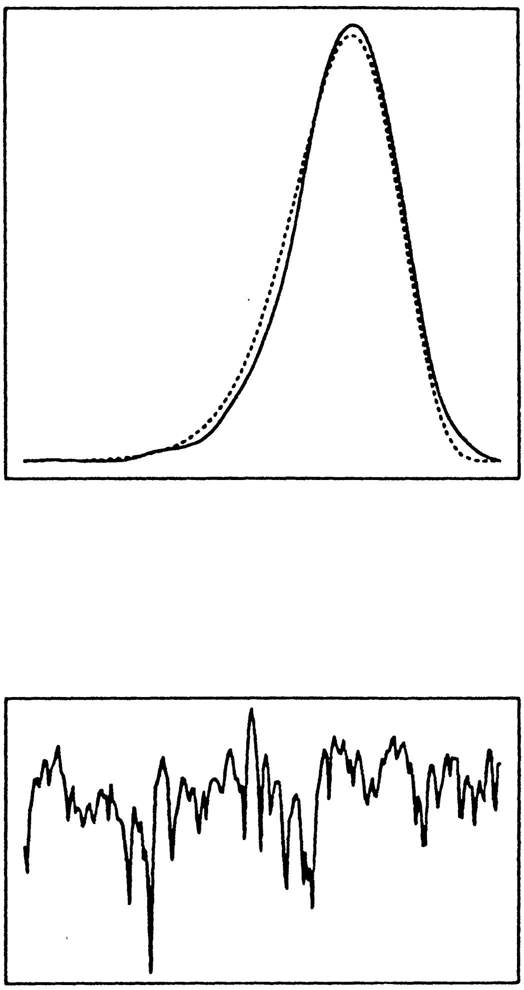

FIG. 2. *Surface trace and height density for worn roller. The roller was the same one studied in Fig.* 1, *except that it was measured after a period of seating. All other specifications, including bandwidth and kernel, are the same as for Figure* 1, *except that the broken curve represents a* χ^2_{20} *density with fitted location and scale. Fractal index and fractal dimension for these data were estimated to be* $\hat{\alpha} = 0.5$ *and* $\hat{D} = 2 - \frac{1}{2}\hat{\alpha} = 1.75$, *respectively.*

things, it provides a means of assessing the validity of formula (1.1), on which an estimate of D would most likely be based, and yields valuable information about proportions of the roller surface that exceed various levels. Of course, modelling real data traces by deterministic functions of Gaussian processes is restrictive, but there are few alternatives if the data are, for all practical purposes, continuous. This is particularly so when it is desired to draw inferences about surface roughness from properties of covariance, as is frequently the case when fractal dimension is being estimated. An assumption close to that of "Gaussianity" is necessary to draw a connection between moment and sample path properties.

If the recorded line transect were exactly Gaussian, or a known function of a Gaussian process, then both the covariance function and the fractal dimension of sample paths could, with probability 1, be determined without error from an arbitrarily short section of trace. However, the assumption that an empirical data trace is a realization of a function of a Gaussian process is only a convenient mathematical abstraction, as also is the notion that the process is genuinely self-similar. Nevertheless, provided we do not analyse the data at a level that reveals the imperfections, for example, by examining the trace on too fine a scale, the prescriptions for model-fitting outlined in the foregoing text can produce valuable statistical information about the structure of a roller surface and the effect of wear on that surface.

We shall show that formula (1.1) fails in some instances, but that there do exist simple conditions under which it is valid. In particular, it is valid under the chi-squared process model suggested for the data in Figure 2. Our results also demonstrate that (1.1) is valid for convolutions of squares of Gaussian processes with different centers and scales; such convolutions are used in practice to model heavily worn roller surfaces, whose marginal densities have "shoulders" on the left-hand side. These results are stated in Section 3 and proved in Section 4. Brief definitions of fractal dimension and fractal index are given in Section 2. All our results have direct analogs in higher dimensions, with virtually identical proofs. However, since we do not have strong practical motivation for the higher dimensional case, we shall not treat it here.

The notion of a fractal or Hausdorff dimension for a stochastic function goes back at least to work of Taylor (1955). Its applications in the physical and engineering sciences have been discussed by, among others, Berry and Hannay (1978), Sayles and Thomas (1978), Coster and Chermant (1983), Mandelbrot, Passoja and Paullay (1984) and Thomas and Thomas (1988). Statistical properties have been the subject of increasing recent interest; see, for example, Taylor and Taylor (1991) and Smith (1992).

2. Definitions of fractal dimension and fractal index. Let f denote a function defined on an interval $\mathscr{I}$, tracing out the path $\mathscr{S} = \{(t, f(t)): t \in \mathscr{I}\}$. The fractal or Hausdorff dimension of $\mathscr{S}$ may be defined as follows. Given $\varepsilon > 0$, an ε-covering of $\mathscr{S}$ is defined to be a countable collection $\mathscr{C}$ of disks S_i with respective diameters $\delta_i > \varepsilon$, whose union covers $\mathscr{S}$. For $d > 0$, define

$$A(d) = \lim_{\varepsilon \downarrow 0} \inf_{\mathscr{C}} \sum_i \delta_i^d,$$

where the infimum is taken over all ε-coverings of $\mathscr{S}$. It may be shown that there exists a unique number D with the property that $A(d) = \infty$ for all $d < D$ and $A(d) = 0$ for all $d > D$. Necessarily, $1 \le D \le 2$. We call D the fractal dimension of $\mathscr{S}$.

There is a variety of other definitions of fractal dimension, for example, that based on capacity. However, in the context of the stochastic sample path examples considered in this paper, the definitions produce identical values of D with probability 1.

A function f is said to satisfy a Lipschitz condition of order L on $\mathscr{I}$ if

$$\sup_{s, t \in \mathscr{I}: |s-t| \le u} |f(s) - f(t)| = O(u^L), \tag{2.1}$$

as $u \to 0$. In this event the fractal dimension of $\mathscr{S}$ satisfies $D \le 2 - L$. Furthermore, $D \ge d$ if

$$\int_{\mathscr{I}}\int_{\mathscr{I}} \{|s - t| + |f(s) - f(t)|\}^{-d}\, ds\, dt < \infty. \tag{2.2}$$

Thus, the notion of fractal dimension is closely related to that of Lipschitz continuity.

Let X_t denote a stationary square-integrable stochastic process, with variogram

$$v_t = E(X_t - X_0)^2 = 2\{1 - \mathrm{cov}(X_0, X_t)\}. \tag{2.3}$$

If there exists $\alpha \in (0, 2]$ satisfying

$$\alpha = \sup\{\beta : v_t = O(t^\beta) \text{ as } t \downarrow 0\} = \inf\{\beta : t^\beta = O(v_t) \text{ as } t \downarrow 0\},$$

then α is called the fractal index or fractional index of the process X_t.

Formula (1.1) may be proved by noting that if X_t is Gaussian with fractal index α, then with probability 1 the sample paths of $f(t) = X_t$ satisfy (2.1) for each $L < \frac{1}{2}\alpha$ and satisfy (2.2) for each $d < 2 - \frac{1}{2}\alpha$. All results mentioned in this section are elucidated by Adler [(1981), Chapter 8].

In many cases of practical or theoretical interest, the variogram of a process X_t with fractal index α satisfies $v_t \sim \text{const.}|t|^\alpha$ as $t \to 0$. Of course, two processes X_t and Z_t whose variograms satisfy $E(X_t - X_0)^2 \sim$ const. $E(Z_t - Z_0)^2$, will share the same fractal index. However, this close relationship between the processes' covariances is only sufficient, not necessary, for commonality of fractal dimension, as is plain from the definition of dimension.

3. Relationship between fractal dimension and fractal index.

Generally speaking, the sample paths of a smooth function g of a stochastic process Z have the same fractal dimension as the paths of Z. This may be seen by simple Taylor expansion, as follows. If g has a continuous derivative, then there exists a point $t^* \in (0, t)$ such that

$$g(Z_t) - g(Z_0) = g'(Z_{t^*})(Z_t - Z_0) \sim g'(Z_0)(Z_t - Z_0) \tag{3.1}$$

as $t \to 0$. It follows that, provided $P\{g'(Z_0) = 0\} = 0$, the process $X_t = g(Z_t)$ has the same Lipschitz behaviour, and hence the same fractal dimension D, as Z_t. For example, it may be shown from (3.1) that if the process Z_t has fractal index $\beta \in (0, 2)$ and if we define $f(t) = X_t$, then with probability 1, (2.1) holds for all $L < \frac{1}{2}\beta$ and (2.2) holds for all $d < 2 - \frac{1}{2}\beta$. This ensures that sample paths of X_t share the fractal dimension $D = 2 - \frac{1}{2}\beta$ of the paths of Z_t. However, the fractal index α of the process X_t can be different from its counterpart for Z_t, so that the relationship $D = 2 - \frac{1}{2}\alpha$ may not be valid.

It is often possible to express a non-Gaussian process, say X_t, in terms of a stationary Gaussian process Z_t by the relationship

$$X_t = g(Z_t), \tag{3.2}$$

where g is a smooth function. For example, if X is a stationary process with a continuous marginal distribution F, then we may choose to define $Z_t = (\Phi^{-1}F)(X_t)$, where Φ denotes the standard normal distribution function. This guarantees that at least the one-dimensional marginals of Z_t are Gaussian and that (3.2) holds with $g = F^{-1}\Phi$, but of course not that Z_t is actually Gaussian. We may assume without loss of generality that $E(Z_t) = 0$ and $E(Z_t^2) = 1$, and we do so in the following text.

We suppose that

(3.3) for some $\xi > 0$, $E(X_t - X_0)^2 = O(t^{\xi})$ as $t \to 0$, g' exists and is continuous almost everywhere, g' is not identically zero and, with probability 1, Z_t is a continuous function of t.

Our first result states conditions on g that are sufficient to ensure that the processes X_t and Z_t have identical covariance behaviour near the origin, and so have identical fractal index. Since the sample paths of X_t and Z_t have identical fractal dimension [see the argument following (3.1)], then the relationship between fractal dimension and fractal index for X is, under the conditions of Theorem 3.1, the same as for a Gaussian process:

$$\text{fractal dimension of } X_t \text{ paths} = 2 - \tfrac{1}{2}\,(\text{fractal index of } X_t). \tag{3.4}$$

Let N denote a random variable having the standard normal distribution.

THEOREM 3.1. *Assume condition* (3.3) *and that for some* $\xi > 0$ *and all* $\lambda > 0$,

$$E\{g'(N)^2N^2\} < \infty, \qquad E\Big\{\sup_{|z| \le \lambda|N|} |g'(z)|^{2+\xi}\Big\} < \infty. \tag{3.5}$$

Then

$$E(X_t - X_0)^2 = E(Z_t - Z_0)^2 E\{g'(Z_0)^2\} + o\{E(Z_t - Z_0)^2\} \tag{3.6}$$

as $t \to 0$.

We noted in Section 2 that a very close relationship, such as that in (3.6), between the covariances of X_t and Z_t is not essential for us to be able to equate the fractal indices of these processes. Our next theorem shows that condition (3.5) may be slightly relaxed, to such an extent that (3.6) may fail to hold [see the case $r = \frac{1}{2}$ in (3.10)], yet not so much as to invalidate (3.4).

THEOREM 3.2. *Assume condition* (3.3) *and that for all* $0 < \xi < 2$ *and all* $\lambda > 0$,

$$E\left\{ \sup_{|z| \le \lambda |N|} |g'(z)|^{2-\xi} \right\} < \infty. \tag{3.7}$$

Suppose too that all moments of X_t *are finite. Then*

$$\liminf_{t \to 0} E(X_t - X_0)^2 / E(Z_t - Z_0)^2 > 0 \tag{3.8}$$

and for all $0 < \xi < 1$,

$$\lim_{t \to 0} E(X_t - X_0)^2 / \left\{ E(Z_t - Z_0)^2 \right\}^{1-\xi} = 0. \tag{3.9}$$

In view of our definition of fractal index in Section 2, formulae (3.8) and (3.9) imply that the processes X_t and Z_t have the same fractal index. Since, by (3.1), their sample paths have the same fractal dimension, then the classical Gaussian process relationship between fractal index and fractal dimension is valid; that is, the identity (3.4) holds.

Theorem 3.2 has a corollary that applies to convolutions of functions of Gaussian processes, such as the chi-squared process used to model the data depicted in Figure 2. To appreciate this point, let $Z_{1t}, \dots, Z_{mt}$ denote independent, stationary, continuous Gaussian processes and let $g_1, \dots, g_m$ be functions such that each g_j' exists and is continuous almost everywhere, g_j' is not identically zero,

$$\max_{1 \le j \le m} E\left\{ \sup_{|z| < \lambda |N|} |g_j'(z)|^{2-\xi} \right\} < \infty \qquad \text{for all } 0 < \xi < 2,$$

and

$$\max_{1 \le j \le m} E\{g_j(Z_{jt}) - g_j(Z_{j0})\}^2 = O(t^{\xi}) \quad \text{for all } \xi > 0,$$

as $t \to 0$. Put $X_t = \sum_j g_j(Z_{jt})$. Then by Theorem 3.2,

$$\liminf_{t \to 0} E(X_t - X_0)^2 \Bigg/ \left\{ \sum_{j=1}^{m} E(Z_{jt} - Z_{j0})^2 \right\} > 0,$$

and for all $0 < \xi < 1$,

$$\lim_{t \to 0} E(X_t - X_0)^2 \Bigg/ \left\{ \sum_{j=1}^{m} E(Z_{jt} - Z_{j0})^2 \right\}^{1-\xi} = 0.$$

It follows that (3.4) holds, and the fractal dimension of X equals the greatest of the fractal dimensions of $Z_1, \dots, Z_m$. Hence, if the data in Figure 2 are chi-squared, then an estimate of the fractal dimension, D, of the trace may be obtained as 2 minus half the value of an estimate of the fractal index of the process generating the trace. The value of D quoted for Figure 2 was obtained in precisely this manner, using an estimate of fractal index based on the periodogram.

Finally, we consider a number of examples that elucidate our main results and show that condition (3.7) is close to being necessary and sufficient for (3.4). In broad terms, each of (3.4), (3.5) and (3.7) is very nearly equivalent to the condition that $E\{g'(Z_0)^2\} < \infty$. The exceptional cases are those where $E\{g'(Z_0)^2\}$ is either "just finite" or "just infinite." To appreciate this point, let $g(x) = |x + a|^r h(x)$ or $g(x) = \operatorname{sgn}(x + a)|x + a|^r h(x)$, where a is any real number, $r > -\frac{1}{2}$ and the function h satisfies $h(a) \neq 0$ and has two derivatives that are bounded on compact intervals and increase no faster than polynomially on unbounded intervals. [The assumption that $r > -\frac{1}{2}$ is necessary to ensure that $X_t = g(Z_t)$ has finite variance.] Then it may be proved by elementary calculus that as $t \to 0$,

$$(3.10) \qquad E(X_t - X_0)^2 \sim \text{const.}\begin{cases} E(Z_t - Z_0)^2, & \text{if } r > \frac{1}{2}, \\ E(Z_t - Z_0)^2 \left|\log E(Z_t - Z_0)^2\right|, & \text{if } r = \frac{1}{2}, \\ \left\{E(Z_t - Z_0)^2\right\}^{r + (1/2)}, & \text{if } r < \frac{1}{2}, \end{cases}$$

where the constant is positive and depends on g. Similarly it may be shown that (3.5) holds if and only if $r > \frac{1}{2}$, and (3.7) is true (for all $0 < \xi < 2$) if and only if $r \geq \frac{1}{2}$. From these facts it may be seen that the three conditions (3.4), (3.7) and $r \geq \frac{1}{2}$ are all equivalent, and that the three conditions (3.6), $E\{g'(Z_0)^2\} < \infty$ and $r > \frac{1}{2}$ are all equivalent. We may deduce from (3.10) that when $r < \frac{1}{2}$, the fractal index of X_t equals $r + \frac{1}{2}$ times the fractal index of Z_t. Therefore, (3.4), which is not true for $r < \frac{1}{2}$, should be replaced by the formula

$$(3.11) \qquad \begin{aligned} &\text{fractal dimension of } X_t \text{ paths} \\ &\quad = 2 - (2r + 1)^{-1}\,(\text{fractal index of } X_t). \end{aligned}$$

The examples just treated include many instances where $g = F^{-1}\Phi$, and F and Φ are the respective marginal distributions of X_t and Z_t. In particular, this is the case when g is increasing, for example, if $g(x) = \operatorname{sgn}(x + a)|x + a|^r h(x)$, where $r > 0$ and h is symmetric about a and increasing on (a, ∞).

In conclusion, we briefly refer to the issue of scaling laws discussed in Section 1. The reader will remember our point that formula (3.4) is crucial to statistical analyses based on those laws, and our claim that if (3.4) is invalid (as it is, for example, if $r < \frac{1}{2}$), then many traditional estimators of fractal properties are not statistically consistent. We are now in a position to be explicit about this matter. Suppose that $r < \frac{1}{2}$, so that (3.11) holds instead of (3.4). Then it may be proved that in place of the fundamental equations (1.2) and (1.3), we have, respectively, the following formulae:

$$\log l(s) = \left\{\tfrac{1}{2}(2r + 1)D - 2r\right\}|\log s| + \text{constant} + o(1),$$

$$|\log v(s)| = (2r + 1)(2 - D)|\log s| + \text{constant} + o(1).$$

If one ignores the differences between these formulae and their counterparts when (3.4) holds, and tries to estimate D by linear regression of $\log l(s)$ or $\log v(s)$ on $|\log s|$, then the estimates will be inconsistent.

4. Proofs of Theorems 3.1 and 3.2.

PROOF OF THEOREM 3.1. There exists $t^* \in (0, t)$ with the property that $X_t - X_0 = g'(Z_t^*)(Z_t - Z_0)$. Let $\mathscr{I}' = (r', s')$ denote any interval that does not contain a zero of g' and let $\mathscr{I} = (r, s)$ be any interval interior to $\mathscr{I}'$: $r' < r < s < s'$. Put

$$C_1 = \inf_{z \in \mathscr{I}} g'(z)^2 > 0.$$

Then

$$\begin{aligned} C_1^{-1} E(X_t - X_0)^2 &= C_1^{-1} E\{g'(Z_{t^*})^2 (Z_t - Z_0)^2\} \\ &\geq E\{(Z_t - Z_0)^2 I(Z_u \in \mathscr{I}, 0 \leq u \leq t)\} \\ &= E\big[I(Z_0 \in \mathscr{I}) E\{(Z_t - Z_0)^2 | Z_0\}\big] \\ &\quad - E\big[I(Z_0 \in \mathscr{I}) \\ &\qquad \times E\{(Z_t - Z_0)^2 I(Z_u \notin \mathscr{I}, \text{some } 0 < u \leq t) | Z_0\}\big] \\ &\geq E\big[I(Z_0 \in \mathscr{I}) E\{(Z_t - Z_0)^2 | Z_0\}\big] \\ &\quad - E\Big(I(Z_0 \in \mathscr{I}) \big[E\{(Z_t - Z_0)^4 | Z_0\} \\ &\qquad \times P(Z_u \notin \mathscr{I}, \text{some } 0 < u \leq t | Z_0)\big]^{1/2}\Big) \\ &\geq E\big[I(Z_0 \in \mathscr{I}) E\{(Z_t - Z_0)^2 | Z_0\}\big] \\ &\quad - \Big(E\big[I(Z_0 \in \mathscr{I}) E\{(Z_t - Z_0)^4 | Z_0\}\big] \\ &\qquad \times P(Z_0 \in \mathscr{I}; Z_u \notin \mathscr{I}, \text{some } 0 < u \leq t)\Big)^{1/2}. \end{aligned} \tag{4.1}$$

Since the process Z is continuous, then

$$P(Z_0 \in \mathscr{I}; Z_u \notin \mathscr{I}, \text{some } 0 < u \leq t) \to 0$$

as $t \to 0$. There exist constants $C_2, C_3 > 0$, depending on $\mathscr{I}$, with the property that for all sufficiently small t,

$$E\{(Z_t - Z_0)^2 | Z_0 = z\} \geq C_2 E(Z_t - Z_0)^2,$$

$$E\{(Z_t - Z_0)^4 | Z_0 = z\} \leq C_3 \{E(Z_t - Z_0)^2\}^2$$

uniformly in $z \in \mathscr{I}$. Hence, by (4.1),

$$E(X_t - X_0)^2 \geq C_1 C_2 \{1 + o(1)\} E(Z_t - Z_0)^2 \tag{4.2}$$

as $t \to 0$.

Result (4.2) implies that $E(Z_t - Z_0)^2 = O(t^{\xi})$ as $t \to 0$, for some $\xi > 0$. Hence, by Fernique's lemma [e.g., Leadbetter, Lindgren and Rootzén (1983), page 219], for constants $C_4, C_5 > 0$ and all $x > 0$,

$$P\Big(\sup_{0<u<1} |Z_u| > x\Big) \le C_4 \exp\big(-C_5 x^2\big). \tag{4.3}$$

Therefore, by the assumptions on g,

$$E\Big\{\sup_{0<u<1} |g'(Z_u)|^{2+2\eta}\Big\} < \infty$$

for some $\eta > 0$. The dominated convergence theorem and continuity of Z now imply that

$$\delta_t \equiv E\Big\{\sup_{0<u<t} \big|g'(Z_u)^2 - g'(Z_0)^2\big|^{1+\eta}\Big\} \to 0$$

as $t \to 0$. Hence,

$$\begin{aligned} E(X_t - X_0)^2 &= E\big\{g'(Z_{t^*})^2 (Z_t - Z_0)^2\big\} \\ &= E\big\{g'(Z_0)^2 (Z_t - Z_0)^2\big\} + R, \end{aligned} \tag{4.4}$$

where, with $p = 1 + \eta$ and $q = 1 + \eta^{-1}$,

$$\begin{aligned} |R| &= \Big|E\Big[\big\{g'(Z_{t^*})^2 - g'(Z_0)^2\big\}(Z_t - Z_0)^2\Big]\Big| \\ &\le \delta_t^{1/p}\big(E|Z_t - Z_0|^{2q}\big)^{1/q} \\ &= o\Big\{\big(E|Z_t - Z_0|^{2q}\big)^{1/q}\Big\} = o\big\{E(Z_t - Z_0)^2\big\}. \end{aligned}$$

With $\gamma = \gamma_t = \operatorname{cov}(Z_0, Z_t)$, we have

$$\begin{aligned} E\big\{g'(Z_0)^2 (Z_t - Z_0)^2\big\} &= E\Big[g'(Z_0)^2 E\big\{(Z_t - Z_0)^2 | Z_0\big\}\Big] \\ &= (1 - \gamma)\Big[(1 + \gamma) E\big\{g'(Z_0)^2\big\} \\ &\qquad + (1 - \gamma) E\big\{g'(Z_0)^2 Z_0^2\big\}\Big] \\ &\sim 2(1 - \gamma) E\big\{g'(Z_0)^2\big\} \end{aligned}$$

as $t \to 0$. More simply, $E(Z_t - Z_0)^2 = 2(1 - \gamma)$. Combining the results from (4.4) down, we conclude that, as $t \to 0$,

$$E(X_t - X_0)^2 \sim 2(1 - \gamma) E\big\{g'(Z_0)^2\big\}. \qquad \square$$

PROOF OF THEOREM 3.2. The proof of Theorem 3.1, up to (4.2) and (4.3), may be followed as before. In particular, we have

$$\liminf_{t \to 0} E(X_t - X_0)^2 / E(Z_t - Z_0)^2 > 0.$$

In place of the argument leading to (4.4), we note that, if $0 < \zeta < 2$ and ζ is very small, then for each $p_1, p_2 > 1$ and $q_j = (1 - p_j^{-1})^{-1}$,

$$
\begin{aligned}
E(X_t - X_0)^2 &= E\Big\{|g'(Z_{t^*})|^{2-\zeta}|Z_t - Z_0|^{2-\zeta}|X_t - X_0|^{\zeta}\Big\} \\
&\le \Big\{E|g'(Z_{t^*})(Z_t - Z_0)|^{(2-\zeta)p_1}\Big\}^{1/p_1}\Big(E|X_t - X_0|^{\zeta q_1}\Big)^{1/q_1} \\
&\le \Big\{E|g'(Z_{t^*})|^{(2-\zeta)p_1p_2}\Big\}^{1/(p_1p_2)}\Big(E|Z_t - Z_0|^{(2-\zeta)p_1q_2}\Big)^{1/(p_1q_2)} \\
&\quad \times\Big(E|X_t - X_0|^{\zeta q_1}\Big)^{1/q_1}.
\end{aligned} \tag{4.5}
$$

Choose p_1, p_2 so close to 1 that $p_1p_2 = (2 - \frac{1}{2}\zeta)(2 - \zeta)^{-1}$. In view of (4.3) and the assumptions on g,

$$\sup_{0<t<1} E\Big\{|g'(Z_{t^*})|^{2-(1/2)\zeta}\Big\} < \infty$$

for each $0 < \zeta < 4$. Therefore, by (4.5),

$$
\begin{aligned}
E(X_t - X_0)^2 &= O\Big\{\Big(E|Z_t - Z_0|^{(2-\zeta)p_1q_2}\Big)^{1/p_1q_2}\Big\} \\
&= O\Big[\Big\{E(Z_t - Z_0)^2\Big\}^{1-(1/2)\zeta}\Big].
\end{aligned}
$$

Since $E(Z_t - Z_0)^2 \to 0$, then for all $\zeta > 0$,

$$E(X_t - X_0)^2/\Big\{E(Z_t - Z_0)^2\Big\}^{1-\zeta} \to 0$$

as $t \to 0$. □

Acknowledgments. We are grateful to Dr. A. G. Constantine of CSIRO for supplying the data that are the subject of Figures 1 and 2, to R. J. Murison for assistance in preparing the figures and to two anonymous reviewers for their very constructive comments on an earlier version of the paper.

REFERENCES

ADLER, R. J. (1981). *The Geometry of Random Fields*. Wiley, New York.

BERRY, M. V. and HANNAY, J. H. (1978). Topography of random surfaces. *Nature* **273** 573.

BROWN, C. A., CHARLES, P. D., JOHNSEN, W. A. and CHESTER, S. (1993). Fractal analysis of topographic data by the patchwork method. *Wear* **161** 61–67.

BURROUGH, P. A. (1981). Fractal dimensions of landscapes and other environmental data. *Nature* **19** 240–242.

CARTER, P. H., CAWLEY, R. and MAULDIN, R. D. (1988). Mathematics of dimension measurement for graphs of functions. In *Fractal Aspects of Materials* (D. A. Weitz, L. M. Sander and B. B. Mandelbrot, eds.). Materials Research Society, Pittsburgh.

CONSTANTINE, A. G. and HALL, P. (1993). Characterizing surface roughness by estimation of effective fractal dimension. *J. Roy. Statist. Soc. Ser. B*. To appear.

COSTER, M. and CHERMANT, J. L. (1983). Recent developments in quantitative fractography. *Internat. Metals Rev.* **28** 234–238.

DELFINER, P. and DELHOMME, J. P. (1975). Optimal interpolation by kriging. In *Display and Analysis of Spatial Data* (J. C. Davis and M. J. McCullagh, eds.) 96–114. Wiley, London.

JOURNEL, A. G. and HUIJBREGTS, CH. J. (1978). *Mining Geostatistics*. Academic, London.

LEADBETTER, M. R., LINDGREN, G. and ROOTZÉN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, New York.

LING, F. F. (1987). Scaling law for contoured length of engineering surfaces. *J. Appl. Phys.* **62** 2570–2572.

LING, F. F. (1989). The possible role of fractal geometry in tribology. *Tribology Trans.* **32** 497–505.

LING, F. F. (1990). Fractals, engineering surfaces and tribology. *Wear* **136** 141–156.

MANDELBROT, B. B., PASSOJA, D. E. and PAULLAY, A. J. (1984). Fractal character of fracture surfaces of metals. *Nature* **308** 721–722.

MAJUMDAR, A. and BHUSHAN, B. (1991). Fractal model of elastic–plastic contact between rough surfaces. *J. Tribology* **113** 1–11.

SAYLES, R. S. and THOMAS, T. R. (1978). Surface topography as a nonstationary random process. *Nature* **271** 431–434.

SERRA, J. (1968). Les structures gigogles—morphologie, mathematique et interpretation metallogenique. *Miner. Deposita* **3** 135–154.

SMITH, R. L. (1992). Estimating dimension in noisy chaotic time series. *J. Roy. Statist. Soc. Ser. B* **54** 329–351.

TAYLOR, C. C. and TAYLOR, S. J. (1991). Estimating the dimension of a fractal. *J. Roy. Statist. Soc. Ser. B* **53** 353–364.

TAYLOR, S. J. (1955). The α-dimensional measure of the graph and set of zeros of a Brownian path. *Proc. Cambridge Philos. Soc.* **51** 265–274.

THOMAS, T. R. and THOMAS, A. P. (1988). Fractals and engineering surface roughness. *Surface Topography* **1** 143–152.

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