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REGULAR VARIATION IN THE TAIL BEHAVIOUR OF SOLUTIONS OF RANDOM DIFFERENCE EQUATIONS

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Let Q and M be random variables with given joint distribution. Under some conditions on this joint distribution, there will be exactly one distribution for another random variable R, independent of (Q, M), with the property that $Q + MR$ has the same distribution as R. When M is nonnegative and satisfies some moment conditions, we give an improved proof that if the upper tail of the distribution of Q is regularly varying, then the upper tail of the distribution of R behaves similarly; this proof also yields a converse. We also give an application to random environment branching processes, and consider extensions to cases where $Q + MR$ is replaced by $\Psi(R)$ for random but nonlinear Ψ and where M may be negative.

1. Introduction. A random first order difference equation takes the form

$$R_{n+1} = Q_{n+1} + M_{n+1}R_n, \qquad n = 0, 1, 2, \dots,$$

where $\{(Q_n, M_n);\ n = 1, 2, 3, \dots\}$ are independent and identically distributed (i.i.d.) random pairs with some given joint distribution and R_0 is independent of these with some given starting distribution. If the joint distribution of Q_n and M_n (denoted by Q and M when there is no ambiguity) satisfies appropriate conditions, the distribution of R_n will converge as $n \to \infty$ to a limit that does not depend upon R_0, and that will be the unique solution to the random functional equation

$$R =_D Q + MR,$$

where $=_D$ denotes equality in distribution and, on the right-hand side, R is independent of (Q, M).

Kesten (1973) (generally working in many dimensions), Grincevičius (1975) and Goldie (1991) (working with a more general functional equation) have studied how the tail behaviour of the distribution of R is determined by the joint distribution of Q and M. They show essentially that if there exists $\kappa > 0$ such that $E|M|^\kappa = 1$, $E|M|^\kappa \log^+|M| < \infty$ and $E|Q|^\kappa < \infty$, then

$$P(|R| > t) \sim ct^{-\kappa} \quad \text{as } t \to \infty$$

for some $c > 0$. The key to this result is that when $|R_n|$ is large, $\{\log|R_n|\}$ may be compared to a random walk with negative mean increment $E \log|M|$, for

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which the asymptotic behaviour of the probability of reaching a high level is well known from renewal theory.

It will be noted that if the foregoing regularity conditions hold, then it is the distribution of M that is important in determining the tail behaviour of R (by defining κ). Theorem 1 of Grincevičius (1975) covers perhaps the most tractable case where κ does not exist, namely, where M is nonnegative, there exist $\beta > \alpha > 0$ such that $EM^{\alpha} < 1$ and $EM^{\beta} < \infty$, and the upper tail of the distribution of Q is regularly varying with index $-\alpha$. (Here κ cannot exist because to satisfy $EM^{\kappa} = 1$ it would have to be greater than α, and then $E|Q|^{\kappa} < \infty$ would not hold.) Grincevičius' Theorem 1 states that the upper tail of the distribution of R is also regularly varying and comparable to that of Q. Thus it is now the distribution of Q that is important in determining the tail behaviour of R.

More recently, Resnick and Willekens (1991) have generalised this result to the multidimensional case, although under the assumption that Q and M are independent. It may be possible using their definition of multivariate regular variation to extend the results of the present paper to more than one dimension, but this possibility is not pursued here.

Grincevičius' theorem is an elegant result but the proof given is unnecessarily complicated in places and appears flawed in its final stages. The theorem deserves a more streamlined and accurate proof. This is given in Section 2, attention being drawn to the point at which Grincevičius' proof seems lacking, and the technique used yields the added bonus of a converse to the theorem with little extra effort. It is a synthesis of the ideas of Grincevičius and those developed (in temporary ignorance of the earlier result) by the author.

In Section 3 we give an explicit example that shows that it is possible for both Q and M to play comparable roles, and in Section 4 we consider an application to random environment branching processes. In Section 5 we look at the possibility of extension to cases where $Q + MR$ is replaced by a random and not necessarily linear transformation $\Psi(R)$, as studied in some detail by Goldie (1991). In Section 6 we indicate how our basic result may be adapted to cases where M may be negative.

2. The basic result.

THEOREM 1. *Let (Q, M) be jointly distributed random variables with $E\log^{+}|Q| < \infty$, $P(M \geq 0) = 1$, $EM^{\alpha} < 1$ and $EM^{\beta} < \infty$ for some $\beta > \alpha > 0$. Let R be a random variable independent of (Q, M). Then there exists exactly one distribution for R such that $Q + MR$ has the same distribution as R. If R has this distribution and L is a function slowly varying at infinity, then the following two statements are equivalent*:

$$P(Q > t) \sim t^{-\alpha}L(t) \quad \text{as } t \to \infty$$

and

$$P(R > t) \sim \frac{1}{1 - EM^{\alpha}} t^{-\alpha} L(t) \quad \text{as } t \to \infty.$$

In order to prove this theorem, we first prove three preliminary lemmas.

LEMMA 1. *Let X be a random variable with*

$$P(X > t) = t^{-\alpha} L_1(t) \quad \text{for } t > 0,$$

where L_1 is slowly varying at infinity. Then given $\delta > 0$, there exists $K > 1$ such that

$$\frac{L_1(\lambda t)}{L_1(t)} \le \max\{\lambda^{\alpha}, K\lambda^{-\delta}\} \quad \text{for all } t > 0,\ \lambda > 0.$$

PROOF. If $\lambda \ge 1$, the fact that $L_1(\lambda t) \le \lambda^{\alpha} L_1(t)$ follows immediately from the fact that $P(X > t)$ is nonincreasing in t. By Potter's theorem [Bingham, Goldie and Teugels (1987), Theorem 1.5.6], given $A > 1$ and $\delta > 0$, there exists t_0 such that if $\lambda < 1$, then

$$\frac{L_1(\lambda t)}{L_1(t)} \le A\lambda^{-\delta} \quad \text{provided } \lambda t \ge t_0.$$

Also, since $1 \ge t^{-\alpha} L_1(t) \ge t_0^{-\alpha} L_1(t_0)$ on $(0, t_0]$, if $0 < \lambda t < t \le t_0$, then

$$\frac{L_1(\lambda t)}{L_1(t)} \le \frac{\lambda^{\alpha} t^{\alpha}}{t_0^{-\alpha} L_1(t_0) t^{\alpha}} = B\lambda^{\alpha} \le B\lambda^{-\delta},$$

say. Finally if $\lambda t < t_0 < t$, then

$$\begin{aligned} \frac{L_1(\lambda t)}{L_1(t)} &= \frac{L_1(\lambda t)}{L_1(t_0)} \cdot \frac{L_1(t_0)}{L_1(t)} \\ &\le B\left(\frac{\lambda t}{t_0}\right)^{-\delta} A\left(\frac{t_0}{t}\right)^{-\delta} = AB\lambda^{-\delta}. \end{aligned}$$ □

LEMMA 2 [An improved version of Grincevičius (1975), Lemma 1]. *If Y is a random variable with*

$$P(Y > t) \sim ct^{-\alpha} L(t) \quad \text{as } t \to \infty$$

for some constant $c > 0$ and function L slowly varying at infinity, and (Q, M) is as in Theorem 1, independent of Y, then

$$P(Q > t) \sim t^{-\alpha} L(t) \quad \text{as } t \to \infty$$

if and only if

$$P(Q + MY > t) \sim (1 + cEM^{\alpha}) t^{-\alpha} L(t) \quad \text{as } t \to \infty.$$

PROOF. For $t > 0$ write $P(Y > t) = t^{-\alpha}L_2(t)$, where L_2 is slowly varying and $L_2(t) \sim cL(t)$. We first choose $\varepsilon \in (0, 1)$ and start with the identity

$$\begin{aligned} P(Q + MY > t) &= P(Q > (1+\varepsilon)t) - P(Q > (1+\varepsilon)t, Q + MY \le t) \\ &\quad + P((1-\varepsilon)t < Q \le (1+\varepsilon)t, Q + MY > t) \\ &\quad + P(Q \le (1-\varepsilon)t, Q + MY > t). \end{aligned}$$

Write this as

$$J(t) = I_1(t) - I_2(t) + I_3(t) + I_4(t), \quad \text{say.}$$

Now estimate some terms on the right-hand side. First,

$$\begin{aligned} 0 \le I_2(t) &\le P(Q > (1+\varepsilon)t, MY < -\varepsilon t) \\ &\le P(M > t^{(\alpha+\beta)/2\beta}) + P(Q > (1+\varepsilon)t)P(Y < -\varepsilon t^{(\beta-\alpha)/2\beta}) \\ &= o(t^{-(\alpha+\beta)/2}) + o(P(Q > (1+\varepsilon)t)), \end{aligned}$$

using in the first term the fact that $EM^\beta < \infty$ and therefore $P(M > m) = o(m^{-\beta})$. Next,

$$I_4(t) = t^{-\alpha}L_2(t)E\left\{\frac{P(Y > M^{-1}(t-Q))}{t^{-\alpha}L_2(t)}; M > 0, Q \le (1-\varepsilon)t\right\},$$

where the expectation is taken over Q and M. The random variable in the brackets converges pointwise to M^α as $t \to \infty$ and is dominated by

$$\frac{P(Y > M^{-1}\varepsilon t)}{t^{-\alpha}L_2(t)},$$

which by Lemma 1 with $\delta = \beta - \alpha$ is in turn dominated by

$$\max\{1, K\varepsilon^{-\beta}M^\beta\}$$

for some $K > 0$. This random variable is integrable and so, by dominated convergence, we conclude that

$$I_4(t) \sim EM^\alpha t^{-\alpha}L_2(t) \sim cEM^\alpha t^{-\alpha}L(t) \quad \text{as } t \to \infty.$$

We are now in a position to prove the "only if" part of the lemma. Suppose that

$$P(Q > t) \sim t^{-\alpha}L(t).$$

Then

$$\begin{aligned} &I_1(t) \sim (1+\varepsilon)^{-\alpha}t^{-\alpha}L(t), \\ &I_2(t) = o(t^{-\alpha}L(t)) \quad \text{since } t^{-(\alpha+\beta)/2} = o(t^{-\alpha}L(t)), \\ &\quad 0 \le I_3(t) \le P((1-\varepsilon)t < Q \le (1+\varepsilon)t) \\ &\qquad \sim ((1-\varepsilon)t)^{-\alpha}L((1-\varepsilon)t) - ((1+\varepsilon)t)^{-\alpha}L((1+\varepsilon)t) \\ &\qquad \sim \{(1-\varepsilon)^{-\alpha} - (1+\varepsilon)^{-\alpha}\}t^{-\alpha}L(t). \end{aligned}$$

Thus

$$\limsup_{t\to\infty} \frac{J(t)}{t^{-\alpha}L(t)} \le (1+\varepsilon)^{-\alpha} - 0 + \left\{(1-\varepsilon)^{-\alpha} - (1+\varepsilon)^{-\alpha}\right\} + cEM^{\alpha}$$

and

$$\liminf_{t\to\infty} \frac{J(t)}{t^{-\alpha}L(t)} \ge (1+\varepsilon)^{-\alpha} - 0 + 0 + cEM^{\alpha}.$$

Now letting $\varepsilon \downarrow 0$ yields the required result.

Now consider the "if" part of the lemma. Suppose that

$$J(t) \sim (1 + cEM^{\alpha})t^{-\alpha}L(t).$$

Then subtracting off our known estimate of $I_4(t)$ yields that

$$I_1(t) - I_2(t) + I_3(t) \sim t^{-\alpha}L(t).$$

Since $I_1(t) + I_3(t) \ge P(Q > (1+\varepsilon)t)$, whereas $I_2(t)$ is bounded above by the sum of two terms, one of which is $o(t^{-\alpha}L(t))$ and the other of which is $o(P(Q > (1+\varepsilon)t))$, this simplifies to

$$I_1(t) + I_3(t) \sim t^{-\alpha}L(t).$$

However, $I_1(t) + I_3(t)$ is bounded between $P(Q > (1+\varepsilon)t)$ and $P(Q > (1-\varepsilon)t)$. It follows that

$$\limsup_{t\to\infty} \frac{P(Q > (1+\varepsilon)t)}{t^{-\alpha}L(t)} \le 1$$

and

$$\liminf_{t\to\infty} \frac{P(Q > (1-\varepsilon)t)}{t^{-\alpha}L(t)} \ge 1.$$

Changing variables yields easily that

$$(1-\varepsilon)^{\alpha} \le \liminf_{t\to\infty} \frac{P(Q>t)}{t^{-\alpha}L(t)} \le \limsup_{t\to\infty} \frac{P(Q>t)}{t^{-\alpha}L(t)} \le (1+\varepsilon)^{\alpha}.$$

Letting $\varepsilon \downarrow 0$ now completes the proof. □

LEMMA 3. *Under the conditions of Theorem* 1, *if*

$$P(Q > t) \sim t^{-\alpha}L(t) \quad \text{as } t \to \infty,$$

then there exists a random variable Z, *independent of* (Q, M), *with*

$$P(Z > t) \sim ct^{-\alpha}L(t) \quad \text{as } t \to \infty$$

for some $c > 0$, *such that*

$$Q + MZ \le_D Z,$$

where $\le_D$ *denotes "is stochastically not greater than."*

PROOF. Choose $c^* > (1 - EM^\alpha)^{-1}$ and let Y be any random variable satisfying

$$P(Y > t) \sim c^* t^{-\alpha} L(t) \quad \text{as } t \to \infty.$$

Then by Lemma 2,

$$P(Q + MY > t) \sim (1 + c^* EM^\alpha) t^{-\alpha} L(t) \quad \text{as } t \to \infty.$$

Hence, since $c^* > 1 + c^* EM^\alpha$, the inequality

$$P(Q + MY > t) \le P(Y > t)$$

is certainly true for $t \ge t_0$, say. Now let Z have the distribution of Y conditional on $Y \ge t_0$. Then Z possesses the required tail behaviour, where $c = c^*/P(Y \ge t_0)$. Also, for $t \ge t_0$,

$$\begin{aligned} P(Q + MZ > t) &= P(Q + MY > t \mid Y \ge t_0) \\ &\le \frac{P(Q + MY > t)}{P(Y \ge t_0)} \\ &\le \frac{P(Y > t)}{P(Y \ge t_0)} \\ &= P(Z > t). \end{aligned}$$

This inequality is also trivially true for $t < t_0$, since then $P(Z > t) = 1$. This completes the proof. □

PROOF OF THEOREM 1. It is known [Kesten (1973), Goldie (1991); see also the review of this topic in Vervaat (1979)] that if $\{(Q_n, M_n);\ n = 1, 2, 3, \dots\}$ are i.i.d. pairs with the given joint distribution of (Q, M) and the sequence $\{R_n;\ n = 1, 2, 3, \dots\}$ of random variables is defined recursively by

$$R_{n+1} = Q_{n+1} + M_{n+1} R_n, \qquad n = 0, 1, 2, \dots,$$

where R_0 is arbitrarily chosen independently of $\{(Q_n, M_n)\}$, then under the weak condition $E \log^+ |Q| < \infty$, the sequence $\{R_n\}$ converges in distribution and the limit distribution does not depend on R_0. Moreover, if R has the limit distribution, then the equation

$$R =_D Q + MR$$

is satisfied. This easily yields the existence and uniqueness.

The implication from R to Q in the theorem now follows immediately from putting $Y = R$ and $c = (1 - EM^\alpha)^{-1}$ in the "if" part of Lemma 2. It remains to prove the implication from Q to R.

The idea here is to bound R stochastically above and below by sequences of random variables with known tail behaviour. (It is here that Grincevičius' proof is unnecessarily complicated in the bounding above, and apparently incomplete in the bounding below.)

It is clear in the foregoing recursion that if R_0 can be found such that $R_0 \ge_D R_1$, then $\{R_n\}$ is a stochastically nonincreasing sequence, so that

$R_n \geq_D R$ for all n. We shall use this fact and the preceding results to bound the tail of the distribution of R from above.

Choose Z distributed as in Lemma 3 and let $R_0 = Z$. Then

$$R_1 = Q_1 + M_1 R_0 \leq_D R_0$$

and so $\{R_n\}$ is stochastically nonincreasing. By Lemma 2 and induction we have that

$$P(R_n > t) \sim c_n t^{-\alpha} L(t) \quad \text{for all } n$$

where $c_0 = c$ and $\{c_n\}$ is a sequence satisfying the linear recurrence relation

$$c_{n+1} = 1 + c_n E M^{\alpha} \quad \text{for } n = 0, 1, 2, \ldots$$

and having the limit

$$c_\infty = \frac{1}{1 - EM^{\alpha}}.$$

It follows from $P(R > t) \leq P(R_n > t)$ that

$$\limsup_{t \to \infty} \frac{P(R > t)}{t^{-\alpha} L(t)} \leq \frac{1}{1 - EM^{\alpha}}.$$

To obtain a lower bound we use the slightly different argument that if $R_0 \leq_D R$, then $R_n \leq_D R$ for all n. (It does not seem easy to find a stochastically nondecreasing sequence.)

Note that $P(R > 0) > 0$. This is because, if it were not true, since R is certainly not degenerate at zero, there would be some negative t such that $P(Q + Mt \leq 0) = 1$, contradicting the fact, implicit in our assumptions, that the upper tail of the distribution of M is of smaller order than that of Q. Hence, the following statement is nonvacuous: for all t,

$$P(R > t) = P(Q + MR > t) \geq P(R > 0) P(Q > t).$$

If we choose R_0 such that

$$P(R_0 > t) = P(R > 0) P(Q > t) \quad \text{for } t \geq 0$$

and

$$P(R_0 > t) = P(R > t) \quad \text{for } t < 0$$

[which makes $1 - P(R_0 > t)$ a legitimate distribution function], then clearly $R_0 \leq_D R$. By Lemma 2 and induction we again have

$$P(R_n > t) \sim c_n t^{-\alpha} L(t) \quad \text{for all } n,$$

where now $c_0 = P(R > 0)$, but the sequence $\{c_n\}$ satisfies the same linear recurrence relation as before and has the same limit

$$c_\infty = \frac{1}{1 - EM^{\alpha}}.$$

It follows from $P(R > t) \geq P(R_n > t)$ that

$$\liminf_{t\to\infty} \frac{P(R > t)}{t^{-\alpha}L(t)} \geq \frac{1}{1 - EM^{\alpha}}.$$

Putting these two bounds together now completes the proof that

$$P(R > t) \sim \frac{1}{1 - EM^{\alpha}} t^{-\alpha}L(t). \qquad \square$$

3. Another example. Rarely will it be possible to obtain an explicit solution to the functional equation

$$R =_D Q + MR,$$

but there is one obvious nontrivial case where it is possible: If Q and M are independent with, respectively, gamma distribution $\mathrm{Ga}(\alpha_2, \beta)$ and beta distribution $\mathrm{Be}(\alpha_1, \alpha_2)$, then it is easy to check using standard distribution theory that the preceding equation is satisfied when R has gamma distribution $\mathrm{Ga}(\alpha_1 + \alpha_2, \beta)$. [See Tollar (1988) for further properties of this model; also Letac (1986).] This distribution has the property that

$$P(R > t) \sim \beta^{-1} t^{\alpha_1 + \alpha_2 - 1} e^{-\beta t} \quad \text{as } t \to \infty$$

and so the presence of all three parameters α_1, α_2 and β (assumed positive) in the exponents indicates that the distributions of Q and M both play a role here. Note also that the tails of the distributions of Q and R are not comparable as they were in Section 2. This is a case where $P(0 < M < 1) = 1$, which is the reason for the nonexistence of $\kappa > 0$ satisfying $EM^{\kappa} = 1$.

4. An application to random environment branching processes. In a branching process in i.i.d. environments [Smith and Wilkinson (1969)], suppose that all possible family size distributions are modified geometric with fractional linear probability generating functions of the form

$$f(s) = \frac{A + (1 - A - B)s}{1 - Bs} \quad \text{for } s \in [0, 1],$$

where $A, B \in [0, 1)$. We have written the parameters A and B in capital letters to emphasize that in random environments they become jointly distributed random variables. In Grey and Lu (1994) it was shown that if $q(\bar{\zeta})$ is the random variable denoting the probability of ultimate extinction starting with a single ancestor conditional on environment sequence $\bar{\zeta}$, then $R = q(\bar{\zeta})/(1 - q(\bar{\zeta}))$ satisfies the random functional equation

$$R =_D Q + MR,$$

where

$$(Q, M) =_D \left(\frac{A}{1 - A}, \frac{1 - B}{1 - A}\right).$$

Since the mean family size and probability of zero family size are given, respectively, by

$$f'(1) = \frac{1-A}{1-B} \quad \text{and} \quad f(0) = A,$$

it follows that the conditions $EM^{\alpha} < 1$, $EM^{\beta} < \infty$ and $P(Q > t) > t^{-\alpha}L(t)$ translate into

$$Ef'(1)^{-\alpha} < 1,$$
$$Ef'(1)^{-\beta} < \infty$$

and

$$P(f(0) > 1 - v) \sim v^{\alpha}L(v^{-1}) \quad \text{as } v \downarrow 0.$$

If these conditions hold then it follows from Theorem 1 that

$$P\big(q(\bar{\zeta}) > 1 - v\big) \sim \{1 - Ef'(1)^{-\alpha}\}^{-1} v^{\alpha}L(v^{-1}) \quad \text{as } v \downarrow 0.$$

If $q_k = Eq(\bar{\zeta})^k$ denotes the unconditional probability of extinction starting with k ancestors, it then follows from an Abelian theorem that

$$q_k \sim \Gamma(\alpha + 1)\{1 - Ef'(1)^{-\alpha}\}^{-1} k^{-\alpha}L(k) \quad \text{as } k \to \infty.$$

The qualitative message of this result is that if the frequency of dangerously bad environments (where the probability of zero family size is close to 1) is sufficiently high, then it is this frequency that is important in determining the behaviour of the extinction probability for large initial population sizes.

5. Extension to nonlinear transformations. The main development of Goldie (1991) is to show that under the "existence of κ" condition we can replace the random linear transformation $t \to Q + Mt$ by a much more general one, $t \to \Psi(t)$, provided that in a carefully defined sense $\Psi(t) \simeq Mt$ for large t. It is worthwhile considering whether a similar thing can be done under our conditions. One possibility is the following.

THEOREM 2. *Let Ψ be a random nondecreasing function that may be expressed as*

$$\Psi(t) = Q + Mt + N(t),$$

where Q, M and $N(t)$ are random variables determined by Ψ, (Q, M) satisfies the conditions of Theorem 1, *the range of $\Psi(t)$ is unbounded above for all real t and*

$$|N(t)| \le N\phi(t),$$

where N is a nonnegative random variable satisfying $EN^{\beta} < \infty$ and ϕ is a fixed nondecreasing nonnegative function satisfying $\phi(t) = o(t)$ as $t \to \infty$. Let R be a random variable independent of Ψ. Then there exists exactly one distribution for R such that $\Psi(R)$ has the same distribution as R. If R has *this distribution and L is a function slowly varying at infinity, then*

$$P(Q > t) \sim t^{-\alpha}L(t) \quad \text{as } t \to \infty$$

if and only if

$$P(R > t) \sim \frac{1}{1 - EM^{\alpha}} t^{-\alpha}L(t) \quad \text{as } t \to \infty.$$

PROOF. We first show that if $\{\Psi_1, \Psi_2, \ldots\}$ is a sequence of i.i.d. random functions distributed as Ψ, and R_0 is suitably chosen and independent of these, then the sequence $\{R_0, R_1, R_2, \ldots\}$ defined recursively by

$$R_{n+1} = \Psi_{n+1}(R_n) \quad \text{for } n = 0, 1, 2, \ldots$$

converges in distribution and the limit distribution is unique.

Choose $c > 0$ such that $E(M + cN)^{\alpha} < 1$, and then t_0 such that

$$\phi(t) \le ct \quad \text{for } t \ge t_0.$$

Then

$$\Psi(t) \le Q + (M + cN)t \quad \text{for } t \ge t_0$$

and

$$\Psi(t) \le Q + N\phi(t_0) + Mt \quad \text{for } t \le t_0.$$

Hence

$$\Psi(t) \le Q^{+} + N\phi(t_0) + (M + cN)t \quad \text{for all } t \ge 0.$$

So if R_0 is chosen having the distribution (known to exist by Theorem 1) satisfying the random functional equation

$$R_0 =_D Q^{+} + N\phi(t_0) + (M + cN)R_0,$$

then R_0 is obviously nonnegative and so from the foregoing statements we have that

$$R_1 = \Psi_1(R_0) \le_D R_0.$$

Therefore, since $\Psi_1, \Psi_2, \ldots$ are nondecreasing functions, it follows easily that $\{R_n\}$ is a stochastically nonincreasing sequence.

In a similar way we may use

$$\Psi(t) \ge -\{Q + N\phi(0)\}^{-} + Mt \quad \text{for } t \le 0$$

to construct nonpositive R_0^* satisfying

$$R_0^* =_D -\{Q - N\phi(0)\}^{-} + MR_0^*$$

such that

$$R_1^* = \Psi_1(R_0^*) \ge_D R_0^*$$

and the continuing sequence $\{R_n^*\}$ is stochastically nondecreasing.

Since $R_0^* \le_D R_0$, it follows that

$$R_0^* \le_D R_n^* \le_D R_n \le_D R_0 \quad \text{for all } n$$

and therefore that each of $\{R_n\}$ and $\{R_n^*\}$ converges in distribution to a limit that is an equilibrium distribution for the Markov process with transition functions given by those of $\{R_n\}$ or $\{R_n^*\}$.

To show that such a distribution is unique, we use a coupling argument [see, for instance, Pitman (1974)]. Suppose π and π^* are any two equilibrium distributions and redefine $\{R_n\}$ and $\{R_n^*\}$ as processes with R_0 and R_0^* independent with distributions π and π^*, respectively, governed by sequences $\{\Psi_n\}$ and $\{\Psi_n^*\}$, which are independent of each other and of R_0 and R_0^*, up to the random time

$$T = \inf\{n \geq 0; R_n \leq R_n^*\}$$

and governed by a common sequence (say $\{\Psi_n\}$) thereafter. If we can show that $P(T < \infty) = 1$, since $R_n \leq R_n^*$ for $n \geq T$, it follows, letting $n \to \infty$, that $\pi \leq_D \pi^*$. Similarly $\pi \geq_D \pi^*$. To show that $P(T < \infty) = 1$, we note that up to time T the bivariate process $\{(R_n, R_n^*)\}$ is a Markov process with equilibrium distribution $\pi \times \pi^*$. Also, from the tail conditions imposed on Q, M and N, it is not hard to see that for any real s and t,

$$P(\Psi(s) \leq \Psi^*(t)) > 0,$$

where Ψ and Ψ^* are independent. Hence any invariant class of states for the bivariate process contains points of the form (x, y) with $x \leq y$, and so almost surely one such state is eventually visited.

NOTE. Some conditions, such as our tail conditions, are crucial here. Goldie has pointed out, using the example $\Psi(t) = [Q + Mt]$, where square brackets denote integer part, Q is degenerate at $\frac{1}{2}$ and M takes the values $\frac{1}{2}$ and $\frac{5}{4}$ each with probability $\frac{1}{2}$, that uniqueness of equilibrium distribution does not hold more generally. This corrects his assertion [Goldie (1991), Section 8] of uniqueness for the example $\Psi(t) = [Q + Mt]$.

It remains to establish the relationship between the tail behaviour of Q and that of R. We first note that an exact analog of Lemma 2 holds with $Q + MY$ replaced by $\Psi(Y)$. To show this in the "only if" direction, one may use the "only if" part of Lemma 2 and then the inequality

$$\begin{aligned}|P(\Psi(Y) > t) - P(Q + MY > t)|\\ \leq P((1-\varepsilon)t \leq Q + MY \leq (1+\varepsilon)t) + P(|N(Y)| > \varepsilon t),\end{aligned}$$

which holds for $\varepsilon \in (0, 1)$. Dividing by $t^{-\alpha}L(t)$, letting $t \to \infty$ and then $\varepsilon \downarrow 0$ establishes that $\Psi(Y)$ and $Q + MY$ have asymptotically equivalent tail behaviour. For this we need to know that

$$\begin{aligned}P(|N(Y)| > \varepsilon t) &\leq P(N\phi(Y) > \varepsilon t)\\ &= o(t^{-\alpha}L(t)),\end{aligned}$$

which follows easily from the way in which N and ϕ have been constructed. In the converse direction, use the similar inequality

$$|P(\Psi(Y) > t) - P(Q + MY > t)|$$
$$\le P((1 - \varepsilon)t \le \Psi(Y) \le (1 + \varepsilon)t) + P(|N(Y)| > \varepsilon t)$$

and then the "if" part of Lemma 2.

In Theorem 2, the implication from R to Q now follows immediately, as in the proof of Theorem 1. The implication from Q to R also follows similarly. For upper bounding, the exact analog of Lemma 3 [replacing $Q + MZ$ by $\Psi(Z)$] works. For lower bounding, we may choose R_0 such that

$$P(R_0 > t) = P(R > 0)P(\Psi(0) > t) \quad \text{for } t \ge 0$$

and

$$P(R_0 > t) = P(R > t) \quad \text{for } t < 0,$$

noting (which is an easy exercise) that $\Psi(0) = Q + N(0)$ has asymptotically the same upper tail behaviour as Q.

This completes the proof. □

EXAMPLE 1. The transformation

$$\Psi(t) = Q + Mt + N\sqrt{t},$$

considered in some detail by Goldie (1991) under his regularity conditions, fits easily in an obvious way into our model.

EXAMPLE 2. A transformation such as

$$\Psi(t) = \max\{Q, Mt\}$$

does not fit the generalisation that we have chosen, but it is not hard to see that this transformation yields to a direct proof of the equivalent of Theorem 1, using similar but easier techniques. The details are omitted.

6. Extension to not necessarily positive M. We have so far assumed that M is nonnegative mainly in order to be able to employ various monotonicity arguments in the proofs. However, it is possible to obtain similar results if we drop the assumption $P(M \ge 0) = 1$ and merely assume that $E|M|^{\alpha} < 1$ and $E|M|^{\beta} < \infty$. We now have to take into account the simultaneous effects of the upper and lower tails of the distribution of Q on those of R. Leaving aside the question of whether converses exist, we here state analogs of Lemma 2 and Theorem 1, and in each case give an indication of how the proof proceeds.

LEMMA 4. *Let Q, M be random variables with $E|M|^{\alpha} < 1$, $E|M|^{\beta} < \infty$ for some $\beta > \alpha > 0$, and the tails of the distribution of Q satisfying*

$$P(Q > t) \sim t^{-\alpha}L(t),$$

and

$$P(-Q > t) \sim bt^{-\alpha}L(t) \quad \text{as } t \to \infty$$

for some $b \geq 0$. Let Y be independent of (Q, M) with

$$\limsup_{t \to \infty} \frac{P(Y > t)}{t^{-\alpha}L(t)} = c_+$$

and

$$\limsup_{t \to \infty} \frac{P(-Y > t)}{t^{-\alpha}L(t)} = c_-$$

for some $c_+, c_- \geq 0$. Then, writing $\mu_+ = E(M^+)^\alpha$ and $\mu_- = E(M^-)^\alpha$,

$$\limsup_{t \to \infty} \frac{P(Q + MY > t)}{t^{-\alpha}L(t)} \leq 1 + c_+\mu_+ + c_-\mu_-$$

and

$$\limsup_{t \to \infty} \frac{P(-Q - MY > t)}{t^{-\alpha}L(t)} \leq b + c_-\mu_+ + c_+\mu_-.$$

A similar result holds with limit superior replaced throughout by limit inferior, and the inequalities in the conclusion reversed.

SKETCH OF THE PROOF. We consider the upper tail only; the lower tail follows similarly. The decomposition of $P(Q + MY > t)$ used in the proof of Lemma 2 still works; $I_4(t)$ must now be split up into two terms, corresponding to the events $\{M > 0\}$ and $\{M < 0\}$, and a similar analysis leads to

$$\limsup_{t \to \infty} \frac{I_4(t)}{t^{-\alpha}L(t)} \leq c_+\mu_+ + c_-\mu_-.$$

The remaining terms in the decomposition are dealt with in a similar manner to that used in the proof of Lemma 2. The corresponding result for limits inferior follows in an analogous way. □

THEOREM 3. *Let Q and M be as in Lemma 4, and let R be a random variable with the unique distribution satisfying*

$$R =_D Q + MR.$$

Then

$$P(R > t) \sim C_+ t^{-\alpha}L(t)$$

and

$$P(-R > t) \sim C_- t^{-\alpha}L(t) \quad \text{as } t \to \infty,$$

where

$$C_+ = \frac{1}{2}\left\{\frac{1 + b}{1 - \mu_+ - \mu_-} + \frac{1 - b}{1 - \mu_+ + \mu_-}\right\}$$

and

$$C_- = \frac{1}{2}\left\{\frac{1+b}{1-\mu_+-\mu_-} - \frac{1-b}{1-\mu_++\mu_-}\right\}.$$

SKETCH OF THE PROOF. Define

$$C_+ = \limsup_{t\to\infty} \frac{P(R>t)}{t^{-\alpha}L(t)}$$

and

$$C_- = \limsup_{t\to\infty} \frac{P(-R>t)}{t^{-\alpha}L(t)}.$$

We know that these are finite from the easily verifiable fact that

$$-R^* \le_D R \le_D R^*,$$

where R^* uniquely satisfies the equation

$$R^* =_D |Q| + |M|R^*.$$

Applying Lemma 4 yields

$$C_+ \le 1 + C_+\mu_+ + C_-\mu_-$$

and

$$C_- \le b + C_-\mu_+ + C_+\mu_-,$$

whence, in particular, after adding and rearranging,

$$C_+ + C_- \le \frac{1+b}{1-\mu_+-\mu_-}.$$

By a similar application of Lemma 4, the sum of the corresponding limits inferior satisfies the opposite inequality. It follows that C_+ and C_- are limits rather than merely limits superior, and that they satisfy two simultaneous linear equations that may be solved to give the expressions in the statement of the theorem. □

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