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STATE-DEPENDENT CRITERIA FOR CONVERGENCE OF MARKOV CHAINS[1]

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The standard Foster–Lyapunov approach to establishing recurrence and ergodicity of Markov chains requires that the one-step mean drift of the chain be negative outside some appropriately finite set. Malyšhev and Men'šikov developed a refinement of this approach for countable state space chains, allowing the drift to be negative after a number of steps depending on the starting state. We show that these countable space results are special cases of those in the wider context of φ-irreducible chains, and we give sample-path proofs natural for such chains which are rather more transparent than the original proofs of Malyšhev and Men'šikov. We also develop an associated random-step approach giving similar conclusions. We further find state-dependent drift conditions sufficient to show that the chain is actually geometrically ergodic; that is, it has n-step transition probabilities which converge to their limits geometrically quickly. We apply these methods to a model of antibody activity and to a nonlinear threshold autoregressive model; they are also applicable to the analysis of complex queueing models.

1. Background. We consider a time-homogeneous Markov chain $\mathbf{\Phi} = \{\Phi_n,\ n \in \mathbb{Z}_+\}$ evolving on a general space X, equipped with a σ-algebra $\mathscr{B}(\mathsf{X})$. The transition probabilities of $\mathbf{\Phi}$ are defined by a Markov transition function denoted by $P = \{P(x,\ A),\ x \in \mathsf{X},\ A \in \mathscr{B}(\mathsf{X})\}$, with iterates

$$P^n(x,\ A) = \mathsf{P}_x(\Phi_n \in A), \qquad n \in \mathbb{Z}_+,\ x \in \mathsf{X},\ A \in \mathscr{B}(\mathsf{X}).$$

We will assume that $\mathbf{\Phi}$ is φ-irreducible, that is, there exists a finite measure φ such that $\sum_n P^n(x,\ A) > 0$ for all $x \in \mathsf{X}$ whenever $\varphi\{A\} > 0$. Such Markov chains in the general state space setting are discussed in [12] or [10].

Our goal is to develop new criteria for such a chain to be Harris recurrent, positive Harris recurrent (which is often also called, under slight extra conditions, Harris ergodic) and geometrically ergodic. The following are minimal definitions of these stability concepts.

If the stopping time σ_A is defined for a set $A \in \mathscr{B}(\mathsf{X})$ by $\sigma_A = \inf\{k \geq 0\colon \Phi_k \in A\}$, then the φ-irreducible chain $\mathbf{\Phi}$ is called *Harris recurrent* if $\mathsf{P}_x(\sigma_A < \infty) \equiv 1$ for all $x \in \mathsf{X}$ whenever $\varphi\{A\} > 0$. It is easy to see that this is

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equivalent to $\mathsf{P}_x(\tau_A < \infty) \equiv 1$, where $\tau_A = \inf\{k \geq 1 \colon \Phi_k \in A\}$, and this is often used as an alternative definition.

A σ-finite measure π on $\mathscr{B}(\mathsf{X})$ with the property

$$\pi\{A\} = \pi P\{A\} \triangleq \int \pi(dx) P(x, A), \qquad A \in \mathscr{B}(\mathsf{X}),$$

is called *invariant*. It is well-known (see Chapter 10 of [10]) that if $\mathbf{\Phi}$ is Harris recurrent, then a unique (up to constant multiples) invariant measure π exists.

We define a Harris recurrent chain to be *positive Harris recurrent* if the measure π is finite.

If we also assume that the chain is *aperiodic* ([10], Chapter 5), then the existence of such a finite measure is equivalent (see Theorem 13.0.1 of [10]) to

$$\|P^n(x, \cdot) - \pi\| \to 0, \tag{1}$$

for all x as $n \to \infty$, where $\|\cdot\|$ denotes the total variation norm.

Finally, an aperiodic φ-irreducible positive Harris chain is called *geometrically ergodic* if the convergence in (1) occurs geometrically quickly, in which case there exists a function $M(x)$ and a $\rho < 1$ independent of x such that

$$\|P^n(x, \cdot) - \pi\| \leq M(x) \rho^n, \tag{2}$$

for all x and all $n \in \mathbb{Z}_+$. (See Chapters 15 and 16 of [10] for details.)

In this paper we develop state-dependent drift criteria for such forms of stability. These extend a number of state-independent forms which have proved of wide application.

For φ-irreducible chains it is known that recurrence is implied by the existence of a suitably unbounded function V and a suitably finite set C (where these finiteness conditions are made more explicit below) satisfying

$$\int P(x, dy) V(y) \leq V(x), \qquad x \in C^c, \tag{3}$$

and indeed the existence of such a solution to (3) is equivalent to Harris recurrence under some extra conditions (see, e.g., [7] and [10]).

Positive recurrence is equivalent [18, 10] to the existence of a solution to Foster's criterion: there exist some $V \geq 0$ and some suitably finite set C such that

$$\int P(x, dy) V(y) \leq \begin{cases} V(x) - 1, & x \in C^c, \\ b < \infty, & x \in C. \end{cases} \tag{4}$$

It is perhaps less well known that the existence of a solution $V \geq 1$ to

$$\int P(x, dy) V(y) \leq \begin{cases} \lambda V(x), & x \in C^c, \\ b < \infty, & x \in C \end{cases} \tag{5}$$

for some $\lambda < 1$ and some suitably finite set C, first developed by Popov [14] for countable space chains, is equivalent to geometric ergodicity, and that $M = RV$ then provides the bound in (2) for some constant $R < \infty$; see [19], [5],

[11] or [10] for detailed approaches to these conditions and for considerable strengthenings of this brief description.

Specifically, these criteria are defined for general chains in terms of sets C which are *petite*. These play the part taken by finite sets in the countable space context, in the sense that drift toward petite sets suffices for recurrence and ergodicity conditions to hold.

A set $C \in \mathscr{B}(\mathsf{X})$ is called petite if there exist some nontrivial measure ν and some probability distribution a on $\mathbb{Z}_+$ such that

$$\sum_n a(n) P^n(x, B) \geq \nu(B), \qquad x \in C,\ B \in \mathscr{B}(\mathsf{X}). \tag{6}$$

Petite sets generalise the "small sets" of Nummelin [12]. For φ-irreducible chains, it is known ([12] or Proposition 5.5.5 of [10]) that there is a countable cover of X by small sets; and in [11] or Chapter 6 of [10], relatively weak continuity conditions are found under which all compact sets are petite if the space X admits a suitable topology. Thus it is not unnatural that drift toward petite sets might provide the results we seek.

We will call a function V *unbounded off petite sets* if the sublevel sets $\{V \leq v\}$ are petite for each $v \in \mathbb{R}_+$: such functions generalise functions which tend to infinity in the case $\mathsf{X} = \mathbb{Z}_+$.

Although the equivalences in (3)–(5) have proved powerful in analyzing many practical models, for more complex models the analysis of the one-step drift

$$\Delta V(x) = \int P(x, dy)[V(y) - V(x)]$$

toward petite sets is often somewhat difficult; see [9], where drift functions which depend on the whole history of the chain are used, or [17] or [3], where m-step skeletons $\{\Phi_{mk}\}$ are analysed.

In this paper, we consider consequences of state-dependent drift conditions of the form

$$\int P^{n(x)}(x, dy) V(y) \leq g[V(x), n(x)], \qquad x \in C^c, \tag{7}$$

where $n(x)$ is a measurable function from X to $\mathbb{Z}_+$, g is a function depending on which type of stability we seek to establish and C is a petite set. We also consider state-dependent drift conditions of the form

$$\int \sum_n a_x(n) P^n(x, dy) V(y) \leq g[V(x)], \qquad x \in C^c, \tag{8}$$

where $a_x(n)$ is a distribution on $\mathbb{Z}_+$, and again g is an appropriate function and C is a petite set. This allows the analysis of models by using drifts at times which are not only state dependent, but which may in fact be random rather than fixed.

In Section 4 these approaches enable the analysis of different types of models, including competition models, nonlinear SETAR autoregressions and multidimensional queueing models.

2. State-dependent drifts and the embedded chain $\overline{\mathbf{\Phi}}$. The results we prove for fixed state-dependent times are summarised in Theorem 2.1, and those for random state-dependent times are in Theorem 2.2. In the countable space context, Theorem 2.1(i) was first shown as Theorem 1.3 and Theorem 2.1(ii) as Theorem 1.4 of [6]. The proofs there, especially of Theorem 2.1(ii), are somewhat more complex than those we exhibit below, which are based on sample path arguments. Theorem 2.1(iii) is new, even for countable space chains.

THEOREM 2.1. *Suppose that $\mathbf{\Phi}$ is a φ-irreducible chain on X, and let $n(x)$ be a measurable function from X to $\mathbb{Z}_+$.*

(i) *The chain is Harris recurrent if there exist a nonnegative function V unbounded off petite sets and some petite set C satisfying*

$$\int P^{n(x)}(x, dy)V(y) \le V(x), \qquad x \in C^c. \tag{9}$$

(ii) *The chain is positive Harris recurrent if there exist some petite set C, a nonnegative function V bounded on C and a positive constant b satisfying*

$$\int P^{n(x)}(x, dy)V(y) \le V(x) - n(x) + b\mathbf{1}_C(x), \qquad x \in \mathsf{X}, \tag{10}$$

in which case, for all x,

$$\mathsf{E}_x[\tau_C] \le V(x) + b. \tag{11}$$

(iii) *The chain is geometrically ergodic if it is aperiodic and there exist some petite set C, a nonnegative function $V \ge 1$ and bounded on C, and positive constants $\lambda < 1$ and b satisfying*

$$\int P^{n(x)}(x, dy)V(y) \le \lambda^{n(x)}[V(x) + b\mathbf{1}_C(x)]. \tag{12}$$

When (12) *holds,*

$$\sum_n r^n \|P^n(x, \cdot) - \pi\| \le RV(x), \qquad x \in \mathsf{X}, \tag{13}$$

for some constants $R < \infty$ and $r > 1$.

The proof of this theorem is in Section 3.

In order to prove these results we work with an "embedded" chain $\overline{\mathbf{\Phi}}$. Let $n(x)$ be a measurable function from X to $\mathbb{Z}_+$, and define the new transition law $\{\overline{P}(x, A)\}$ by

$$\overline{P}(x, A) = P^{n(x)}(x, A), \qquad x \in \mathsf{X},\ A \in \mathscr{B}(\mathsf{X}), \tag{14}$$

and let $\bar{\Phi}$ be the corresponding Markov chain. This Markov chain may be constructed explicitly as follows. The time $n(x)$ is a (trivial) stopping time. Let $s(k)$ denote its iterates; that is, along any sample path, $s(0) = 0$, $s(1) = n(x)$ and

$$s(k+1) = s(k) + n(\Phi_{s(k)}).$$

Then it follows from the strong Markov property that

$$\bar{\Phi}_k = \Phi_{s(k)}, \qquad k \geq 0, \tag{15}$$

is a Markov chain with transition law $\bar{P}$.

Let $\bar{\mathscr{F}}_k = \mathscr{F}_{s(k)}$ be the σ-field generated by the events "before $s(k)$"; that is,

$$\bar{\mathscr{F}}_k \triangleq \{A: A \cap \{s(k) \leq n\} \in \mathscr{F}_n, n \geq 0\}.$$

We let $\bar{\tau}_A$ and $\bar{\sigma}_A$ denote the first return and first entry index to A, respectively, for the chain $\bar{\Phi}$. Clearly $s(k)$ and the events $\{\bar{\sigma}_A \geq k)$ and $\{\bar{\tau}_A \geq k\}$ are $\bar{\mathscr{F}}_{k-1}$-measurable for any $A \in \mathscr{B}(\mathsf{X})$.

Note that $s(\bar{\tau}_C)$ denotes the time of first return to C by the original chain Φ along an embedded path; that is,

$$s(\bar{\tau}_C) = \sum_{0}^{\bar{\tau}_C - 1} n(\bar{\Phi}_k) \tag{16}$$

and so from (15) we have for every y,

$$s(\bar{\tau}_C) \geq \tau_C, \qquad s(\bar{\sigma}_C) \geq \sigma_C, \quad \text{a.s. } [\mathsf{P}_y]. \tag{17}$$

These relations will enable us to use the drift equations (7), with which we will bound the index at which $\bar{\Phi}$ reaches C, to bound the hitting times on C by the original chain.

For random state-dependent times we will consider the chain whose next position when $\Phi_0 = x$ is given by Φ_{ζ_x}, where ζ_x is a random variable with distribution a_x on $\mathbb{Z}_+$; the results above are thus for the special case where ζ_x is degenerate at $n(x)$. We define the transition probabilities for this randomly sampled chain by

$$K_a(x, A) \triangleq \mathsf{P}_x(\Phi_{\zeta_x} \in A) = \sum_n a_x(n) P^n(x, A).$$

For any nonnegative function V let us define the function W by

$$W(x) = \mathsf{E}_x\left[\sum_{0}^{\zeta_x - 1} V(\Phi_k)\right].$$

In establishing state-dependent random time criteria for stability it will be necessary to have conditions such that

$$W(x) < \infty, \qquad x \in \mathsf{X}. \tag{18}$$

We will show one such common condition to be that

$$\sum_{i>k} a_x(i) \leq B_x a_x(k), \qquad k \in \mathbb{Z}_+; \tag{19}$$

if ζ_x is geometric or is uniform on $\{0, 1, \dots, M\}$ for some M, then (19) is satisfied, and so our results will hold for all such distributions. Note, however, that if ζ_x is degenerate at $n(x)$ (the situation in Theorem 2.1), then in fact (19) does not hold.

The random time result we will prove in Section 3 is as follows.

THEOREM 2.2. *Suppose that $\mathbf{\Phi}$ is a φ-irreducible chain on X, and let ζ_x be a random variable on $\mathbb{Z}_+$ for each x, independent of the chain.*

(i) *The chain is Harris recurrent if there exist a nonnegative function V unbounded off petite sets and some petite set C satisfying*

$$\int K_a(x, dy)V(y) \le V(x), \qquad x \in C^c. \tag{20}$$

(ii) *The chain is positive Harris recurrent if there exist some petite set C, a nonnegative function V bounded on C and a positive constant b satisfying*

$$\int K_a(x, dy)V(y) \le V(x) - 1 + b\mathbf{1}_C(x), \qquad x \in \mathsf{X}, \tag{21}$$

provided (18) *holds*; *in which case, for all* x,

$$\mathsf{E}_x[\tau_C] \le W(x) + b. \tag{22}$$

When (21) *holds, then* (18) *holds as required if a_x satisfies* (19).

(iii) *The chain is geometrically ergodic if it is aperiodic and there exist some petite set C, a nonnegative function $V \ge 1$ and bounded on C and positive constants $\lambda < 1$ and b satisfying*

$$\int K_a(x, dy)V(y) \le \lambda V(x) + b\mathbf{1}_C(x), \tag{23}$$

provided a_x satisfies (19) *with $B_x \le B < \infty$ independent of x. In this case*

$$\sum_n r^n \|P^n(x, \cdot) - \pi\|_V \le RV(x), \qquad x \in \mathsf{X}, \tag{24}$$

for some constants $R < \infty$ and $r > 1$, where the V-norm $\|\cdot\|_V$ is defined for any signed measure μ by

$$\|\mu\|_V \triangleq \sup_{|g| \le V} \int \mu(dy) g(y).$$

It seems reasonable to expect that both Theorems 2.1 and 2.2 are special cases of a more general result covering random times not necessarily meeting the growth condition (19). The correct generalization is not, however, obvious. In the light of the random time result one might well hope, for example, to replace (10) with

$$\int P^{n(x)}(x, dy)V(y) \le V(x) - 1 + b\mathbf{1}_C(x), \qquad x \in \mathsf{X}, \tag{25}$$

and thus have a less stringent criterion for ergodicity.

It is, however, not the case that this can be done. For consider a chain on the countable quadrant $\mathbb{Z}_+ \times \mathbb{Z}_+$, with deterministic motion up the jth column for $m(j)$ steps followed by a deterministic drop to (0, 0), and with such a drop from every height above $m(j)$ also, for specificity. That is, for each $j > 0$,

$$P((j,0),(j,1)) = 1, \qquad P((j,1),(j,2)) = 1, \dots,$$
$$P((j,m(j)-1),(j,m(j))) = 1,$$
$$P((j,m(j)),(0,0)) = 1, \quad P((j,m(j)+1),(0,0)) = 1, \dots.$$

Also assume that $P((0, k), (0, 0)) = 1$ for all $k > 0$.

Let $n(x)$ denote the (deterministic) first hitting time to (0, 0) for $x \neq (0, 0)$, and let $n(0, 0) = 1$. If we choose $V(j, k) = 1$ for all $(j, k) \neq (0, 0)$ and $V(0, 0) = 0$, then clearly for all $(j, k) \neq (0, 0)$ we have (25) satisfied.

Now choose

$$P((0,0),(j,0)) = a_j,$$

for some distribution a_j on $\mathbb{Z}_+$ with $a_0 = 0$. Then again (25) is satisfied, with $b = 2$.

However, we have

$$\mathsf{E}_{(0,0)}[\tau_{(0,0)}] = 1 + \sum_{j \geq 1} a_j [m(j) + 1],$$

so if $\sum_{j \geq 1} a_j m(j) = \infty$, the chain is not positive.

Although it seems that there should be a general result covering both the random time and the fixed time approaches, the form of such a result remains an interesting open question.

3. Hitting times and drift criteria for stability. In this section we give the form of some general hitting time results on which the proofs of Theorems 2.1 and 2.2 rely, and then give those proofs for each type of stability separately.

(*a*) *Harris recurrence.* The results that enable Harris recurrence to be verified in terms of hitting times on petite sets are as follows.

THEOREM 3.1. (i) *If* $\mathbf{\Phi}$ *is a* φ*-irreducible chain on* X, *then* $\mathbf{\Phi}$ *is Harris recurrent if there exists a petite set* C *with* $\mathsf{P}_y(\tau_C < \infty) = 1$ *for all* y.

(ii) *If* $\mathbf{\Phi}$ *is a* φ*-irreducible chain on* X, *then*

$$\mathsf{X} = H \cup N, \tag{26}$$

where $H = N^c$ *is either empty or absorbing, and in the latter case* $\mathbf{\Phi}$ *restricted to* H *is Harris recurrent and* $\varphi(N) = 0$. *If* $\mathbf{\Phi}$ *is not Harris recurrent, then the set* $N \in \mathscr{B}(\mathsf{X})$ *is nonempty and, for any petite set* $C \subseteq N$ *and every* $x \in N$,

$$\mathsf{P}_x(\Phi_k \in N, k = 0, 1, \dots) > 0, \tag{27}$$

$$\mathsf{P}_x(\Phi_k \in C \; i.o.) = 0. \tag{28}$$

PROOF. The first result follows from Proposition 9.1.6 of [10].

If the chain is φ-irreducible, then from Theorem 8.0.1 of [10] it is either transient or recurrent, and in the former case we can take $N = \mathsf{X}$; that theorem also shows the expected number of visits to every petite set is then uniformly bounded so that (28) follows immediately.

In the latter case, where the chain is recurrent but not Harris recurrent, Theorem 9.1.4 of [10] shows that a nonempty N satisfying (27) exists. If C is any petite set contained in N, then (28) follows from Proposition 9.1.6 of [10]. □

In the case of a countable space X, the alternative to Harris recurrence for irreducible chains is simply transience, of course, so in this case we can take $N = \mathsf{X}$ in this proposition, and every finite set is petite.

PROOF OF THEOREM 2.1(i) Define the chain $\overline{\Phi}$ as in (15). We can write (9), for every y and every k, as

$$\mathsf{E}\left[V\left(\overline{\Phi}_{k+1}\right)|\overline{\mathscr{F}}_k\right] \le V\left(\overline{\Phi}_k\right) \quad \text{a.s. } \left[\mathsf{P}_y\right],$$

when $\overline{\sigma}_C > k$, $k \in \mathbb{Z}_+$.

Let $U_i = V(\overline{\Phi}_i)\mathbf{1}\{\overline{\sigma}_C \ge i\}$. Using the fact that $\{\overline{\sigma}_C \ge k\} \in \overline{\mathscr{F}}_{k-1}$, we have that

$$\mathsf{E}\left[U_k|\overline{\mathscr{F}}_{k-1}\right] = \mathbf{1}\{\overline{\sigma}_C \ge k\}\mathsf{E}\left[V\left(\overline{\Phi}_k\right)|\overline{\mathscr{F}}_{k-1}\right] \le \mathbf{1}\{\overline{\sigma}_C \ge k\}V\left(\overline{\Phi}_{k-1}\right) \le U_{k-1}.$$

Hence $(U_k, \overline{\mathscr{F}}_k)$ is a positive supermartingale, so that there exists an almost surely finite random variable U_∞ such that $U_k \to U_\infty$ as $k \to \infty$. From the construction of U_i, either $\overline{\sigma}_C < \infty$, in which case $U_\infty = 0$, or $\overline{\sigma}_C = \infty$, in which case $\limsup_{k\to\infty} V(\overline{\Phi}_k) = U_\infty < \infty$ a.s.

Suppose that N in (26) is nonempty. Then on the set $N_\infty = \{\Phi_k \in N, k = 0, 1, \dots\}$, we have $V(\Phi_k) \to \infty$ a.s. for any initial measure; this follows from (28) and the fact that the sublevel sets of V are assumed petite. Necessarily, then, on this set $\lim_{k\to\infty} V(\overline{\Phi}_k) = \infty$ a.s., and so $\overline{\sigma}_C < \infty$ on N_∞. Hence also $s(\overline{\sigma}_C) < \infty$ on N_∞ and therefore $\sigma_C < \infty$ on N_∞.

Thus the chain is not transient with H empty, for if it were we would have, in contradiction, $\sigma_C < \infty$ P_x-a.s. for all $x \in N = \mathsf{X}$ and since C is petite the chain would be Harris recurrent from Theorem 3.1(i).

Now, since H is nonempty, there exists a petite subset D of H with $\varphi(D) > 0$, for $\varphi(N) = 0$ from Theorem 3.1(ii). For $x \in H$ we know from Harris recurrence on H that $\mathsf{P}_x(\sigma_D < \infty) = 1$. However, since H is absorbing, the event $N_\infty^c = \bigcup_m\{\Phi_k \in H, k = m, m+1, \dots\}$ and thus on N_∞^c we have $\sigma_D < \infty$ P_x-a.s. for all x.

Finally, we have that $E = D \cup C$ is itself petite from Proposition 5.5.5 of [10], and we have shown that $\sigma_E < \infty$ P_x-a.s. for all x; so the chain is Harris recurrent as required. □

The last two steps in this proof are required by the fact that we do not know that petite sets for $\mathbf{\Phi}$ are also petite for $\overline{\mathbf{\Phi}}$ in general. In the case of a countable space, we have (following the proof of Theorem 8.4.3 of [10]) that (9) immediately implies that $\bar{\sigma}_C < \infty$ since $V(j) \to \infty$ as $j \to \infty$; the result then follows immediately from the fact that $s(\bar{\sigma}_C) \geq \sigma_C$ a.s.

PROOF OF THEOREM 2.2(i) This follows exactly as in the previous proof, using the chain with transition law K_a. □

(*b*) *Positive recurrence.* The state-dependent drift criterion for positive recurrence is a direct consequence of the following results, taken from Theorems 13.0.1 and 14.2.2 of [10].

THEOREM 3.2. (i) *Suppose that* $\mathbf{\Phi}$ *is* φ*-irreducible. Then the chain is positive Harris recurrent with invariant probability* π *if and only if there exist some petite set* $C \in \mathscr{B}(\mathsf{X})$ *with* $\mathsf{P}_x(\tau_C < \infty) = 1$, *for all* x, *and* $M_C < \infty$ *such that*

$$\sup_{x \in C} \mathsf{E}_x[\tau_C] \leq M_C. \tag{29}$$

(ii) *Without any irreducibility or other conditions on* $\mathbf{\Phi}$, *if* V *and* f *are nonnegative measurable functions and*

$$\int P(x, dy) V(y) \leq V(x) - f(x) + b\mathbf{1}_C(x), \qquad x \in \mathsf{X}, \tag{30}$$

for some set C, *and some* $b < \infty$, *then for all* $x \in \mathsf{X}$,

$$\mathsf{E}_x\left[\sum_{k=0}^{\tau_C - 1} f(\Phi_k)\right] \leq V(x) + b. \tag{31}$$

PROOF OF THEOREM 2.1(ii) Again define the chain $\overline{\mathbf{\Phi}}$ as in (15). From (10) we can use Theorem 3.2(ii) for $\overline{\mathbf{\Phi}}$, with $f(x)$ taken as $n(x)$, to deduce that

$$\mathsf{E}_x\left[\sum_{k=0}^{\bar{\tau}_C - 1} n(\overline{\Phi}_k)\right] \leq V(x) + b. \tag{32}$$

However, we have by adding the lengths of the embedded times $n(x)$ along any sample path that from (16)

$$\sum_{k=0}^{\bar{\tau}_C - 1} n(\overline{\Phi}_k) = s(\bar{\tau}_C) \geq \tau_C.$$

Thus, from (32) and the fact that V is bounded on the petite set C, we have that $\mathbf{\Phi}$ is positive Harris using Theorem 3.2(i), and the bound (11) follows. □

It is worth noting explicitly that we have not had to establish the positive recurrence, or indeed even the irreducibility, of the chain $\overline{\mathbf{\Phi}}$ in order to prove positivity of $\mathbf{\Phi}$. The step that is crucial in our proof is the so-called f-regularity, for $f = n(x)$, given by the bound in Theorem 3.2(ii).

PROOF OF THEOREM 2.2(ii) When (18) holds we have

$$(33)\qquad \begin{aligned} \int P(x, dy)W(x) &= \mathsf{E}_x\left[\sum_{k=1}^{\zeta_x} V(\Phi_k)\right] \\ &= W(x) - V(x) + \mathsf{E}_x\left[V(\Phi_{\zeta_x})\right] \\ &\le W(x) - V(x) + \left[V(x) - 1 + b\mathbf{1}_C(x)\right], \end{aligned}$$

since (21) holds. Thus the chain is positive recurrent from Theorem 3.2, using (30) with $f \equiv 1$, and (22) holds from (31).

Suppose further that a_x satisfies (19). Then we have

$$(34)\qquad \begin{aligned} W(x) &= \sum_{i=0}^{\infty} a_x(i) \sum_{k=0}^{i-1} P^k V(x) \\ &= \sum_{k=0}^{\infty} P^k V(x)\left[\sum_{i=k+1}^{\infty} a_x(i)\right] \\ &\le B_x \sum_{k=0}^{\infty} P^k V(x) a_x(k) \\ &= B_x \mathsf{E}_x\left[V(\Phi_{\zeta_x})\right] \\ &\le B_x\left[V(x) + b\right], \end{aligned}$$

so that (18) holds as required. □

Note that if $B_x \le B$ independent of x in (19), then we have the bound

$$(35)\qquad \mathsf{E}_x[\tau_C] \le BV(x) + b.$$

Note also that this same proof establishes that if there exist some petite set C, nonnegative functions V, bounded on C, and f, and a positive constant b satisfying

$$(36)\qquad \int K_a(x, dy)V(y) \le V(x) - f(x) + b\mathbf{1}_C(x), \qquad x \in \mathsf{X},$$

then, provided (18) holds, we have f-regularity in the sense that (31) holds, and then (Theorem 14.0.1 of [10]) in the aperiodic case

$$\int P^n(x, dy)f(y) \to \int \pi(dy)f(y) < \infty, \qquad x \in \mathsf{X}.$$

This gives us a state-dependent criterion for the existence of moments of the stationary distribution π, and convergence of the time-dependent moments to $\int \pi(dy)f(y)$.

(c) *Geometric ergodicity.* We turn thirdly to geometric ergodicity. In the general state space case we have from Theorem 15.0.1 of [10] the following criteria for geometric ergodicity, which we shall use somewhat as we used Theorem 3.2 in proving positive recurrence.

THEOREM 3.3. *Suppose that* $\mathbf{\Phi}$ *is* φ*-irreducible and aperiodic. Then the following conditions are equivalent, and the chain is then called geometrically ergodic*:

(i) *The chain* $\mathbf{\Phi}$ *is positive Harris recurrent with invariant probability* π, *and, for some* $r > 1$, *some* $R(x) < \infty$ *and all* x,

$$\sum_n r^n \|P^n(x, \cdot) - \pi\| \le R(x). \tag{37}$$

(ii) *There exist some petite set* $C \in \mathscr{B}(\mathsf{X})$ *and* $\kappa > 1$ *such that, for all* $x \in \mathsf{X}$,

$$\mathsf{E}_x[\kappa^{\tau_C}] < \infty \tag{38}$$

and, for some $M_C < \infty$,

$$\sup_{y \in C} \mathsf{E}_y[\kappa^{\tau_C}] \le M_C. \tag{39}$$

(iii) *There exist a petite set* C, *constants* $b < \infty$ *and* $\beta > 0$ *and a finite function* $V \ge 1$ *satisfying*

$$\int P(x, dy)[V(y) - V(x)] \le -\beta V(x) + b\mathbf{1}_C(x), \qquad x \in \mathsf{X}. \tag{40}$$

PROOF OF THEOREM 2.1(iii) Suppose that (12) holds, and define

$$V'(x) = 2\left(V(x) - \tfrac{1}{2}\right) \ge 1.$$

Then we can write (12) as

$$\begin{aligned} \int \overline{P}(x, dy) V'(y) &\le \lambda^{n(x)}[2V(x) + 2b\mathbf{1}_C(x)] - 1 \\ &= \lambda^{n(x)}[V'(x) + 1 + 2b\mathbf{1}_C(x)] - 1. \end{aligned} \tag{41}$$

Thus without loss of generality we will assume that V itself satisfies

$$\int \overline{P}(x, dy) V(y) \le \lambda^{n(x)}[V(x) + 1 + b\mathbf{1}_C(x)] - 1. \tag{42}$$

Define the random variables

$$Z_k = \kappa^{s(k)} V(\overline{\Phi}_k)$$

for $k \in \mathbb{Z}_+$. It follows from (42) that, for $\kappa = \lambda^{-1}$, since $\kappa^{s(k+1)}$ is $\overline{\mathscr{F}}_k$-measurable,

$$\begin{aligned} \mathsf{E}\left[Z_{k+1} | \overline{\mathscr{F}}_k\right] &= \kappa^{s(k+1)} \mathsf{E}\left[V(\overline{\Phi}_{k+1}) | \overline{\mathscr{F}}_k\right] \\ &\le \kappa^{s(k+1)} \left\{\kappa^{-n(\Phi_k)}[V(\Phi_k) + 1 + b\mathbf{1}_C(\Phi_k)] - 1\right\} \\ &= Z_k - \kappa^{s(k+1)} + \kappa^{s(k)} + \kappa^{s(k)} b\mathbf{1}_C(\Phi_k). \end{aligned}$$

Using Dynkin's formula (see Proposition 11.3.2 of [10] for details) this gives

$$\mathsf{E}_x\left[\sum_{k=0}^{\bar\tau_C-1}\left[\kappa^{s(k+1)}-\kappa^{s(k)}\right]\right]\le Z_0(x)+\mathsf{E}_x\left[\sum_{k=0}^{\bar\tau_C-1}\kappa^{s(k)}b\mathbf{1}_C(\overline{\Phi}_k)\right].$$

Collapsing the sum on the left-hand side and using the fact that only the first term in the sum on the right-hand side is nonzero, we get

$$\mathsf{E}_x[\kappa^{s(\bar\tau_C)}-1]\le V(x)+b\mathbf{1}_C(x). \tag{43}$$

Since $V<\infty$ and V is assumed bounded on C, and again using the fact that $s(\bar\tau_C)\ge\tau_C$, we have from Theorem 3.3(ii) that the chain is geometrically ergodic.

The final bound in (13) comes from the fact that, for some r, an upper bound on the state-dependent constant term in the rate of convergence (37) is shown in Theorem 15.4.1 of [10] to be given by

$$\mathsf{E}_x[\kappa^{\tau_C}]\le\mathsf{E}_x[\kappa^{s(\bar\tau_C)}]\le(2+b)V(x)$$

since $V\ge 1$. □

For the random time result the proof is not so subtle, because of the conditions imposed on the growth of the distributions a_x.

Proof of Theorem 2.2(iii) If (19) holds with $B_x\le B$, then from (34) we have, for some B',

$$V(x)\le W(x)\le B'V(x), \tag{44}$$

since (23) is stronger than (21) and $V\ge 1$. Write (23) as

$$\int K_a(x,dy)V(y)\le V(x)-\varepsilon V(x)+b\mathbf{1}_C(x).$$

Then we have, as in (33),

$$\begin{aligned}\int P(x,dy)W(x)&=W(x)-V(x)+\mathsf{E}_x\big[V(\Phi_{\zeta_x})\big]\\&\le W(x)-V(x)+\big[V(x)-\varepsilon V(x)+b\mathbf{1}_C(x)\big]\\&\le W(x)[1-\varepsilon/B']+b\mathbf{1}_C(x),\end{aligned}\tag{45}$$

and thus the chain is geometrically ergodic, from Theorem 3.3.

It is then the case that the chain is W-uniformly ergodic in the sense that

$$\sum_n r^n\|P^n(x,\cdot)-\pi\|_W\le RW(x),\qquad x\in\mathsf{X}, \tag{46}$$

for some constants $R<\infty$ and $r>1$, from Theorem 15.0.1 of [10]; the V-uniform ergodicity then follows directly from the bounds in (44). □

4. Various applications. For the countable space results developed in [6], the applications we know of have not used the full power of the results of Theorem 2.1, but have typically been on orthants where negative drift can be established easily in the interior of the space and drift with a fixed number of steps established on the boundaries of the space. Thus for these models only a finite set of values of $n(x)$ is required, essentially corresponding to one-step drift in the interior of the orthant and multistep drift at each edge. In principal a single n-step test could be constructed using such results, and the correspondence [17] between stability of the n-skeleton and the chain invoked.

However, even in such cases much simplification may well be afforded by the use of state-dependent drift criteria here.

We develop two more detailed applications below, one for a countable space model and one for a continuous space model. Both of these seem difficult to analyse directly with a fixed step criterion. We also mention two rather more complex applications in queueing models.

(*a*) *An invasion/antibody model.* We first analyze the positive recurrence of an invasion/antibody model on a countable space.

Models for competition between two groups can be modelled as bivariate processes on the integer-valued quadrant $\mathbb{Z}_+^2 = \{i, j \in \mathbb{Z}_+\}$. Consider such a process in discrete time with the first coordinate process $\Phi_n(1)$ denoting the numbers of invaders and the second coordinate process $\Phi_n(2)$ denoting the numbers of defenders.

Here we have in mind the situation where the defenders and invaders mutually tend to steadily reduce the numbers of the opposition when both groups are present, even though "reinforcements" may join either side; so we have (at least for $i, j > 1$) that, for some $\varepsilon_i, \varepsilon_j \geq \varepsilon > \frac{1}{2}$,

$$\mathsf{E}_{i,j}[\Phi_1(1) + \Phi_1(2)] \leq (i - \varepsilon_i) + (j - \varepsilon_j) \leq i + j - 2\varepsilon, \qquad i, j > 1. \tag{47}$$

Such a behaviour might model, for example, antibody action against invasive bodies where there is physical attachment of at least one antibody to each invader and then both die; in such a context we would have $\varepsilon_i = \varepsilon_j = 1$.

Analysis of this model in the interior of the space is not difficult. By using (4) with $V(i, j) = [i + j]/\varepsilon$ on $I = \{i, j \geq 1\}$, we have [18] that $\mathsf{E}_{i,j}[\tau_{I^c}] < (i + j)/\varepsilon$. The difficulty with such multidimensional models is that even though they reach I^c in a finite mean time, they may then "escape" along one or both of the boundaries. It is in this region that the tools of Theorem 2.1 are useful in assisting with the classification of the model.

We define the boundary action of the invader/antibody model by assuming the following.

(A1) When the defender numbers drop to 0, if the invaders are above a threshold level d the body dies, in which case the invaders also die and the chain drops to (0, 0), so that

$$P((i, 0), (0, 0)) = 1, \qquad i > d; \tag{48}$$

otherwise a new population of antibodies or defenders of finite mean size is generated. These assumptions are of course somewhat unrealistic and clearly with more delicate arguments can be made much more general if required.

(A2) Much more critically, when the invaders fall to level 0, and the defenders are of size $j > 0$, a new "invading army" is raised to bring the invaders to size N, where N is a random variable concentrated on $\{j + 1, j + 2, \ldots, j + d\}$ for the same threshold d, so that

$$\sum_{k=1}^{d} P((0, j), (j + k, j)) = 1; \tag{49}$$

this distribution being concentrated above j represents the physically realistic concept that a new invasion will fail if the invading population is not at least the size of the defending population. The bounded size of the increment is purely for convenience of exposition.

Note that the chain is $\delta_{(0,0)}$-irreducible under the assumptions A1 and A2, regardless of the behaviour at zero. Thus the model can allow for a stationary distribution at (0, 0) (i.e., extinction) or for rebirth and a more generally distributed stationary distribution over the whole of $\mathbb{Z}_2^+$. The only restriction we place in general is that the increments from (0, 0) have finite mean.

Let us, to avoid unrewarding complexities, add to (47) the additional condition that the model is "left-continuous," that is, has bounded negative increments defined by

$$P((i, j), (i - l, j - k)) = 0, \qquad i, j > 0, \quad k, l \geq 1; \tag{50}$$

this may be appropriate if the chain is embedded at the jumps of a continuous time process, for example.

The marginal behaviour of the "invaders" is modelled here to a large extent on the rabies model developed in [1], although the need to be the same order of magnitude as the antibody group is a weaker assumption than that implicit in the continuous time continuous space model there.

We use the test function $V(i, j) = [i + j]/\beta$, where β is to be chosen. Then as remarked above, on the interior $I = \{i, j \geq 1\}$ we have that (10) holds with $n = 1$ in the usual way, provided $\beta < \varepsilon$.

Starting at $B_1(c) = \{(i, 0), i > c\}$, the infinite boundary edge above c, we have that the value of $V(\Phi_1)$ is zero if $c > d$, so that (10) also holds with $n = 1$, provided we choose $c > \max(d, \beta^{-1})$.

On the other infinite boundary edge, denoted $B_2(c) = \{(0, j), j > c\}$, however, we have *positive* one-step drift of the function V. Now from the starting point $(0, j)$, let us consider the $(j + 1)$-step drift. This is bounded above by $[j + d - 2j\varepsilon]/\beta$ and so we have that (10) also holds with $n(j) = j + 1$, provided

$$[j + d - 2j\varepsilon]/\beta < -j - 1,$$

which will hold provided $\beta < 2\varepsilon - 1$, and we then choose $c > (d + \beta)/(2\varepsilon - 1 - \beta)$.

Consequently we can assert that, writing $C = I \cup B_2(c) \cup B_1(c)$ with c satisfying both these constraints, the mean time

$$\mathsf{E}_{(i,j)}[\tau_C] \leq [i + j]/\beta$$

regardless of the threshold level d.

Thus by irreducibility the mean hitting time on (0, 0) is also finite and in this sense the invading strategy is successful in overcoming the defense.

Note that in this model there is no fixed time at which the drift from all points on the boundary $B_2(c)$ is uniformly negative, no matter what the value of c chosen. Thus, state-dependent drift conditions appear needed to analyse this model.

Under an assumption of uniformly bounded geometrically decreasing tails on the increment distributions of the invaders and defenders, it is possible to show that the chain is geometrically ergodic also, using Theorem 2.1(iii). Alternatively, this can be shown using the methods in [16]. We omit the details.

(*b*) *A nonlinear SETAR model.* Second, we illustrate the state-dependent methods by the analysis of geometric ergodicity of a nonlinear SETAR (self-exciting threshold autoregressive) model, where it is also not obvious what the structure of a one-step test function should be.

This model is defined by

$$X_n = \phi_j + \theta_j(X_{n-1})X_{n-1} + W_n(j), \qquad r_j < X_{n-1} \leq r_{j+1} \tag{51}$$

where $-\infty = r_1 < \cdots < r_M < r_{M+1} = \infty$ and $\{W_n(j)\}$ forms an i.i.d. zero-mean sequence for each j, independent of $\{W_n(i)\}$, for $i \neq j$.

The SETAR model with constant coefficients in each region is analyzed in increasing detail in a series of recent papers. Positive recurrence and transience results are essentially covered in [13] [2]; nonpositivity is analysed by [4].

Here we will illustrate the use of state-dependent drift conditions when the coefficients $\theta_j(x)$ are not necessarily constant. Our result will extend the positivity result of [13] and [2] to the geometrically ergodic situation, even in the constant coefficient case. For ease of exposition we will make a number of simplifying assumptions. For models not satisfying these assumptions a more detailed analysis will be needed, although the results will in general be valid. Let us then assume the following.

(S1) The "end" noise variables $W(1)$ and $W(M)$ are of finite range to the left.

(S2) For each $j = 1, \ldots, M$, the noise variable $W(j)$ has a density positive on the whole of its range, which is a nontrivial interval around zero.

(S3) The functions $\theta_j(x)$ are continuous functions of x, and the constant terms $\phi_j \equiv 0$.

We need to identify the petite sets for this model. We have the following proposition, in the nomenclature of [11] or [10].

PROPOSITION 4.1. *Under* (S2) *and* (S3) *the SETAR model* (51) *is φ-irreducible with φ taken as Lebesgue measure μ^{Leb} on $\mathbb{R}$, and the model admits an everywhere nontrivial continuous component. Thus if $\mathsf{P}_x(\tau_C < \infty) > 0$ for some compact set C and some x, then every compact set is petite.*

PROOF. The μ^{Leb}-irreducibility is immediate from the assumption of positive densities for each of the $W(j)$, and the fact that with $\phi_j \equiv 0$ and zero-mean noise variables, the chain can move from one region to the next with no constraints.

We next prove the existence of an everywhere nontrivial continuous component.

It is obvious from the existence of the densities and the continuity of the $\theta_j(x)$ that at any point in the interior of any of the regions (r_i, r_{i+1}) the transition function is strongly continuous. We do not necessarily have this continuity at the boundaries r_i themselves. However, as $x \uparrow r_i$ we have strong continuity of $P(x, \cdot)$ to $P(r_i, \cdot)$, while the limits as $x \downarrow r_i$ of $P(x, A)$ always exist, giving a limit measure $P'(r_i, \cdot)$ which may differ from $P(r_i, \cdot)$.

If we take $T_i(x, \cdot) = \min(P'(r_i, \cdot), P(r_i, \cdot), P(x, \cdot))$, then T_i is a continuous component of P at least in some neighborhood of r_i; the assumption that the densities of both $W(i)$ and $W(i+1)$ are positive everywhere guarantees that T_i is nontrivial.

Now we may put these components together using Proposition 3.2 of [11] and we have that the nonlinear SETAR model admits an everywhere nontrivial continuous component.

If the hitting times on one compact set are then finite with positive probability from some x, the chain is nonevanescent [11]; so from Theorem 3.2 of [11] we have that every compact set is petite. □

Note that this proof is rather simpler than that in [2] for linear SETAR models, to which it is also applicable.

We shall prove the following theorem.

THEOREM 4.2. *If the nonlinear chain* (51) *satisfies* (S1)–(S3) *and if also*

$$\theta_M(x) > 0, \qquad \sup_{x > r_M} \theta_M(x) < 1, \tag{52}$$

$$\sup_{x < r_2} \theta_1(x) < 0, \tag{53}$$

then the chain is geometrically ergodic.

PROOF. We will prove that the hitting times $\tau_{[-R_1,R_M]}$ have geometric tails using (12), for some sufficiently large R_1 and R_M. It will be convenient below to choose a nonsymmetric interval in applying this theorem. This establishes finiteness of $\tau_{[-R_1,R_M]}$ *a fortiori*, and in view of Proposition 4.1 this then further establishes geometric ergodicity as in Theorem 2.1(iii).

There are a number of simple stochastic comparison arguments which make our computations less messy.

First, we may assume that $\theta_M(x)$ is constant and is given by $\theta_M \triangleq \sup_{x>r_M}\theta_M(x) < 1$; clearly a chain with nonconstant $\theta_M(x)$ will hit the half-line $(-\infty, r_M]$ more rapidly than the chain with a constant coefficient and hence will hit the interval $[-R, R]$ more rapidly also for large R given our bounded noise assumption.

We may assume also that as $x \to -\infty$, we have $\liminf \theta_1(x) = -\infty$; otherwise we may use a stochastic monotonicity argument to show that the chain hits some set $[-R, R]$ at least as rapidly as a chain with fixed parameters

$$\theta_M = \sup_{x>r_M} \theta_M(x) < 1, \tag{54}$$

$$\theta_1 = \inf_{x<r_2} \theta_1(x) < 0, \tag{55}$$

and we know from [2] that this chain is geometrically ergodic [although note that while they establish that Theorem 2.1(iii) holds with $n(x) \equiv 1$, in this case they only claim the weaker ergodicity result of Theorem 2.1(ii) with $n(x) \equiv 1$].

Similarly, by stochastic monotonicity we may assume that $\theta_1(x)$ is monotone, so that as $x \to -\infty$, we have $\lim \theta_1(x) = -\infty$.

Note that this justifies the assumption that $\phi(1) = \phi(M) = 0$, for if the constant terms are nonzero, we can always choose different θ_1 and θ_M which satisfy the assumptions of the theorem and for which (outside some larger interval $[-R, R]$) the chain is always stochastically closer to $[-R, R]$. We may of course have to assume more then on the noise distributions to ensure irreducibility.

Lastly, since for any set C, changing the transition probabilities for $x \in C$ does not change the distribution of the hitting times on C from points outside C, let us assume that $\theta_j(x)$ is also given by θ_M for all $x > 0$, and that if h_M is the lower bound of the support of the variable $W(M)$, then for all $0 < x \le h_M/(1-\theta_M)$ the noise variable is degenerate at zero; that is, for such points we have $P(x, \theta_M x) = 1$.

Observe then that for each $x > h_M/(1-\theta_M)$ we have that $P(x,\cdot)$ is supported on $[\theta_M x - h_M, \infty)$. By iterating this we see that, defining

$$x_n = \theta_M^n x - h_M\left[\theta_M^{n-1} + \cdots + 1\right],$$

$P^n(x,\cdot)$ is supported on $[x_n, \infty)$ and in particular is therefore supported on $[\theta_M^n x - h_M/(1-\theta_M), \infty)$. By the degeneracy of the noise variables on $(0, h_M/(1-\theta_M))$ we thus ensure that for all $x > 0$ the motion of the chain is supported by $(0, \infty)$.

Having made these simplifications, we can choose the simple test function,

$$V(x) = |x|.$$

Now let us choose the test interval $[-R_1, R_M]$ so large that $-R_1 < r_1$, and also so that $-R_1 < 0$. Then, for any $R_M > 0$ and $x > R_M$, we have, for a fixed $\lambda \in (\theta_M, 1)$,

$$\int P(x, dy)V(y) = \theta_M V(x) < \lambda V(x). \tag{56}$$

If the noise variable $W(1)$ has support bounded below by h_1 then, for initial values $x < -R_1$, we have that $P(x, \cdot)$ is supported on $[|\theta_1(x)|\,|x| - h_1, \infty)$; we can obviously choose R_1 large enough that $|\theta_1(x)|\,|x| - h_1 > h_M$ for each such x, by our monotonicity assumptions.

Let us now choose

$$n(x) \geq 1 + \log\left(\frac{\theta_1(x)}{\theta_M}\right) \Big/ \log\left(\frac{\lambda}{\theta_M}\right). \tag{57}$$

Then for initial values $x < -R_1$, $P^{n(x)}(x, \cdot)$ is supported on $(0, \infty)$ and we have from (57) that

$$\begin{aligned} \int P^{n(x)}(x, dy)V(y) &= \left[\theta_M^{n(x)-1}\right]\theta_1(x)|x| \\ &\leq \lambda^{n(x)}|x|. \end{aligned} \tag{58}$$

Thus from (56) and (58) we have that (12) is satisfied and the chain is geometrically ergodic as claimed. □

By symmetry, under the parameter combinations

$$\theta_1(x) > 0, \qquad \sup_{x<r_2} \theta_1(x) < 1, \tag{59}$$

$$\sup_{x>r_M} \theta_M(x) < 0, \tag{60}$$

the chain is also geometrically ergodic if the drift to the right is bounded.

In [2] it is shown that if

$$\theta_1(x) \equiv \theta(1) < 1, \qquad \theta_M(x) \equiv \theta(M) < 1, \qquad \theta(1)\theta(M) < 1, \tag{61}$$

then since there exist positive constants a, b such that

$$\begin{aligned} 1 &> \theta(1) > -(b/a), \\ 1 &> \theta(M) > -(a/b), \end{aligned}$$

then by taking

$$V(x) = \begin{cases} ax, & x > 0, \\ b|x|, & x \leq 0, \end{cases}$$

it is possible to show that (12) holds with $n(x) = 1$ under (61) for all $|x|$ sufficiently large.

We have only developed the analogue of this with nonconstant coefficients when the signs of the coefficients are opposite; the situation if both coefficients are positive is obvious, but if both are negative so the chain essentially oscillates in sign, then the general situation with nonconstant coefficients is not so clear.

(*c*) *Operations research applications.* We conclude by indicating two other applications of the methods in this paper.

The random time criteria of Theorem 2.2 take on a simple form when ζ_x is independent of x and uniform on $\{0, 1, \dots, M\}$ for some M. In this case we have that (10), for example, can be written as

$$\int \frac{1}{M+1} \sum_{k=0}^{M} P^k(x, dy)V(y) \le V(x) - 1 + b\mathbf{1}_C(x). \tag{62}$$

A continuous time analogue of this specific formulation has been used in Meyn and Down [8] to show that the Jackson network model is ergodic in appropriate circumstances.

Although it is too complex to spell out here, we mention also that Sadowski and Szpankowski [15] have recently applied the methods developed here to analyse complex operations research models. By applying Theorem 2.1(ii), they establish positive Harris recurrence of a multiserver queueing model where there are i.i.d. interarrival times and i.i.d. batch sizes for arriving customers, and where each server may have a different service time distribution.

Thus in operations research the analysis of stability can be made simpler by the use of state-dependent drift conditions.

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