

EXTREME VALUE THEORY FOR A CLASS OF NONSTATIONARY TIME SERIES WITH APPLICATIONS¹

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Consider a class of nonstationary time series with the form $Y_t = \mu_t + \xi_t$ where $\{\xi_t\}$ is a sequence of infinite moving averages of independent random variables with regularly varying tail probabilities and different scale parameters. In this article, the extreme value theory of $\{Y_t\}$ is studied. Under mild conditions, convergence results for a point process based on the moving averages are proved, and extremal properties of the nonstationary time series, including the convergence of maxima to extremal processes and the limit point process of exceedances, are derived. The results are applied to the analysis of tropospheric ozone data in the Chicago area. Probabilities of monthly maximum ozone concentrations exceeding some specific levels are estimated, and the mean rate of exceedances of daily maximum ozone over the national standard 120 ppb is also assessed.

1. Introduction. In recent years, the study of extreme values in environmental time series has been receiving growing interest because of its wide applicability to the analysis of phenomena such as extreme ozone observations, floods, storm winds and extreme temperatures. Three types of extreme value limit distributions, first identified by Fisher and Tippett (1928), play a fundamental role in the analysis of extremes of environmental data. The extreme value theory of independent and identically distributed (iid) random variables is well documented. For example, a sequence of iid random variables with distribution function F belongs to the domain of attraction of the Type II extreme value distribution if and only if $1 - F(x)$ is regularly varying at ∞ with index $-\alpha$ (written $1 - F(x) \in RV_{-\alpha}$); that is, $\lim_{t \rightarrow \infty} (1 - F(tx))/(1 - F(t)) = x^{-\alpha}$ for $x > 0$.

The extreme value theory of stationary random sequences has been extensively studied, where most of the work has focused primarily on the extension of classical results to stationary settings [see, e.g., Leadbetter (1974), Hsing, Hüsler and Leadbetter (1988), Leadbetter and Hsing (1990), Davis and Resnick (1985, 1988, 1991)]. In practice, however, many environmental time series vary systematically in response to meteorological conditions, and therefore are often not stationary. Extreme value theory of nonstationary processes has been discussed under certain conditions. For instance, Horowitz (1980) considered the following model for daily ozone maxima $\{Y_t\}$:

$$(1.1) \quad \log(Y_t) = f(t) + \xi_t,$$

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where $f(t)$ is a deterministic part, such as a seasonal component or trend, and $\{\xi_t\}$ is a normal stationary autoregressive process. Ballerini and McCormick (1989) studied the limit theory for processes of the form $Y_t = f(t) + h(t)\xi_t$, where $h(\cdot)$ is positive and periodic and $\{\xi_t\}$ is a stationary process satisfying certain mixing conditions.

In this article, we study the limit theory for extreme values of a class of nonstationary time series with the form

$$(1.2) \quad Y_t = \mu_t + \xi_t, \quad \xi_t = \sum_{j=0}^{\infty} c_j Z_{t-j},$$

where $\{Z_t = \sigma_t \eta_t; -\infty < t < \infty\}$ and $\{\eta_t; -\infty < t < \infty\}$ is a sequence of iid random variables with regularly varying tail probabilities.

The rest of this article is organized as follows. In Section 2, background on point processes is briefly reviewed and some basic convergence results for a sequence of point processes based on moving average processes are proved. Extremal properties of the nonstationary moving-average processes, including the convergence of maxima to extremal processes and the limit point process of exceedances, are then derived from the convergence results. In Section 3, we apply the results to analyze ground-level ozone concentrations in the Chicago area. The probabilities of monthly ozone maxima exceeding some specific thresholds are estimated based on the limit distribution for maxima, and the mean rate of exceedances of daily maximum ozone over the national standard 120 ppb (parts per billion) is also assessed.

2. Convergence results for extremes of random variables with regularly varying tails. We now study limit theory for moving averages of independent random variables $\{Z_t = \sigma_t \eta_t; -\infty < t < \infty\}$ with regularly varying tail probabilities. Some convergence results for point processes based on one-sided moving averages $\{\xi_t = \sum_{j=0}^{\infty} c_j Z_{t-j}\}$ are derived. Extreme properties of the nonstationary sequence $\{Y_t\}$ are then obtained from these convergence results. In particular, we will show that the maximum of the sequence $\{Y_t\}$ converges weakly to an extremal process generated by a given extreme value distribution and that the exceedance point process converges weakly to a compound Poisson process. The techniques used in this section are similar to those used by Davis and Resnick (1985, 1988); see also Adler (1978) and Hsing, Hüsler and Leadbetter (1988).

For notation and background of point process theory, we follow Neveu (1976); see also Kallenberg (1983) and Resnick (1987). In this article the state space E is taken to be a subset of a compactified Euclidean space such as $\mathbb{R}^d = [-\infty, \infty]^d$. Let \mathcal{E} be the Borel σ -field of subsets of E . For $x \in E$ and $A \in \mathcal{E}$, define the measure ε_x on \mathcal{E} by

$$\varepsilon_x(A) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Let $\{x_i, i \geq 1\}$ be a countable collection of (not necessarily distinct) points of the space E . A point measure m_p is defined to be a measure of the form $m_p = \sum_{i=1}^{\infty} \varepsilon_{x_i}$ which is nonnegative integer-valued and finite on relatively compact subsets of E . The class of point measures is denoted by $M_p(E)$.

Poisson processes on (E, \mathcal{E}) play an important role in the study of extreme theory of random sequences. Let μ be a Radon measure on \mathcal{E} ; that is, $\mu(F) < \infty$ for every compact set $F \in \mathcal{E}$. A *Poisson process* or a *Poisson random measure* with mean measure μ is denoted by $PRM(\mu)$. If $E = \mathbb{R}^d$ and the mean measure $\mu = \lambda\nu_0$, where ν_0 is the Lebesgue measure on E and $\lambda > 0$, then $PRM(\mu)$ is called a *homogeneous* Poisson process and λ is called *rate* of the Poisson process.

Let $C_K^+(E)$ be the class of continuous and nonnegative-valued functions on E (i.e., $E \rightarrow [0, \infty]$) with compact support. For each $f \in C_K^+(E)$ and $\mu \in M_p(E)$, define $\mu(f) = \int f d\mu$. For a sequence of point measures $\{\mu_n\} \in M_p(E)$, we say that μ_n converges vaguely to $\mu \in M_p(E)$ (written $\mu_n \rightarrow_v \mu$) if $\mu_n(f) \rightarrow \mu(f)$ for all $f \in C_K^+(E)$.

We now state a basic convergence result for a point process based on independent nonidentically distributed random variables, which is an extension of a similar theorem for iid random variables and stationary processes [see, e.g., Resnick (1986), Hsing (1987)]. This result provides the link between nonstationary processes considered in this article and point processes. The proof of the following lemma is almost identical to that of Proposition 3.1 of Resnick (1986) and hence is omitted.

LEMMA 2.1. *Suppose for each $n \geq 1$, $\{W_{n,j}; j \geq 1\}$ is a sequence of independent nonidentically distributed random elements of (E, \mathcal{E}) and $\nu \times \mu$ is a Radon measure on the product space $([0, \infty) \times E, \mathcal{B} \times \mathcal{E})$. Define $N_n = \sum_{j=1}^{\infty} \varepsilon_{(j/n, W_{n,j})}$. Assume that N is a PRM on $[0, \infty) \times E$ with mean measure $\nu \times \mu$ and that $\sup_{j \geq 1} P\{W_{n,j} \in A\} \rightarrow 0$ for all compact sets A in \mathcal{E} . Then $N_n \Rightarrow N$ if and only if*

$$(2.1) \quad \sum_{j=1}^{\infty} \varepsilon_{j/n}(\cdot)P\{W_{n,j} \in \cdot\} \rightarrow_v \nu \times \mu.$$

Throughout this article, we assume that the scale parameter σ_t satisfies the condition $0 < \sigma_t \leq \delta_0 < \infty$. Similarly to the setting in Davis and Resnick (1985), we assume that

$$(2.2) \quad P\{|\eta_1| > x\} \in RV_{-\alpha}, \quad \alpha > 0$$

and

$$(2.3) \quad \lim_{x \rightarrow \infty} \frac{P\{\eta_1 > x\}}{P\{|\eta_1| > x\}} = \pi_0, \quad \lim_{x \rightarrow \infty} \frac{P\{\eta_1 < -x\}}{P\{|\eta_1| > x\}} = 1 - \pi_0,$$

where $0 \leq \pi_0 \leq 1$. Consider one-sided moving averages of the form

$$(2.4) \quad \xi_t = \sum_{j=0}^{\infty} c_j Z_{t-j} = \sum_{j=0}^{\infty} c_j \sigma_{t-j} \eta_{t-j}, \quad -\infty < t < \infty,$$

where $\{c_j\}$ is a sequence of real constants with $c_0 = 1$ and

$$(2.5) \quad \sum_{j=0}^{\infty} |c_j|^\gamma < \infty \quad \text{for some } 0 < \gamma < 1 \wedge \alpha.$$

Furthermore we assume that

$$(2.6) \quad \frac{1}{n} \sum_{t=1}^n \sigma_t^\alpha \rightarrow \sigma^\alpha \quad \text{as } n \rightarrow \infty,$$

where $\sigma > 0$.

The regular variation conditions in (2.2) and (2.3) lead to consideration of such state spaces as $[0, \infty]^d \setminus \{\mathbf{0}\}$ and $[-\infty, \infty]^d \setminus \{\mathbf{0}\}$ for some $d \geq 1$ where $\{\mathbf{0}\}$ is understood as the origin of the space \mathbb{R}^d . The compact sets of the space $[-\infty, \infty]^d \setminus \{\mathbf{0}\}$ are those compact sets in \mathbb{R}^d which are bounded away from $\{\mathbf{0}\}$. Let a_n be the $(1 - n^{-1})$ quantile of $|\eta_1|$; that is, let

$$(2.7) \quad a_n = \inf\{x: P\{|\eta_1| \leq x\} \geq 1 - n^{-1}\}$$

and let $W_{n,j} = a_n^{-1} Z_j$. We first prove the following lemma which is related to condition (2.1).

LEMMA 2.2. *Suppose that the independent random variables $\{\eta_t; -\infty < t < \infty\}$ satisfy the regular variation conditions specified in (2.2) and (2.3) and that the scale parameters σ_t satisfy the condition (2.6). Then*

$$(2.8) \quad \sum_{j=1}^{\infty} \varepsilon_{j/n}(\cdot) P\{W_{n,j} \in \cdot\} \rightarrow_v \nu_0 \times \mu$$

in the space $[0, \infty) \times [-\infty, \infty] \setminus \{0\}$, where ν_0 is Lebesgue measure on $[0, \infty)$ and

$$\mu(dx) = \sigma^\alpha (\pi_0 \alpha x^{-\alpha-1} dx \mathbf{1}_{(0, \infty)}(x) + (1 - \pi_0) \alpha (-x)^{-\alpha-1} dx \mathbf{1}_{[-\infty, 0)}(x)).$$

PROOF. For any $b > 0$ and $x > 0$, it suffices to show that

$$\sum_{j=1}^{\infty} \varepsilon_{j/n}([0, b)) P\{W_{n,j} \in (x, \infty]\} \rightarrow_v \nu_0([0, b)) \times \mu((x, \infty]).$$

By (2.2) and the definition of a_n , it is easy to show that

$$(2.9) \quad P\{|\eta_1| > a_n\} \sim 1/n \quad \text{only as } n \rightarrow \infty,$$

where $P\{|\eta_1| > a_n\} \sim 1/n$ means that $\lim_{n \rightarrow \infty} n P\{|\eta_1| > a_n\} = 1$. Since $0 < \sigma_t \leq \delta_0 < \infty$, from (2.3) we have

$$(2.10) \quad \lim_{n \rightarrow \infty} \frac{P\{|\eta_1| > a_n x / \sigma_j\}}{P\{|\eta_1| > a_n\} (x / \sigma_j)^{-\alpha}} = 1 \quad \text{uniformly for } j \geq 1.$$

Similarly by (2.3) we have

$$(2.11) \quad \lim_{n \rightarrow \infty} \frac{P\{\eta_1 > a_n x / \sigma_j\}}{P\{|\eta_1| > a_n x / \sigma_j\}} = \pi_0 \quad \text{uniformly for } j \geq 1.$$

Notice that

$$\sum_{j=1}^{\infty} \varepsilon_{j/n}([0, b])P\{W_{n,j} \in (x, \infty)\} = \sum_{j=1}^{[nb]} P\{\eta_1 > a_n x / \sigma_j\},$$

where $[nb]$ denotes the integer part of nb . Equations (2.9), (2.10) and (2.11) imply that

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{j=1}^{[nb]} P\{\eta_1 > a_n x / \sigma_j\} &= \lim_{n \rightarrow \infty} \sum_{j=1}^{[nb]} \frac{\pi_0(x/\sigma_j)^{-\alpha}}{n} \\ (2.12) \qquad \qquad \qquad &= \left(\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{[nb]} \sigma_j^\alpha \right) (\pi_0 x^{-\alpha}) \\ &= \nu_0([0, b]) \times \mu((x, \infty]). \end{aligned}$$

Similarly, for any $b > 0$ and $x < 0$, it can be shown that

$$(2.13) \quad \sum_{j=1}^{\infty} \varepsilon_{j/n}([0, b])P\{W_{n,j} \in [\infty, x]\} \rightarrow_{\nu} \nu_0([0, b])\mu([-\infty, x]).$$

Result (2.8) follows from (2.12) and (2.13). \square

Next we state a convergence result for a point process based on moving averages $\{\xi_t\}$ of the regularly varying random variables $\{Z_t = \sigma_t \eta_t; -\infty < t < \infty\}$.

THEOREM 2.1. *Suppose the independent random variables $\{\eta_t; -\infty < t < \infty\}$ satisfy the regular variation conditions specified in (2.2) and (2.3) and that the scale parameters σ_t satisfy the condition (2.6). Then in the space $M_p([0, \infty) \times [-\infty, \infty] \setminus \{0\})$,*

$$(2.14) \quad \sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1} Z_k)} \Rightarrow \sum_{k=1}^{\infty} \varepsilon_{(t_k, U_k)} \quad \text{as } n \rightarrow \infty,$$

where a_n is defined in (2.7) and $\sum_{j=1}^{\infty} \varepsilon_{(t_j, U_j)}$ is a PRM($\nu_0 \times \mu$) with the mean measure $\nu_0 \times \mu$ specified in Lemma 2.2. Furthermore, suppose that $\{\xi_t\}$ is the moving average sequence defined in (2.4) and that $\{c_j\}$ satisfies (2.5). Then in $M_p([0, \infty) \times [-\infty, \infty] \setminus \{0\})$, we have

$$(2.15) \quad \sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1} \xi_t)} \Rightarrow \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \varepsilon_{(t_k, c_j U_k)} \quad \text{as } n \rightarrow \infty.$$

PROOF. For any compact set $A \subset [-\infty, \infty] \setminus \{0\}$, there is a $\delta > 0$ such that $A \subset [-\infty, \delta] \cup [\delta, \infty]$. Hence we have

$$\sup_{j \geq 1} P\{W_{n,j} \in A\} \leq \sup_{j \geq 1} P\{|\eta_1| \geq a_n \delta / \sigma_j\}.$$

Since $a_n \rightarrow \infty$ and $0 < \sigma_t \leq \delta_0 < \infty$, by (2.2) we have $\sup_{j \geq 1} P\{W_{n,j} \in A\} \rightarrow 0$. Now (2.14) follows directly from Lemma 2.1 and Lemma 2.2. The proof of (2.15) is omitted. It is similar to that of Theorem 2.4 in Davis and Resnick (1985) with some minor changes since we consider the heteroscedastic sequence $\{Z_t = \sigma_t \eta_t; -\infty < t < \infty\}$ instead of iid random variables. \square

REMARK 1. To evaluate the structures of the limit point processes in (2.14) and (2.15), we may consider the following. Let $\{V_j, j \geq 1\}$ be the points of a homogeneous $PRM(\mu)$ on $[0, \infty)$ with rate σ^α , that is, with $\mu = \sigma^\alpha \nu_0$, and let $\{U_j^*, j \geq 1\}$ be iid random variables independent of $\{V_j, j \geq 1\}$ and with the density function

$$g_0(x) = (\pi_0 \alpha x^{-\alpha-1} dx \mathbf{1}_{(0, \infty)}(x) + (1 - \pi_0) \alpha (-x)^{-\alpha-1} dx \mathbf{1}_{[-\infty, 0)}(x)).$$

Then by Proposition 3.8 of Resnick (1987), we have

$$\sum_{j=1}^{\infty} \mathcal{E}(t_j, U_j) =_d \sum_{j=1}^{\infty} \mathcal{E}(V_j, U_j^*),$$

which implies that

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \mathcal{E}(t_k, c_j U_k^*) =_d \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \mathcal{E}(V_k, c_j U_k^*).$$

REMARK 2. The nonstationary aspect of the series $\{\xi_t\}$ in this study is reflected by the nonconstant scale parameters $\{\sigma_t\}$. In the limit point process, the scale parameter parameters are represented by the ‘‘average value’’ σ . In fact, the limiting point process in (2.15) is identical to the one obtained from a moving average $\sum_{j=0}^{\infty} (c_j \sigma \eta_{t-j})$ as in Davis and Resnick (1985). In practice, σ^α can be estimated by

$$\widehat{\sigma}^\alpha = \frac{1}{n} \sum_{t=1}^n \hat{\sigma}_t^{\hat{\alpha}},$$

where $\hat{\sigma}_t$ and $\hat{\alpha}$ are estimated based on data.

The following corollary extends the result in (2.15) to the process $\{Y_t\}$, and the proof of this result is omitted.

COROLLARY 2.1. *Suppose that the assumptions in Theorem 2.1 hold and that $\mu_t = E(Y_t)$ is a bounded function on $[0, \infty)$. Then in the space $M_p([0, \infty) \times [-\infty, \infty] \setminus \{0\})$,*

$$(2.16) \quad \sum_{k=1}^{\infty} \mathcal{E}(k/n, a_n^{-1} Y_k) \Rightarrow \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \mathcal{E}(t_k, c_j U_k) \quad \text{as } n \rightarrow \infty,$$

where a_n is defined in (2.7).

For the nonstationary time series Y_t in (1.2), define

$$(2.17) \quad X_n(t) = \begin{cases} a_n^{-1} \max_{1 \leq k \leq [nt]} Y_k, & \text{if } t \geq n^{-1}, \\ a_n^{-1} Y_1, & \text{if } 0 < t < n^{-1}. \end{cases}$$

Furthermore, set

$$c_+ = \max\{c_j \vee 0, 0 \leq j < \infty\}, \quad c_- = \max\{-c_j \vee 0, 0 \leq j < \infty\}.$$

Then based on Corollary 2.1 and the proof of Theorem 3.1 of Davis and Resnick (1985), we have that $X_n(t) \Rightarrow X(t)$ in the space $D(0, \infty)$, where $X(t)$ is an extremal process generated by the extreme value distribution $\exp\{-\sigma^\alpha(c_+^\alpha \pi_0 + c_-^\alpha(1 - \pi_0))x^{-\alpha}\}$ for $x > 0$. In particular, $X_n(1) = a_n^{-1} \max_{1 \leq k \leq n} Y_k \Rightarrow X(1)$, where $X(1)$ has an extreme value distribution.

We now study exceedances over high thresholds by the nonstationary time series $\{Y_{tj}\}$. For a given level u and $B \in \mathcal{B}([0, \infty))$, define

$$N_n(B) = \#\{k/n \in B: a_n^{-1}Y_k > u\}.$$

We will refer to N_n as the *exceedance point process* on $[0, \infty)$. Moreover, N_n can be expressed as

$$(2.18) \quad N_n = \sum_{k=1}^{\infty} \varepsilon_{k/n} \mathbf{1}_{[Y_k > a_n u]} = \sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1}Y_k)}(\cdot \times (u, \infty)).$$

Let $\{V_j, j \geq 1\}$ be the points of a homogeneous $PRM(\mu)$ on $[0, \infty)$ with rate $\sigma^\alpha u^{-\alpha}$ and let $\{U_j^*, j \geq 1\}$ be iid random variables independent of $\{V_j, j \geq 1\}$ and with common density function

$$(2.19) \quad g(x) = (\pi_0 \alpha x^{-\alpha-1} dx \mathbf{1}_{(u, \infty)}(x) + (1 - \pi_0) \alpha (-x)^{-\alpha-1} dx \mathbf{1}_{[-\infty, -u)}(x)) u^\alpha.$$

For convenience, we assume that $|c_j| \leq 1$. In this case $c_j U_k^* \in (x, \infty)$ implies $U_k^* \in (x, \infty)$ or $U_k^* \in (-\infty, -x)$. By Corollary 2.1 and Remark 1 after Theorem 2.1, we have

$$\sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1}Y_k)} \Rightarrow \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \varepsilon_{(V_k, c_j U_k^*)}.$$

Hence by the continuous mapping theorem

$$(2.20) \quad \begin{aligned} N_n &= \sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1}Y_k)}(\cdot \times (u, \infty)) \Rightarrow \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \varepsilon_{(V_k, c_j U_k^*)}(\cdot \times (u, \infty)) \\ &= \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \varepsilon_{V_k} \mathbf{1}_{[c_j U_k^* > u]} = \sum_{k=1}^{\infty} \zeta_k \varepsilon_{V_k}, \end{aligned}$$

where $\zeta_k = \sum_{j=0}^{\infty} \mathbf{1}_{[c_j U_k^* > u]}$ are iid random variables. The point process $\sum_{k=1}^{\infty} \zeta_k \varepsilon_{V_k}$ is a compound Poisson process with the points $\{V_k, k \geq 1\}$ as “centers” of clusters of exceedances and $\{\zeta_k, k \geq 1\}$ as “cluster size” random variables.

3. Applications. In this section, the results in Section 2 are applied to the analysis of tropospheric ozone data. Ground-level ozone arises as a consequence of the emissions of nitrous oxides and hydrocarbons into the atmosphere. Meteorological conditions, including daily temperature, relative humidity and wind speed and direction, also play an important role in determining the severity of ozone concentration.

In recent years, various statistical techniques have been used to estimate the influence of meteorological variables on ozone trends. Cox and Chu (1992) proposed a model for the daily maxima of hourly ozone concentrations based on the Weibull distribution, in which the scale parameter was allowed to vary as a function of meteorological conditions. Niu (1996) introduced a class of nonlinear additive time series models for daily maxima of ozone concentrations in which both mean levels and variances were nonlinear functions of relevant meteorological variables. As an alternative approach to analyze tropospheric ozone data, here we focus on estimating probabilities of monthly maximum ozone observations exceeding some specific levels and calculating the mean rate of exceedances of daily maximum ozone over the national standard level 120 ppb.

Specifically, let $\{\xi_t, 1 \leq t \leq n\}$ be daily maximum values of hourly ozone readings in a specific area. We consider the following autoregressive model for $\{\xi_t\}$:

$$(3.1) \quad \xi_t - \sum_{j=1}^p \phi_j \xi_{t-j} = Z_t,$$

where $\{Z_t = \sigma_t \eta_t\}$ and $\{\eta_t; -\infty < t < \infty\}$ is assumed to be a sequence of iid random variables with the Type II extreme value distribution

$$(3.2) \quad F_{|\eta_t|}(x) = \exp\{-(x)^{-\alpha}\} \quad \text{for } x > 0 \text{ and } \alpha > 0.$$

Furthermore, the scale parameter σ_t is modeled as a nonlinear function of meteorological variables of the form

$$(3.3) \quad \sigma_t = \exp\left\{\beta_0 + \sum_{j=1}^m \beta_j x_{tj}\right\}.$$

Let $\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$. When all roots of $\phi(z) = 0$ lie outside the unit circle, ξ_t can be expressed in the form

$$\xi_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j},$$

where the coefficients ψ_j decrease exponentially as j tends to infinity. Hence for any $\gamma > 0$, $\sum_{j=0}^{\infty} |\psi_j|^\gamma < \infty$ and condition (2.5) is satisfied.

As an illustration, we now apply the model specified in (3.1)–(3.3) to analyze tropospheric ozone concentrations in the Chicago Metropolitan Statistical Area where ozone levels have been historically high. Daily maxima of hourly ozone observations over 42 stations in the area were obtained from the Technical Support Division, U. S. Environmental Protection Agency. The observations

were over the period 1983–1992 and limited to the seven-month ozone “season” of 1 April to 31 October, during which daily maximum ozone levels were likely to be near or above 120 ppb. Figure 1a shows daily maximum ozone series for the ten-year period, from which we can see that the national standard was exceeded many times in each of the ten years. In fact, there were 113 daily maxima of ozone concentrations which exceeded 120 ppb during the ten-year period, and these exceedances are shown in Figure 1b. The average of the daily maximum ozone values for the ten-year period is $\mu = 69$. For this analysis we will use the mean corrected series $\{\xi_t = Y_t - \mu\}$ where $\{Y_t\}$ is the daily maximum ozone series.

There are many meteorological variables that are potentially important predictors of daily ozone levels. In this study, the following eight meteorological variables will be used: daily maximum surface temperature (Temp), morning average wind speed (MWS), afternoon average wind speed (AWS), relative humidity (Humidity), opaque cloud cover (Cloud), morning mixing height (MMH), morning average wind direction (MWD) and afternoon average wind direction (AWD). Among the eight variables, morning mixing height is the height below which the surface pollutants are free to mix up. The original measurements of wind directions were from 0° to 360° , and the *cosine* transformation is performed on the morning and afternoon average wind directions.

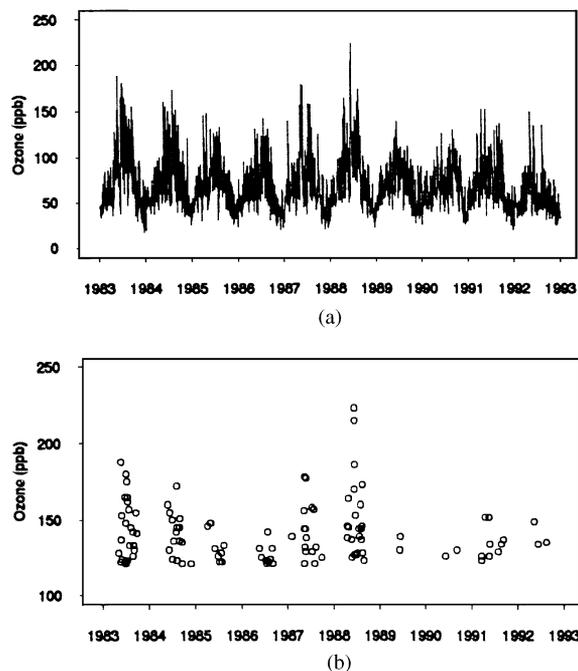


FIG. 1. (a) Daily maximum ground-level ozone observations in the Chicago area; (b) daily maximum ozone values over the national standard 120 ppb.

3.1. *Parameter estimation.* To build models for the daily maximum ozone series, we use the maximum likelihood method to estimate the unknown parameters in (3.1), (3.2) and (3.3). Define

$$\boldsymbol{\phi} = (\phi_1, \dots, \phi_p)', \quad \boldsymbol{\beta} = (\beta_0, \dots, \beta_m)', \quad \boldsymbol{\xi}_n = (\xi_1, \dots, \xi_n)'.$$

The likelihood function of $\boldsymbol{\xi}$ can be written in the form

$$L(\alpha, \boldsymbol{\phi}, \boldsymbol{\beta}|\boldsymbol{\xi}) = p(\xi_1, \dots, \xi_p)L^*(\alpha, \boldsymbol{\phi}, \boldsymbol{\beta}|\boldsymbol{\xi}),$$

where $p(\xi_1, \dots, \xi_p)$ denotes the joint density function of the first p random variables of the series and $L^*(\alpha, \boldsymbol{\phi}, \boldsymbol{\beta}|\boldsymbol{\xi})$ is called the conditional likelihood function, which has the form

$$L^*(\alpha, \boldsymbol{\phi}, \boldsymbol{\beta}|\boldsymbol{\xi}) = \alpha^{n-p} \prod_{t=p+1}^n \left[\left| \xi_t - \sum_{j=1}^p \phi_j \xi_{t-j} \right|^{-\alpha-1} \sigma_t^\alpha \right] \times \exp \left\{ - \sum_{t=p+1}^n \left(\frac{|\xi_t - \sum_{j=1}^p \phi_j \xi_{t-j}|}{\sigma_t} \right)^{-\alpha} \right\}.$$

When the sample size n is large, the likelihood function $L(\alpha, \boldsymbol{\phi}, \boldsymbol{\beta}|\boldsymbol{\xi})$ will be dominated by the conditional likelihood function, and the influence of $p(\xi_1, \dots, \xi_p)$ will be negligible. In this study, the parameters α , $\boldsymbol{\phi}$ and $\boldsymbol{\beta}$ will be estimated by maximizing the conditional log-likelihood function $l^*(\alpha, \boldsymbol{\phi}, \boldsymbol{\beta}|\boldsymbol{\xi})$.

For $0 < \alpha < 2$, the variance of η_t is infinite. Asymptotic properties of various types of parameter estimators in time series models with infinite variance have been discussed extensively in literature. For example, when $\{Z_1, \dots, Z_n\}$ in model (3.1) are iid random variables and in the domain of attraction of a stable law with index $\alpha \in (0, 2)$, Knight (1987) proved that least squares estimates of the autoregressive parameters are strongly consistent and have a very fast rate of convergence, and Davis, Knight and Liu (1992) showed that nondegenerate limit laws exist for M -estimates if the loss function is sufficiently smooth. For the model specified in (3.1)–(3.3), asymptotic properties of the maximum likelihood estimates of parameters, including consistency, limit distribution and robustness, have not been addressed yet; these will be studied in another paper.

For the daily maxima ozone series in the Chicago area, the model defined in (3.1)–(3.3) is fitted. The number of parameters in the model is chosen by using the Schwarz Bayesian criterion [Schwarz (1978)] which is similar to the well-known Bayesian criterion (BIC). Assume that a statistical model of M parameters is fitted to a data set. Then the Schwarz Bayesian criterion for the fitted model is defined as

$$SBC(M) = -2 \log(L(\hat{\alpha}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\beta}}|\boldsymbol{\xi})) + M \log(n).$$

In this study, the conditional likelihood function $L^*(\alpha, \boldsymbol{\phi}, \boldsymbol{\beta}|\boldsymbol{\xi})$ is used in $SBC(M)$ instead of $L(\alpha, \boldsymbol{\phi}, \boldsymbol{\beta}|\boldsymbol{\xi})$. The final selected model for $\{\xi_t\}$ is the following AR(1) model:

$$(3.4) \quad \xi_t = 0.804 \xi_{t-1} + Z_t \quad \text{for } 2 \leq t \leq 2140,$$

TABLE 1
Estimated tail probabilities $1 - \hat{F}(x)$ based on $\hat{F}_{|\eta_t|}(x)$ and $\{\hat{\eta}_t\}$

x	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0
Based on $\hat{F}_{ \eta_t }(x)$	0.632	0.277	0.154	0.100	0.071	0.053	0.042	0.034
Based on $\{\hat{\eta}_t\}$	0.689	0.294	0.163	0.107	0.077	0.057	0.048	0.035

where the estimate value for α is $\hat{\alpha} = 1.625$. Four out of the eight meteorological predictors, Temp, Cloud, MWS and AWS, are found to have significant impacts on the scale parameters. Based on the fitted model, the estimated scale parameters $\{\hat{\sigma}_t\}$ are calculated. The limit $\sigma^\alpha = \lim_{n \rightarrow \infty} \sum_{t=1}^n \sigma_t^\alpha / n$ is estimated by $\hat{\sigma}^{\hat{\alpha}} = \sum_{t=1}^{2140} \hat{\sigma}_t^{\hat{\alpha}} / 2140 = 17.11$.

The standardized residuals are defined by

$$\hat{\eta}_t = \hat{Z}_t / \hat{\sigma}_t \quad \text{for } 2 \leq t \leq 2140.$$

The estimated distribution function for η_t based on the fitted model is

$$(3.5) \quad \hat{F}_{|\eta_t|}(x) = \exp\{-(x)^{-1.625}\}.$$

Tail probabilities $1 - F(x)$ for some given x values are estimated based on $\hat{F}_{|\eta_t|}(x)$ and the standardized residuals. The two sets of probability estimates, listed in Table 1, coincide with one another well, especially for $x \geq 2$. Therefore we conclude that the assumption in (3.2) about the distribution of η_t is reasonable.

3.2. Exceedance probabilities. One important problem in tropospheric ozone data analysis is to estimate probabilities that some specific levels are exceeded by maximum ozone concentrations. Based on the fitted model in (3.4), the mean-corrected daily maximum ozone series $\{\xi_t\}$ in the Chicago area can be approximated by the following process:

$$(3.6) \quad \hat{\xi}_t = \sum_{j=0}^{\infty} \hat{\phi}_1^j \hat{Z}_{t-j}.$$

It is obvious that the series $\{\hat{\xi}_t\}$ satisfies the conditions specified in Section 2. Since $\hat{\phi}_1 = 0.804$, we have $\hat{c}_j = 0.804^j$, $\hat{c}_+ = 1$ and $\hat{c}_- = 0$.

For a given level x , the probabilities of η_1 and $|\eta_1|$ exceeding x can be estimated by the relative frequencies

$$\hat{P}\{\eta_1 > x\} = \frac{\text{Number of } \{t: \hat{\eta}_t > x\}}{2139}$$

and

$$\hat{P}\{|\eta_1| > x\} = \frac{\text{Number of } \{t: |\hat{\eta}_t| > x\}}{2139},$$

TABLE 2
 Frequencies of some given high levels exceeded by the sequences $\{\hat{\eta}_t\}$ and $\{|\hat{\eta}_t|\}$

Level x	Number of $\{t: \hat{\eta}_t > x\}$	Number of $\{t: \hat{\eta}_t > x\}$	Ratio of the frequencies
8.0	44	93	0.473
8.5	37	81	0.457
9.0	32	68	0.471
9.5	26	58	0.448
10.0	23	48	0.479
10.5	18	38	0.474
11.0	16	32	0.500
11.5	14	27	0.519
12.0	13	25	0.520

respectively. The frequencies of some given high levels exceeded by the sequences $\{\hat{\eta}_t\}$ and $\{|\hat{\eta}_t|\}$ are calculated and listed in Table 2. Ratios of the frequencies, which can be used as estimates for the probability ratios $P\{\eta_1 > x\}/P\{|\eta_1| > x\}$, are listed in the fourth column of Table 2.

In this study, the limit probability ratio $\pi_0 = \lim_{x \rightarrow \infty} P\{\eta_1 > x\}/P\{|\eta_1| > x\}$ is estimated by the average of the nine frequency ratios listed in the fourth column of Table 2; that is,

$$\hat{\pi}_0 = \frac{1}{9} \sum \hat{P}\{\eta_1 > x\} / \hat{P}\{|\eta_1| > x\} = 0.482.$$

According to the results in Section 2, $X_n(1) = a_n^{-1} \max_{1 \leq k \leq n} \xi_k$ has approximately the extreme value distribution

$$(3.7) \quad F_0(x) = \exp\{-17.11 \times 0.482 \times x^{-1.625}\} \quad \text{for } x > 0.$$

We now estimate the probabilities of monthly maximum ozone concentrations over the national standard 120 ppb and other thresholds in the Chicago area. From the definition (2.7), the normalizing constant a_n for the extreme value distribution given in (3.2) is

$$a_n = \left[\frac{1}{\log(n/n-1)} \right]^{1/\alpha}.$$

For $n = 30$ and $\hat{\alpha} = 1.625$, we have $\hat{a}_n \approx 9.2$ and

$$(3.8) \quad \begin{aligned} P\left(\max_{1 \leq k \leq n} Y_k > x\right) &= P\left(\max_{1 \leq k \leq n} \xi_k > x - 69\right) \\ &= P\left(a_n^{-1} \max_{1 \leq k \leq n} \xi_k > a_n^{-1}(x - 69)\right) \\ &\approx 1 - \exp\left\{-17.11 \times 0.482 \times \left(\frac{x - 69}{9.2}\right)^{-1.625}\right\}. \end{aligned}$$

The last term in (3.8) can be used to estimate the exceedance probability of monthly maximum ozone over a given level x .

TABLE 3

Estimated probabilities and relative frequencies of some specific thresholds exceeded by monthly maximum ozone in the Chicago area

Threshold (ppb)	Estimated exceedance probability	REF(x)
120	0.400	0.471
125	0.355	0.442
130	0.317	0.386
135	0.285	0.300
140	0.258	0.243
145	0.234	0.214
150	0.214	0.191
155	0.196	0.169
160	0.181	0.140
165	0.167	0.126
170	0.155	0.113
175	0.144	0.100
180	0.134	0.086

Based on the daily maximum ozone observations, 70 monthly maximum values during the ten-year period are calculated. Denote the monthly maximum ozone series by $\{M_t\}$. Then the relative exceedance frequency (REF) of the monthly ozone maxima over a specific threshold x is defined by

$$(3.9) \quad \text{REF}(x) = \frac{\text{Number of } \{t: M_t > x\}}{70}.$$

The quantity $\text{REF}(x)$ can be used as an alternative estimate for the probability of monthly maximum ozone values over the the given threshold x .

Table 3 lists the estimated exceedance probabilities and the relative exceedance frequencies of monthly maximum ozone observations in the Chicago area over some specific thresholds. Compared with the relative exceedance frequencies, the estimated exceedance probabilities based on the fitted model are lower for thresholds 120–135 ppb but slightly higher for thresholds 140 and up. In particular, the estimated probability of monthly ozone maxima over the national standard 120 ppb is about 0.4 based on the fitted model, which implies that about three monthly maxima will exceed the national standard in the seven-month ozone season of a given year.

3.3. Exceedances. We now examine the exceedances of daily maximum ozone observations over the national standard 120 ppb in the Chicago area. The 113 exceedances during the ten-year period are shown in Figure 1b. Consecutive exceedances are said to form a cluster. For convenience, an isolated exceedance is also called a cluster. Based on this definition, the 113 exceedances are divided into 79 clusters, and the average cluster size is about 1.43. The longest cluster of exceedances occurred in October 1988, where six consecutive daily maximum ozone observations exceeded the national standard.

According to the results in Section 2, cluster center points of the exceedances are from a homogeneous $PRM(\mu)$ with rate $\sigma^\alpha u^{-\alpha}$ and cluster sizes $\{\zeta_k = \sum_{j=0}^\infty \mathbf{1}_{[c_j U_k^* > u]}\}$ are iid random variables where $\{U_k^*, k \geq 1\}$ is also a sequence of iid random variables with the common density specified in (2.19). Based on these results and the fitted model, the mean value of the cluster sizes and the mean rate of cluster center points of the exceedances can be estimated. Specifically, the mean value of the cluster size is

$$\begin{aligned} E\zeta_k &= \sum_{j=0}^\infty P(c_j U_k^* > u) = \sum_{j=0}^\infty \int_{u/c_j}^\infty g(x) dx \\ &= \sum_{j=0}^\infty [\pi_0 \alpha u^\alpha] \int_{u/c_j}^\infty x^{-\alpha-1} dx = \pi_0 \sum_{j=0}^\infty c_j^\alpha. \end{aligned}$$

For $\pi_0 = 0.482$, $\hat{c}_j = 0.804^j$ and $\hat{\alpha} = 1.625$, we have $E\zeta_k \approx 0.482/(1 - 0.804^{1.625}) = 1.61$; that is, the mean value of cluster size based on the fitted model is estimated to be about 1.6.

Similarly, the rate at which clusters occur, $\mu = EV_k = \sigma^\alpha u^{-\alpha}$, can be estimated by $\hat{\sigma}^{\hat{\alpha}} u^{-\hat{\alpha}}$. Notice that $Y_t > 120$ is equivalent to $a_n^{-1} \xi_t > a_n^{-1} \times 51$ and $u^{-\alpha} = (a_n^{-1} 51)^{-\alpha} = a_n^\alpha / 51^\alpha$. For $n = 2140$ and $\hat{\alpha} = 1.625$, we have $a_n^{\hat{\alpha}} = 2139.5$ and $u^{-\hat{\alpha}} \approx 3.593$. Since $\hat{\sigma}^{\hat{\alpha}} = 17.11$, the mean rate of cluster center points of exceedances is estimated by $\hat{\sigma}^{\hat{\alpha}} u^{-\hat{\alpha}} = 61.48$. Furthermore, the mean number of exceedances can be estimated by

$$E[N([0, 1])] = \mu E(\zeta_k) \approx 61.48 \times 1.61 \approx 99.0$$

Compared with the observed 113 exceedances over the national standard during the ten-year period, the mean number 99.0 based on the fitted model is a slightly lower estimate.

Extreme value theory is an elegant and mathematically fascinating theory which pervades an enormous variety of applications. In this article, some basic results of extreme value theory for a special class of nonstationary processes are derived and applied to the analysis of tropospheric ozone concentrations in the Chicago area. The results here can be extended to infinite moving averages of random variables which belong to the domain of two other types of extreme value distributions. Applications of these results to the analysis of other environmental time series will also be pursued.

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